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CONVEXITY AND SYMMETRY OF TRANSLATING SOLITONS IN MEAN CURVATURE FLOWS***

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Abstract

This paper proves that any rotationally symmetric translating soliton of mean curvature flow in R^3 is strictly convex if it is not a plane and it intersects its symmetric axis at one point. The authors also study the symmetry of any translating soliton of mean curvature flow in R^n.

Keywords Convexity, Soliton, Mean curvature flow, Rotationally symmetric surface
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§ 1. Introduction

Let $T \in R^{n+1}$ be a unit vector and $f : M \to R^{n+1}$ a smooth immersion. The surface $M_0 = f(M)$ is called a translating solition of mean curvature flow (MCF), if it is an n-dimensional smooth manifold (without boundary) and satisfies the equation

$$-\langle T, \nu(f(p))\rangle = H(f(p)), \qquad p \in M, \tag{1.1}$$

where ν and H are the outer unit normal and the mean curvature, respectively, at the point $f(p)$ on the surface $M_0 = f(M)$, and $\langle\, ,\, \rangle$ denotes the usual Euclidean inner product. Here, the signs are chosen such that the mean curvature of a convex surface is nonnegative and $\vec{H} = -H\nu$ is the mean curvature vector.

Let $F(\cdot, t) = f(\cdot) + tT$. Then it follows from (1.1) that

$$-\langle \partial_t F, \nu(F)\rangle = H(F). \tag{1.2}$$

This means that the one-parameter family of $M_t = F(M, t)$ is a solution of MCF

$$\frac{\partial F}{\partial t}(p, t) = -H(p, t)\nu(p, t), \qquad p \in M,\ t \in (0, T) \tag{1.3}$$

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up to a tangential diffeomorphism. Therefore, a translating soliton is the solution of MCF which moves by vertical translating along T-direction.

It is proved in [1–3] that a type II singularity of a MCF evolved by a mean convex initial surface is characterized by a complete convex translating soliton. Moreover, it is believed that such a soliton should be rotationally symmetric (see [3, Conjecture 2]). In a very recent paper [4], Wang showed the conjecture in the case $n = 2$ and claimed that this conjecture might be wrong if $n \geq 3$. A natural question is that whether any 2-dimensional translating soliton is strictly convex (see the open problem in [4]) and what is the optimal condition for the symmetry of higher dimensional translating soliton.

In this paper, we try to study these problems and will prove the following results.

Theorem 1.1. *Let Σ be a rotationally symmetric translating soliton of MCF in R^3. If Σ intersects its symmetric axis at one point, then either Σ is a plane orthogonal to its symmetric axis, or Σ is strictly convex, and $\Sigma = \{(x, \rho(|x|)) : x \in R^2\}$ up to a rigid motion, where $\rho \in C^\infty[0, \infty)$ satisfies*

$$\frac{\rho''(t)}{1 + (\rho'(t))^2} + \frac{\rho'(t)}{t} = 1, \qquad t \in (0, \infty) \tag{1.4}$$

and

$$\rho''(t) > 0 \quad \text{for } t \in (0, \infty), \qquad \rho(0) = \rho'(0) = 0. \tag{1.5}$$

Moreover, the function $u(x) = \rho(|x|)$ satisfies

$$\frac{|x|}{2} \leq |\nabla u(x)| \leq |x|, \qquad \forall x \in R^2 \tag{1.6}$$

and

$$\frac{|x|^2}{4} \leq u(x) \leq \frac{|x|^2}{2}, \qquad \forall x \in R^2. \tag{1.7}$$

Theorem 1.2. *Suppose $n \geq 2$. Then we have the following conclusions.*

(1) *Any n-dimensional strictly convex and complete translating soliton of MCF can be represented as a graph $\{(x, u(x)) : x \in \Omega\}$ for an unbounded convex domain $\Omega \subset R^n$ and a strictly convex, smooth function u satisfying equation*

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = \frac{1}{\sqrt{1 + |\nabla u|^2}}. \tag{1.8}$$

(2) *Suppose that $u \in C^2(R^n)$ satisfies (1.8) in R^n. If there are a sequence $r_k \to \infty$ and a point $p \in R^n$ such that*

$$\sup_{k \geq 1} [u(x^k) - u(y^k)] \geq 0 \tag{1.9}$$

for any sequences $\{y^k\}$ *and* $\{x^k\}$, $y^k \in B_{r_k}(p) = \{x \in R^n : |x-p| < r_k\}$ *and* $x^k \in \partial B_{r_k}(p)$, *the boundary of* $B_{r_k}(p)$, *which satisfy* $\lim_{k\to\infty} \frac{|y^k|}{|x^k|} = 1$, *then* u *must be rotationally symmetric about the point* p, *i.e.,* $u(x) = U(|x-p|)$ *for all* $x \in R^n$ *and some function* $U \in C^2[0,\infty)$.

Corollary 1.1. *If* (1.9) *is replaced either by*

$$\lim_{k\to\infty}[u(r_k x + p) - u(r_k y + p)] = 0 \tag{1.10}$$

uniformly for $x, y \in S^{n-1} = \partial B_1(0)$, *or by*

$$\lim_{k\to\infty}[u(r_k x + p)) - g(r_k)] = 0 \tag{1.11}$$

uniformly for $x \in S^{n-1}$, *where* g *is some function defined in* $[0,\infty)$, *then Theorem* 1.2(2) *is still true, i.e.,* u *must be rotationally symmetric about the point* p.

Properties of similar problems were studied in [5–8].

§ 2. Proof of Theorem 1.1

Let $\Sigma \subset R^3$ be a rotationally symmetric translating soliton of MCF along the unit direction T. Choose a coordinate system for R^3 such that one of the coordinate axis, say x_1-axis for example, is the symmetric axis of Σ. Let $T = (x_0, y_0, z_0)$ be the unit vector. If Σ is not a plane orthogonal to the x_1-axis, then we can parameterize Σ by

$$f : [a,b) \times [0, 2\pi) \mapsto R^3, \quad f(t,\theta) = (t, r(t)\cos\theta, r(t)\sin\theta),$$

where $-\infty < a < b \le +\infty$, $r \in C^\infty(a,b)$, $r(a) = 0$, $r > 0$ in (a,b). Here we have chosen the line $\overline{ab}$ as the symmetric axis and assumed that $f(a,0)$ is the intersection point of Σ and its symmetric axis. Note that the outer unit normal of Σ

$$\nu(f) = \frac{\dfrac{\partial f}{\partial\theta} \times \dfrac{\partial f}{\partial t}}{\left|\dfrac{\partial f}{\partial\theta} \times \dfrac{\partial f}{\partial t}\right|} = \frac{(-r', \cos\theta, \sin\theta)}{[1+(r')^2]^{\frac{1}{2}}},$$

the metric tensor

$$g = \begin{pmatrix} 1+(r')^2 & 0 \\ 0 & r^2 \end{pmatrix},$$

and the second fundamental form

$$A = \begin{pmatrix} \frac{-r''}{[1+(r')^2]^{\frac{1}{2}}} & 0 \\ 0 & \frac{r}{[1+(r')^2]^{\frac{1}{2}}} \end{pmatrix}.$$

Then we see that the two principal curvatures of Σ are

$$\lambda_1 = \frac{-r''}{[1+(r')^2]^{\frac{3}{2}}}, \qquad \lambda_2 = \frac{1}{r[1+(r')^2]^{\frac{1}{2}}} \tag{2.1}$$

and the mean curvature

$$H=\lambda_1+\lambda_2=\frac{1+(r')^2-rr''}{r[1+(r')^2]^{\frac{3}{2}}}.$$

Therefore, by (1.1) we see that $y_0=z_0=0$ and $|x_0|=1$. Moreover, $H(f(t,p))>0$ when t near a. Thus $x_0=1$ again by (1.1) and, therefore, the equation for soliton Σ is

$$r(t)r''(t)+r(t)[r'(t)]^3-[r'(t)]^2+r(t)r'(t)-1=0, \qquad \forall\, t\in(a,b). \tag{2.2}$$

Furthermore, the strict convexity of the soliton Σ, which we want to prove below, is equivalent to

$$r''(t)<0, \qquad \forall\, t\in(a,b). \tag{2.3}$$

Since $r(a)=0$, $r'(a)=+\infty$ by the smoothness of Σ. Thus we may choose a small interval $[\alpha,\beta]\subset(a,b)$, near a sufficiently, such that $r''(t)<0$, $\forall\, t\in[\alpha,\beta]$.

Lemma 2.1. *Let*

$$\bar a=\inf\{t\in(a,b): r''(t)<0\}. \tag{2.4}$$

Then $r''(t)<0$ for all $t\in(\bar a,b)$.

Proof. Suppose the contrary that there is a zero point of r'' in $(\bar a,b)$. We use t_1 to denote the first zero point. Note that (2.2) can be rewritten as

$$r''=[1+(r')^2]\frac{1-rr'}{r} \qquad \text{in} \ \ (a,b). \tag{2.5}$$

Then

$$r'(t)>0, \qquad \forall\, t\in(\bar a,t_1) \tag{2.6}$$

since $r''(t)<0$ for all $t\in(\bar a,t_1)$. It follows from (2.5) and (2.4) that there exists $t_2\in(\bar a,t_1)$ such that

$$\frac{d(-r'')}{dt}\ge A(t)(-r'') \qquad \text{in} \ \ (t_2,t_1) \ \ \text{for} \ \ A(t)=\frac{r'-1-2r(r')^2}{r}. \tag{2.7}$$

Integration of (2.7) yields

$$\ln(-r''(t_1-\varepsilon))-\ln(-r''(t_2))=\int_{t_2}^{t_1-\varepsilon}\frac{d(-r'')}{-r''}\ge\int_{t_2}^{t_1-\varepsilon}A(t)dt.$$

Letting $\varepsilon\to0^+$ we obtain a contradiction.

Lemma 2.2. *Let $\bar a$ be defined by* (2.4). *Then $-\infty<a=\bar a<b=+\infty$.*

Proof. Suppose the contrary that $a<\bar a$. We conclude that

$$r''(t)>0, \qquad \forall\, t\in(a,\bar a). \tag{2.8}$$

Otherwise, there is $t_2 \in (a, \bar{a})$ such that $r''(t_2) = 0$ since $r''(t) \geq 0$ in $(a, \bar{a})$ by the definition of $\bar{a}$. Using (2.5) we see that

$$1 = r(t_2)r'(t_2) \tag{2.9}$$

and t_2 is a minimum point of the function $1 - rr'$ in the interval $(a, \bar{a})$. Hence

$$0 = (1 - rr')'(t_2) = -[r'(t_2)]^2 - r(t_2)r''(t_2) = -[r'(t_2)]^2,$$

contradicting (2.9).

However, (2.8) is impossible because of the fact $r'(a) = +\infty$ (by the smoothness of Σ at a). Therefore we have proved $-\infty < a = \bar{a}$.

In order to prove $b = +\infty$ we use (2.5) and Lemma 2.1 that

$$r''(t) < 0, \quad r'(t) > 0, \qquad \forall\, t \in (a, b). \tag{2.10}$$

Consequently, we have

$$+\infty \geq A := \lim_{t \to a^+} r'(t) > r'(t) > \lim_{t \to b^-} r'(t) := B \geq 0, \qquad \forall\, t \in (a, b). \tag{2.11}$$

Choose $t_0 \in (a, b)$. (2.10) yields

$$r(t) - r(t_0) \leq r'(t_0)(t - t_0), \qquad \forall\, t \in [t_0, b). \tag{2.12}$$

Now suppose that b is finite. Then $r(b) := \lim_{t \to b^-} r(t)$ is finite by (2.12). Therefore, $r(b) = 0$ and $r'(b) := \lim_{t \to b^-} r'(t) = -\infty$ since Σ is smooth and without boundary, contradicting (2.11). In this way, we have shown Lemma 2.2.

Proof of Theorem 1.1. (Continued). Combining Lemmas 2.1 and 2.2, we see that Σ is strictly convex. Furthermore, it follows from (2.5) that

$$\frac{d}{dt}(2t - r^2) < 0, \qquad \forall\, t \in (a, \infty),$$

which yields

$$r(t) > \sqrt{2(t - a) + r^2(a)} = \sqrt{2(t - a)}, \qquad \forall\, t \in (a, \infty). \tag{2.13}$$

Choosing a new coordinate system such that the translating direction $T = (0, 0, 1)$ and using (2.13) and the strict convexity of Σ, we have $\Sigma = \{(x, \rho(|x|)) : x \in R^2\}$ for some function $\rho \in C^\infty[0, \infty)$. Rewriting (1.1) for ρ, we obtain

$$\frac{\rho''(t)}{1 + (\rho'(t))^2} + \frac{\rho'(t)}{t} = 1, \qquad \forall\, t \in (0, \infty). \tag{2.14}$$

The strict convexity reads as

$$\rho''(t) > 0 \qquad \text{for} \ \ t \in (0, \infty). \tag{2.15}$$

By a rigid motion, we may assume the origin to be the lowest point of the strictly convex surface Σ. Thus

$$\rho(0) = \rho'(0) = 0. \tag{2.16}$$

Therefore, in order to complete the proof of Theorem 1.1, we want only to prove (1.6) and (1.7). It follows from (2.14)–(2.16) that $\frac{\rho'(t)}{t} < 1$ and $\rho'' + \frac{1}{t}\rho' \geq 1$ for all $t \in (0, \infty)$. This fact yields

$$\rho'(t) \leq t \ \text{ and } \ \rho(t) \leq \frac{t^2}{2}, \qquad \forall\, t \in [0, \infty). \tag{2.17}$$

Let $\omega(t) = \rho(t) - \frac{t^2}{4}$. Then

$$\omega'' + \frac{\omega'}{t} \geq 0 \quad \text{for } t \in (0, \infty), \qquad \omega(0) = 0 = \omega'(0). \tag{2.18}$$

Consequently, we conclude that

$$w'(t) \geq 0, \qquad \forall\, t \in (0, \infty). \tag{2.19}$$

Otherwise, we could find $t_0 \geq 0$ and $\delta > 0$ such that $w'(t_0) = 0$ but $w'(t) < 0$ for $t \in (t_0, t_0 + \delta]$. Integrating (2.18), we have

$$0 > w'(t_0 + \delta) \geq -\int_{t_0}^{t_0+\delta} \frac{w'(t)}{t} dt > 0,$$

a contradiction. It follows from (2.18) and (2.19) that $\frac{t}{2} \leq \rho'(t)$ and $\frac{t^2}{4} \leq \rho(t)$ for all $t \in [0, \infty)$, which, together with (2.17), yields the desired (1.6) and (1.7).

§ 3. Proofs of Theorem 1.2 and Corollary 1.1

Let $n \geq 2$. It is easy to see that any n-dimensional strictly convex translating soliton M can be represented as a graph over a domain $\Omega \subset R^n$ orthogonal to the translating direction. In fact, by the definition (see (1.1)), the outer unit normal at every point of such a soliton always has an angle larger than $\frac{\pi}{2}$ with the translating direction. If the soliton can not be represented as a graph over any domain $\Omega \subset R^n$ orthogonal to the translating direction, by the connectedness, one can find a point at which the normal is orthogonal to the translating direction. This yields a contradiction.

Therefore, we may assume $M = \{(x, u(x)) : x \in \Omega\}$ for a $u \in C^\infty(\Omega)$ and a domain $\Omega \subset R^n$. Obviously, Ω is convex since M is strictly convex. We will show that Ω is unbounded, and therefore it is the entire R^n when M is rotationally symmetric.

Note that the induced metric on M is $g = g_{ij} dx^i dx^j$ with

$$g_{ij} = \delta_{ij} + u_i u_j, \tag{3.1}$$

where we have used the Kronecker's symbol, the summation convention and the notations

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \text{etc.}$$

According to the sign choice in Section 1, the outer unit normal

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}.$$

Hence the second fundamental form

$$h_{ij} = -\Big\langle \nu, \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \Big\rangle = \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}} \tag{3.2}$$

and the mean curvature

$$H = g^{ij} h_{ij} = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \Big). \tag{3.3}$$

Since the translating direction is $T = (0, 0, \cdots, 1)$, the translating soliton equation (1.1) is turned to Equation (1.8).

Now, suppose the contrary that Ω is bounded. Since Ω is convex, we may choose an $x_0 \in \partial\Omega$ and a ball $B_{R_0}(y_0) \subset R^n$ such that

$$x_0 \in \overline{B_{R_0}(y_0)} \cap \partial\Omega = \{x_0\} \quad \text{and} \quad B_{R_0}(y_0) \supset \Omega.$$

Hence $M \subset B_{R_0}(y_0) \times R$. This, together with the convexity of M, implies

$$H((x, u(x))) \geq \frac{n}{R_0} \qquad \text{as} \ \ x \to x_0 \ \ \text{with} \ \ x \in \Omega. \tag{3.4}$$

On the other hand, since M is complete and x_0 is on the boundary, $|\nabla u(x)| \to \infty$ as x goes to x_0. It follows from (3.3) and (1.8) that $H((x, u(x))) \to 0$ as $x \to x_0$, which contradicts (3.4).

In order to prove Theorem 1.2(2), we use $u \in C^2(R^n)$ to denote a solution of (1.8) in R^n and rewrite (1.8) as

$$\Big(\delta_{ij} - \frac{u_i(x) u_j(x)}{1+|\nabla u(x)|^2} \Big) u_{ij}(x) = 1, \qquad \forall\, x \in R^n. \tag{3.5}$$

Without loss of generality, we assume that the point p in Theorem 1.2(2) is the origin in R^n. We want only to prove that u is symmetric in any direction with respect to the origin. Since Equation (3.5) is symmetric in every direction, it is sufficient to do this in one direction. Without loss of generality, we will do it in x_1-direction. Hence, for the proof of Theorem 1.2(2), it is enough to prove

$$u(-x_1, x_2, \cdots, x_n) = u(x_1, x_2, \cdots, x_n), \qquad \forall\, x = (x_1, x_2, \cdots, x_n) \in R^n. \tag{3.6}$$

Denote $x = (x_1, x_2, \cdots, x_n) = (x_1, x')$. For any $\lambda \in R$, we define

$$\Sigma(\lambda) = \{x = (x_1, x') \in R^n : x_1 < \lambda\}, \quad w_\lambda(x) = u(x_1, x') - u(2\lambda - x_1, x').$$

Then we conclude that for any $\lambda < 0$,

$$w_\lambda(x) = u(x_1, x') - u(2\lambda - x_1, x') \geq 0, \qquad \forall\, x = (x_1, x') \in \Sigma(\lambda). \tag{3.7}$$

In order to prove (3.7), we need an elliptic equation for the function $w_\lambda(x)$. For any $\lambda \in R$, since both $u(x)$ and $v_\lambda(x) = u(2\lambda - x_1, x')$ satisfy Equation (3.5), the function $w_\lambda(x)$ satisfies

$$A_{ij}(x)(w_\lambda)_{ij} + B_i(x)(w_\lambda)_i = 0 \qquad \text{in} \ \ R^n, \tag{3.8}$$

where

$$A_{ij}(x) = \delta_{ij} - \frac{u_i(x)u_j(x)}{1 + |\nabla u(x)|^2}$$

and

$$B_i(x) = \Big(\frac{u_j(v_\lambda)_{kj}(v_\lambda)_k(u_i + (v_\lambda)_i) - (1 + |\nabla u|^2)(v_\lambda)_j(v_\lambda)_{ij}}{(1 + |\nabla u|^2)(1 + |\nabla v_\lambda|^2)}\Big)(x)$$
$$- \Big(\frac{(1 + |\nabla v_\lambda|^2)u_j(v_\lambda)_{ji}}{(1 + |\nabla u|^2)(1 + |\nabla v_\lambda|^2)}\Big)(x).$$

Since the maximum eigenvalue of the $n \times n$ matrix $(u_i u_j)$ is $|\nabla u|^2$, (3.8) is a linear elliptic equation in R^n and is strictly elliptic on any bounded domain in R^n. Therefore, a strong maximum principle can be applied to Equation (3.8).

Now suppose the contrary that (3.7) is false for some $\lambda_1 < 0$. Then we could find $\hat{x} \in \Sigma(\lambda_1)$ such that $w_{\lambda_1}(\hat{x}) < 0$. Choose k_0 such that $r_k > |\hat{x}|$ for all $k \geq k_0$. Then we claim that for each $k \geq k_0$, there exist

$$x^k = (x_1^k, x_k') \in \partial B_{r_k}(0)) \cap \Sigma(\lambda_1)$$

such that

$$w_{\lambda_1}(x^k) < w_{\lambda_1}(\hat{x}) < 0, \qquad \forall\, k > k_0. \tag{3.9}$$

In fact, suppose the contrary that (3.9) were not true for some k. Then w_{λ_1} would attain its interior minimum at a point in $B_{r_k}(0) \cap \Sigma(\lambda_1)$ since

$$w_{\lambda_1}(x) = 0, \qquad \forall\, x \in \partial\Sigma(\lambda_1). \tag{3.10}$$

Therefore, by the strong maximum principle we obtain that

$$w_{\lambda_1}(x) = w_{\lambda_1}(\hat{x}) < 0, \qquad \forall\, x \in B_{r_{k_0}}(0) \cap \Sigma(\lambda_1), \tag{3.11}$$

contradicting (3.10). In this way, we have obtained (3.9).

However, noticing that $\lambda_1 < 0$ implies the fact

$$|x^k|^2 > |(2\lambda_1 - x_1^k, x_k')|^2$$

and

$$\lim_{k\to\infty} \frac{|(2\lambda_1 - x_1^k, x_k')|}{|x^k|} = 1,$$

and using the assumption (1.9), we have

$$\sup\{w_{\lambda_1}(x^k) : k \geq k_0\} \geq 0,$$

which contradicts (3.9). This proves (3.7).

Letting $\lambda \to 0^-$ in (3.7) we obtain

$$u(x_1, x') \geq u(-x_1, x'), \qquad \forall\, x = (x_1, x') \in R^n \text{ with } x_1 < 0.$$

The opposite inequality is also true, because $V(x) := u(-x_1, x')$ is a solution to (3.5) in R^n. This proves (3.6) and thus Theorem 1.2.

Proof of Corollary 1.1. We want only to show that the assumption (1.9) can be derived from (1.10) or (1.11). Obviously, (1.11) implies (1.10) and (1.10) implies (1.11) by taking $g(r_k) = u(r_k(1,0) + p)$. Thus (1.11) is equivalent to (1.10).

For each $k \geq 1$, choose $z^k \in \partial B_{r_k}(p)$ such that

$$\max_{\partial B_{r_k}(p)} u(x) = u(z^k).$$

Since constants $u(z^k)$ are supersolution of the Dirichlet problem of Equation (1.8) on the ball $B_{r_k}(p)$ with the boundary value $u(x)$, we obtain

$$u(y) \leq u(z^k), \qquad \forall\, y \in B_{r_k}(p)$$

by the comparison principle in [9, Theorem 10.1]. Then (1.9) follows immediately from (1.10).

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