Chin. Ann. Math.
26B:3(2005),423–436.

INEQUALITIES FOR MIXED INTERSECTION BODIES***

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Abstract

In this paper, some properties of mixed intersection bodies are given, and inequalities from the dual Brunn-Minkowski theory (such as the dual Minkowski inequality, the dual Aleksandrov-Fenchel inequalities and the dual Brunn-Minkowski inequalities) are established for mixed intersection bodies.

Keywords Star body, Mixed intersection body, Dual mixed volume, Spherical radon transform
2000 MR Subject Classification 52A40, 52A20

§ 1 . Introduction

Intersection body is a basic concept in the dual Brunn-Minkowski theory. The history of intersection bodies began with Busemann's theorem which has important implications for Busemann's theory of area in Finsler spaces (see [3]). Intersection bodies were first explicitly defined and named by Lutwak in the important paper [14], and played a key role in the ultimate solution of Busemann-Petty problem (see [4, 7, 8, 10, 11, 14, 26–28]). It was in [6, 14] that the duality between intersection bodies and projection bodies was first made clear. Interest in projection bodies was rekindled by three highly influential articles which appeared in the latter half of the 60's by Bolker [1], Petty [19], and Schneider [22]. Projection bodies have been the objects of intense investigation during the past three decades (see [2, 6, 21, 24]), and Lutwak established the inequalities for the mixed projection bodies and their polar bodies (see [13, 15, 16]). Though the notion of mixed intersection bodies has also been raised (see [12, p.251]), their properties have not been studied systematically by now. The corresponding work about the mixed intersection bodies is done in this paper. Basic properties of mixed intersection bodies are given in §2. In §3, the proof of dual Minkowski inequality for mixed intersection bodies is presented and an upper bound estimate about mixed intersection bodies is given. The dual Aleksandrov-Fenchel inequalities and dual Brunn-Minkowski inequalities for mixed intersection bodies are proven in §4 and §5. At the same time the equality conditions of these inequalities are obtained.

Manuscript received December 4, 2003. Revised January 4, 2005.
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***Project supported by the National Natural Science Foundation of China (No.10271071).

As usual, S^{n-1} denotes the unit sphere, B the unit ball, and o the origin in Euclidean n-space $\mathbb{E}^n$. If u is a unit vector, that is, an element of S^{n-1}, we denote by $u^{\perp}$ the $(n-1)$-dimensional subspace orthogonal to u and by l_u the line through the origin parallel to u. We write V_k for k-dimensional Lebesgue measure in $\mathbb{E}^n$, where $0\le k \le n$, and where we identify V_k with k-dimensional Hausdorff measure (V_0 is the counting measure). We also generally write V instead of V_n. κ_i denotes the volume of i-dimensional unit ball, where $1 \le i \le n$.

A set L is star-shaped at o if $L \cap l_u$ is either empty or a (possibly degenerate) closed line segment for each $u \in S^{n-1}$. If L is star-shaped at o, we define its radial function ρ_L by

$$\rho_L(u) = \begin{cases} \max\{c : cu \in L\}, & L \cap l_u \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

A body is a compact set equal to the closure of its interior. By a star body we mean a body L star-shaped at o such that ρ_L, restricted to its support, is continuous. We denote the class of star bodies in $\mathbb{E}^n$ by $\mathcal{L}^n$, and the subclass of star bodies containing the origin in their interiors by $\mathcal{L}^n_o$.

If $x_i \in \mathbb{E}^n$, $1 \le i \le m$, then $x_1 \widetilde{+} \cdots \widetilde{+} x_m$ is defined to be the usual vector sum of the points x_i, if all of them are contained in a line through o, and 0 otherwise. Let L_i be a star body in $\mathbb{E}^n$ with $o \in L_i$, and $t_i \ge 0, 1 \le i \le m$. Then

$$t_1 L_1 \widetilde{+} \cdots \widetilde{+} t_m L_m = \{t_1 x_1 \widetilde{+} \cdots \widetilde{+} t_m x_m : x_i \in L_i, 1 \le i \le m\}$$

is called a radial linear combination. Moreover, for each $u \in S^{n-1}$,

$$\rho_{t_1 L_1 \widetilde{+} t_2 L_2}(u) = t_1 \rho_{L_1}(u) + t_2 \rho_{L_2}(u). \tag{1.1}$$

The dual mixed volume $\widetilde{V}(L_1, \cdots, L_n)$ of star bodies $L_1, \cdots, L_n$ containing the origin in E is defined by

$$\widetilde{V}(L_1, \cdots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{L_1}(u) \cdots \rho_{L_n}(u) du, \tag{1.2}$$

where du signifies integration on S^{n-1} with respect to V_{n-1}, which in S^{n-1} is identified with spherical Lebesgue measure. When $L_1 = L_2 = \cdots = L_{n-i} = K_1,\ L_{n-i+1} = \cdots = L_n = K_2$ in (1.2), we write $\widetilde{V}_i(K_1, K_2)$ for $\widetilde{V}(K_1, n-i; K_2, i)$. For $0 \le i \le n$, the dual volume $\widetilde{V}_i(L)$ and dual quermassintegral $\widetilde{W}_{n-i}(L)$ are defined by

$$\widetilde{V}_i(L) = \widetilde{W}_{n-i}(L) = \widetilde{V}(L, i; B, n-i) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^i du. \tag{1.3}$$

In particular, $\widetilde{V}_n(L) = V(L)$, i.e., the polar coordinate formula for volume of star body L with $o \in L$ is

$$V(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n du. \tag{1.4}$$

Let L_j be a star body in $\mathbb{E}^n$ with $o \in L_j$, $1 \le j \le n$, and suppose that $1 \le i \le n$. Lutwak proved the dual Aleksandrov-Fenchel inequality (see [6, Section B.4])

$$\widetilde{V}(L_1, \cdots, L_n)^i \le \prod_{j=1}^{i} \widetilde{V}(L_j, i; L_{i+1}, \cdots, L_n), \tag{1.5}$$

in which the equality holds if and only if $L_1, \cdots, L_i$ are dilatates of each other. The inequality has the same form as the classical Aleksandrov-Fenchel inequality. Two special cases of (1.5) are worthy of note. If $L_i \in \mathcal{L}^n$, with $o \in L_i$, and $1 \le i \le n$, then

$$\widetilde{V}(L_1, \cdots, L_n)^n \le V(L_1)V(L_2)\cdots V(L_n), \tag{1.6}$$

and

$$\widetilde{V}_i(L_1, L_2)^n \le V(L_1)^{n-i}V(L_2)^i, \tag{1.7}$$

in which the equalities hold if and only if the bodies are dilatates of each other.

If $L \in \mathcal{L}^n$, $n \ge 2$, with $\rho_L \in C(S^{n-1})$, then the intersection body $I(L)$ of L is a star body defined by

$$\rho_{IL}(u) = V_{n-1}(L \cap u^{\perp}) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_L(v)^{n-1}dv. \tag{1.8}$$

Suppose that f is a bounded Borel function on S^{n-1}. The spherical Radon transform Rf of f is defined by

$$(Rf)(u) = \int_{S^{n-1}\cap u^{\perp}} f(v)dv \qquad \text{for all } \; u \in S^{n-1}.$$

The transform R is self-adjoint (see, for example, [6, Theorem C.2.6]), that is,

$$\int_{S^{n-1}} f(u)(Rg)(u)du = \int_{S^{n-1}} (Rf)(u)g(u)du, \tag{1.9}$$

for bounded Borel functions f and g on S^{n-1}. Let $\phi \in SO_n$, where SO_n is the special orthogonal group of rotations about the origin. Then (see [25, p.635])

$$\phi(Rf)(u) = (Rf)(\phi^{-1}u) = \int_{S^{n-1}\cap(\phi^{-1}u)^{\perp}} f(v)dv. \tag{1.10}$$

Suppose $L \in \mathcal{L}_o^n$ and $u \in S^{n-1}$. The i-chord function $\rho_{i,L}$ of L at o is defined by

$$\rho_{i,L}(u) = \begin{cases} \rho_L(u)^i + \rho_L(-u)^i, & i \ne 0, \\ \rho_L(u)\rho_L(-u), & i = 0, \end{cases} \tag{1.11}$$

and the i-chordal symmetral $\widetilde{\nabla}_i L$ of L is a centered set defined by

$$\rho_{i,\widetilde{\nabla}_i L}(u) = \rho_{i,L}(u). \tag{1.12}$$

§ 2 . Properties of Mixed Intersection Bodies

Definition 2.1. (see [12, 14]) *Let $K_1, \cdots, K_{n-1} \in \mathcal{L}^n$, $n \geq 2$. The mixed intersection body $I(K_1, \cdots, K_{n-1})$ of $K_1, \cdots, K_{n-1}$ is defined by*

$$\rho_{I(K_1,\cdots,K_{n-1})}(u) = \tilde{v}(K_1 \cap u^{\perp}, \cdots, K_{n-1} \cap u^{\perp}), \tag{2.1}$$

where $u \in S^{n-1}$, $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

The definition implies that $I(K_1, \cdots, K_{n-1})$ is a centered star body. Taking $K_1 = \cdots = K_{n-1} = K$, we notice that the diagonal form of the mixed intersection body reduces to the intersection body, i.e., $I(K, \cdots, K) = I(K)$. Specially, $I(B, \cdots, B) = \kappa_{n-1}B$. By (1.2), we can rewrite (2.1) as the equivalent integral form, restricted to the star bodies containing the origin in their interiors, involving the spherical Radon transform R:

$$\begin{aligned}\rho_{I(K_1,\cdots,K_{n-1})}(u) &= \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv \\ &= R\Big(\frac{1}{n-1}\rho_{K_1}\cdots\rho_{K_{n-1}}\Big)(u).\end{aligned} \tag{2.2}$$

Now, we develop some basic properties of the mixed intersection operator $I : \underbrace{\mathcal{L}_o^n \times \cdots \times \mathcal{L}_o^n}_{n-1} \to \mathcal{L}_o^n$. Of course, most of the following results remain valid for $I : \underbrace{\mathcal{L}^n \times \cdots \times \mathcal{L}^n}_{n-1} \to \mathcal{L}^n$.

Proposition 2.1. *The mixed intersection operator is positively homogeneous, i.e., if $K_1, \cdots, K_{n-1} \in \mathcal{L}_o^n$, and $\alpha_1, \cdots, \alpha_{n-1} \geq 0$, then*

$$I(\alpha_1 K_1, \cdots, \alpha_{n-1}K_{n-1}) = \alpha_1 \cdots \alpha_{n-1} I(K_1, \cdots, K_{n-1}).$$

Proposition 2.2. *The mixed intersection operator is multilinear with respect to the radial linear combinations, i.e., if $K', K_1, \cdots, K_{n-1} \in \mathcal{L}_o^n$, and $\alpha, \beta \geq 0$, then*

$$I(\alpha K_1 \tilde{+} \beta K', K_2, \cdots, K_{n-1}) = \alpha I(K_1, K_2, \cdots, K_{n-1}) \tilde{+} \beta I(K', K_2, \cdots, K_{n-1}).$$

Proposition 2.3. *The mixed intersection operator is monotone nondecreasing with respect to set inclusion, i.e., if $K', K, K_2, \cdots, K_{n-1} \in \mathcal{L}_o^n$, $K \subset K'$, then*

$$I(K, K_2, \cdots, K_{n-1}) \subset I(K', K_2, \cdots, K_{n-1}).$$

These properties of the mixed intersection operator follow immediately from (2.2) and (1.1).

The intersection bodies of linearly equivalent star bodies are linearly equivalent (see [17]), i.e., if $\phi \in \mathrm{GL}_n$, then $I(\phi L) = |\det\phi|\phi^{-t}(IL)$, where ϕ^{-t} is the transpose of the inverse of ϕ. For the special orthogonal group SO_n (with determinant one) and the mixed intersection operator, we have

Proposition 2.4. *If $K_1, \cdots, K_{n-1} \in \mathcal{L}_o^n, \phi \in \mathrm{SO}_n$, then*

$$I(\phi K_1, \cdots, \phi K_{n-1}) = \phi^{-t} I(K_1, \cdots, K_{n-1}).$$

Proof. Let $u \in S^{n-1}$. By the property of linear transformation: $x \cdot \phi y = \phi^t x \cdot y$, we know that $x \cdot u = 0$ if and only if $\phi^{-1}x \cdot \phi^t u = 0$. Therefore, if ω is the unit vector in the direction of $\phi^t u$, then $\omega^\perp = \phi^{-1}u^\perp$.

If E is a V_{n-1}-measurable subset of $(n-1)$-dimensional subspace $u^\perp$ in $\mathbb{E}^n$, with $o \in E$, and ψ is a linear transformation in $\mathbb{E}^n$, then we have (see [6, p.274])

$$V_{n-1}(\psi E) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{\psi E}^{n-1}(u)du = \|\psi^{-t}u\||\mathrm{det}\psi|V_{n-1}(E).$$

It is easy to see that, if $L_1, \cdots, L_{n-1}$ are V_{n-1}-measurable star bodies contained in $u^\perp$, with $o \in L_1, \cdots, L_{n-1}$, then

$$\begin{aligned}\tilde{v}(\psi L_1, \cdots, \psi L_{n-1}) &= \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{\psi L_1}(v)\cdots\rho_{\psi L_{n-1}}(v)dv \\ &= \|\psi^{-t}u\||\mathrm{det}\psi|\tilde{v}(L_1, \cdots, L_{n-1}).\end{aligned}$$

Hence

$$\begin{aligned}\rho_{I(\phi K_1,\cdots,\phi K_{n-1})}(u) &= \tilde{v}(\phi K_1 \cap u^\perp, \cdots, \phi K_{n-1}\cap u^\perp) \\ &= \tilde{v}(\phi(K_1 \cap \phi^{-1}u^\perp), \cdots, \phi(K_{n-1}\cap \phi^{-1}u^\perp)) \\ &= \|\phi^{-t}u\|\ \tilde{v}(K_1 \cap \phi^{-1}u^\perp, \cdots, K_{n-1}\cap \phi^{-1}u^\perp) \\ &= \frac{1}{n-1}\|\phi^{-t}u\|\int_{S^{n-1}\cap(\phi^{-1}u^\perp)} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv \\ &= \frac{1}{n-1}\|\phi^{-t}u\|\int_{S^{n-1}\cap\left(\frac{\phi^t u}{\|\phi^t u\|}\right)^\perp} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv \\ &= \frac{1}{n-1}\|\phi^{-t}u\|\int_{S^{n-1}\cap((\|\phi^t u\|\phi^{-t})^{-1}u)^\perp} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv.\end{aligned}$$

From the property (1.10) of spherical Radon transform, we have

$$\begin{aligned}\rho_{I(\phi K_1,\cdots,\phi K_{n-1})}(u) &= \|\phi^{-t}u\|\cdot\|\phi^t u\|\cdot\phi^{-t}\Big(\frac{1}{n-1}\int_{s^{n-1}\cap u^\perp}\rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv\Big) \\ &= \phi^{-t}(\rho_{I(K_1,\cdots,K_{n-1})}(u)).\end{aligned}$$

The proof is completed.

Proposition 2.5. *Let $M = (K_1, \cdots, K_{n-i-1})$, $1 \le i \le n-1$, and write $I_i(M,K)$ for $I(M, \underbrace{K, \cdots, K}_{i})$. If $K, L, K_1, \cdots, K_{n-i-1} \in \mathcal{L}_o^n$, then*

$$I_i(M, K\cap L)\tilde{+}I_i(M, K\cup L) = I_i(M,K)\tilde{+}I_i(M,L).$$

Proof. Since $\rho_{K\cup L}(u) = \max\{\rho_K(u), \rho_L(u)\}$, $\rho_{K\cap L}(u) = \min\{\rho_K(u), \rho_L(u)\}$, and together with (1.1), we get

$$\rho_{I_i(M,K\cap L)\tilde{+}I_i(M,K\cup L)}(u) = \rho_{I_i(M,K)\tilde{+}I_i(M,L)}(u), \qquad u \in S^{n-1}.$$

Thus, it is easy to obtain Proposition 2.5.

Take $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$ in $I(K_1, \cdots, K_{n-1})$, and $I(\underbrace{K, \cdots, K}_{n-i-1}, \underbrace{L, \cdots, L}_{i})$ will be written as $I_i(K, L)$. If $L = B$, then $I_i(K, B)$ is called the ith intersection body of K and is usually written as I_iK. Specially, $IK{=}I_0K$.

Proposition 2.6. *If K is a centered star body, and $L \in \mathcal{L}_o^n$, $0 < i \leq n-1$, then*

$$I_i(K, \widetilde{\nabla}_i L) = I_i(K, L).$$

Proof. Let $u \in S^{n-1}$. From (1.11) and (1.12), it follows that

$$\rho_{\widetilde{\nabla}_i L}(u) = \Big(\frac{\rho_L(u)^i + \rho_L(-u)^i}{2}\Big)^{\frac{1}{i}}.$$

Since K is centered, $\rho_K(-u)^i = \rho_K(u)^i$. Thus

$$\begin{aligned}
&\rho_{I_i(K,\widetilde{\nabla}_i L)}(u) = \frac{1}{n-1}\int_{s^{n-1}\cap u^\perp} \rho_K(v)^{n-i-1}\rho_{\widetilde{\nabla}_i L}(v)^i dv\\
&= \frac{1}{2(n-1)}\Big(\int_{s^{n-1}\cap u^\perp} \rho_K(v)^{n-i-1}\rho_L(v)^i dv + \int_{s^{n-1}\cap u^\perp} \rho_K(v)^{n-i-1}\rho_L(-v)^i dv\Big)\\
&= \frac{1}{2(n-1)}\Big(\int_{s^{n-1}\cap u^\perp} \rho_K(v)^{n-i-1}\rho_L(v)^i dv + \int_{s^{n-1}\cap u^\perp} \rho_K(-v)^{n-i-1}\rho_L(-v)^i dv\Big)\\
&= \frac{1}{n-1}\int_{s^{n-1}\cap u^\perp} \rho_K(v)^{n-i-1}\rho_L(v)^i dv\\
&= \rho_{I_i(K,L)}(u).
\end{aligned}$$

The proof is completed .

Proposition 2.7. *If $K_1, \cdots, K_{n-1}, K_1', \cdots, K_{n-1}' \in \mathcal{L}_o^n$, $I(K_1, \cdots, K_{n-1}) = I(K_1', \cdots, K_{n-1}')$, and M is a centered star body, then*

$$\widetilde{V}(K_1, \cdots, K_{n-1}, M) = \widetilde{V}(K_1', \cdots, K_{n-1}', M).$$

Proof. Since M is a centered star body, $\rho_M \in C_e^\infty(S^{n-1})$. Thus, there is $f \in C_e(S^{n-1})$, such that $\rho_M = Rf$ (see [14, 23]).

Since for each $u \in S^{n-1}$, $\rho_{I(K_1,\cdots,K_{n-1})}(u) = \rho_{I(K_1',\cdots,K_{n-1}')}(u)$, we have

$$R(\rho_{K_1}\cdots\rho_{K_{n-1}}) = R(\rho_{K_1'}\cdots\rho_{K_{n-1}'}).$$

Then, by (1.2), (1.9) and the above equalities, it follows that

$$\begin{aligned}
\widetilde{V}(K_1, \cdots, K_{n-1}, M) &= \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\cdots\rho_{K_{n-1}}(u)\rho_M(u)du\\
&= \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\cdots\rho_{K_{n-1}}(u)Rf(u)du\\
&= \frac{1}{n}\int_{S^{n-1}} R(\rho_{K_1}\cdots\rho_{K_{n-1}})(u)f(u)du\\
&= \frac{1}{n}\int_{S^{n-1}} R(\rho_{K_1'}\cdots\rho_{K_{n-1}'})(u)f(u)du
\end{aligned}$$

$$= \frac{1}{n}\int_{S^{n-1}} \rho_{K_1'}(u)\cdots\rho_{K_{n-1}'}(u)\rho_M(u)du$$
$$= \widetilde{V}(K_1', \cdots, K_{n-1}', M).$$

§ 3 . Dual Minkowski Inequality for Mixed Intersection Bodies

In this section, the following dual Minkowski inequalities for mixed intersection bodies will be established.

Theorem 3.1. *If $K, K' \in \mathcal{L}_o^n$, and $0 \le i \le n-1$, then*

$$V(I_i(K, K'))^{n-1} \le V(IK)^{n-i-1}V(IK')^i, \tag{3.1}$$

and the equality holds if and only if K and K' are dilatates.

Proof. Suppose $u \in S^{n-1}$.

Let L_1, L_2 be star bodies contained in $(n-1)$-dimensional subspace $u^\perp$, $o \in L_1, L_2$, and write $\tilde{v}_i(L_1, L_2)$ for $\tilde{v}(L_1, n-i-1; L_2, i)$. From (1.7), we have

$$\tilde{v}_i(L_1, L_2)^{n-1} \le V_{n-1}(L_1)^{n-i-1}V_{n-1}(L_2)^i.$$

Now using the polar coordinate formula for volume (1.4), the above inequality and Hölder integral inequality, we have

$$\begin{aligned}
V(I_i(K, K')) &= \frac{1}{n}\int_{S^{n-1}} \rho_{I_i(K,K')}(u)^n du = \frac{1}{n}\int_{S^{n-1}} \tilde{v}_i(K\cap u^\perp, K'\cap u^\perp)^n du\\
&\le \frac{1}{n}\int_{S^{n-1}} V_{n-1}(K\cap u^\perp)^{(n-i-1)\cdot n/(n-1)} \cdot V_{n-1}(K'\cap u^\perp)^{i\cdot n/(n-1)}du\\
&\le \frac{1}{n}\Big(\int_{S^{n-1}} V_{n-1}(K\cap u^\perp)^n du\Big)^{(n-i-1)/(n-1)}\Big(\int_{S^{n-1}} V_{n-1}(K'\cap u^\perp)^n du\Big)^{i/(n-1)}\\
&= \Big(\frac{1}{n}\int_{S^{n-1}} \rho_{IK}(u)^n du\Big)^{(n-i-1)/(n-1)}\Big(\frac{1}{n}\int_{S^{n-1}} \rho_{IK'}(u)^n du\Big)^{i/(n-1)}\\
&= V(IK)^{(n-i-1)/(n-1)}V(IK')^{i/(n-1)}.
\end{aligned}$$

From the equality conditions of the inequality (1.7) and Hölder integral inequality, this implies that $K\cap u^\perp$ and $K'\cap u^\perp$ must be dilatates. It is well known that if $K\cap u^\perp$ and $K'\cap u^\perp$ are dilatates for all $u \in S^{n-1}$, then K and K' are dilatates (see [20]). Hence, we get that Theorem 3.1 holds with equality if and only if K and K' are dilatates.

Corollary 3.1. *If K is a convex body whose centroid is at the origin, K^* is the polar body of K, and $0 \le i \le n-1$, then*

$$V(I_i(K^*, K)) \cdot V(K)^{n-2i-1} \le \kappa_{n-1}^n \cdot \kappa_n^{n-2i},$$

and the equality holds if and only if K is a centered ellipsoid.

Proof. Taking $K = K^*$, $K' = K$ in Theorem 3.1, we have

$$V(I_i(K^*, K))^{n-1} \le V(I(K^*))^{n-i-1}V(IK)^i.$$

Notice the following two well-known inequalities: Busemann intersection inequality (see [6, p.333]) states that if $K \in \mathcal{L}_o^n$, then

$$V(IK) \le (\kappa_{n-1}^n/\kappa_n^{n-2})V(K)^{n-1},$$

and the equality holds if and only if K is a centered ellipsoid; the Extended Blaschke-Santaló inequality (see [18]) states that if K is a convex body whose centroid is at the origin, then

$$V(K^*)V(K) \le \kappa_n^2,$$

and the equality holds if and only if K is a centered ellipsoid.

Hence, we get

$$\begin{aligned} V(I_i(K^*,K))^{n-1} &\le (\kappa_{n-1}^{n(n-1)}/\kappa_n^{(n-1)(n-2)})V(K^*)^{(n-1)(n-i-1)}V(K)^{(n-1)i} \\ &\le (\kappa_{n-1}^{n(n-1)}/\kappa_n^{(n-1)(n-2)}) \cdot (\kappa_n^{2(n-1)(n-i-1)}/V(K)^{(n-1)(n-i-1)})V(K)^{(n-1)i} \\ &= \kappa_{n-1}^{n(n-1)} \cdot \kappa_n^{(n-1)(n-2i)} \cdot V(K)^{(n-1)(2i-n+1)}. \end{aligned}$$

It is easy to obtain the equality condition of Corollary 3.1.

§ 4. Dual Aleksandrov-Fenchel Inequalities for Mixed Intersection Bodies

The dual Aleksandrov-Fenchel inequality, for mixed intersection bodies, which will be proven is: If $K_1, \cdots, K_{n-1} \in \mathcal{L}_o^n$, $1 \le i \le n-1$, then

$$V(I(K_1,\cdots,K_{n-1}))^i \le \prod_{j=1}^{i} V(I(K_j,i;K_{i+1},\cdots,K_{n-1})), \tag{4.1}$$

and the equality holds if and only if $K_1, \cdots, K_i$ are dilatates.

This is the special case $i = 0$ of Theorem 4.1.

Theorem 4.1. *If $K_1, \cdots, K_{n-1} \in \mathcal{L}_o^n$, $0 \le i \le n$, $0 < m \le n-1$, then*

$$\widetilde{W}_i(I(K_1,\cdots,K_{n-1}))^m \le \prod_{j=1}^{m} \widetilde{W}_i(I(K_j,m;K_{m+1},\cdots,K_{n-1})),$$

and the equality holds if and only if $K_1, \cdots, K_m$ are dilatates.

Proof. By (1.3) and (2.2), we have

$$\begin{aligned} &\widetilde{W}_i(I(K_1,\cdots,K_{n-1})) \\ &= \frac{1}{n}\int_{S^{n-1}} \rho_{I(K_1,\cdots,K_{n-1})}(u)^{n-i}du \\ &= \frac{1}{n}\int_{S^{n-1}} \Big(\frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v)dv\Big)^{n-i}du. \end{aligned}$$

An extension of Hölder inequality states that (see [9, p.140])

$$\int_{\Omega}\prod_{i=1}^{m} f_i(u)du \le \prod_{i=1}^{m}\Big(\int_{\Omega} f_i(u)^m du\Big)^{1/m},$$

and the equality holds if and only if f_i are proportional.

Applying the extension of Hölder inequality, we have

$$\begin{aligned}
&\widetilde{W}_i(I(K_1,\cdots,K_{n-1}))^m\\
&\le \Big(\frac{1}{n}\int_{S^{n-1}}\Big(\prod_{j=1}^{m}\Big(\frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}}\rho_{K_j}(v)^m\rho_{K_{m+1}}(v)\cdots\rho_{K_{n-1}}(v)dv\Big)\Big)^{(n-i)/m}du\Big)^m\\
&= \Big(\frac{1}{n}\int_{S^{n-1}}\Big(\prod_{j=1}^{m}\rho_{I(K_j,m;K_{m+1},\cdots,K_{n-1})}(u)\Big)^{(n-i)/m}du\Big)^m.
\end{aligned}$$

Applying the extension of Hölder inequality again, we obtain

$$\begin{aligned}
\widetilde{W}_i(I(K_1,\cdots,K_{n-1}))^m &\le \prod_{j=1}^{m}\Big(\frac{1}{n}\int_{S^{n-1}}\rho_{I(K_j,m;K_{m+1},\cdots,K_{n-1})}(u)^{n-i}du\Big)\\
&= \prod_{j=1}^{m}\widetilde{W}_i(I(K_j,m;K_{m+1},\cdots,K_{n-1})).
\end{aligned}$$

From the equality condition of the first Hölder inequality, this implies that $\rho_{K_j}(v)^m$ $\rho_{K_{m+1}}(v)\cdots\rho_{K_{n-1}}(v)$ $(j=1,\cdots,m)$ must be proportional. Since $K_i\in\mathcal{L}_o^n$ $(i=1,\cdots,n-1)$, $K_1,\cdots,K_m$ are dilatates. It follows that there exist $a_i\ge 0$ $(i=2,\cdots,m)$ such that $K_1=a_iK_i$. Then

$$\begin{aligned}
\rho_{I(K_1,m;K_{m+1},\cdots,K_{n-1})}(u) &= \tilde{v}(K_1\cap u^{\perp},m;K_{m+1}\cap u^{\perp},\cdots,K_{n-1}\cap u^{\perp})\\
&= \tilde{v}(a_i(K_i\cap u^{\perp}),m;K_{m+1}\cap u^{\perp},\cdots,K_{n-1}\cap u^{\perp})\\
&= a_i^m\tilde{v}((K_i\cap u^{\perp}),m;K_{m+1}\cap u^{\perp},\cdots,K_{n-1}\cap u^{\perp})\\
&= a_i^m\rho_{I(K_i,m;K_{m+1},\cdots,K_{n-1})}(u).
\end{aligned}$$

That $K_1=a_iK_i$ $(i=2,\cdots,m)$ implies that $\rho_{I(K_j,m;K_{m+1},\cdots,K_{n-1})}$ $(j=1,\cdots,m)$ are proportional, which is also the equality condition of the second Hölder inequality. So the equality of Theorem 4.1 holds if and only if $K_1,\cdots,K_m$ are dilatates.

Taking $i=n-1$ in (4.1), we have

Corollary 4.1. *If $K_1,\cdots,K_{n-1}\in\mathcal{L}_o^n$, then*

$$V(I(K_1,\cdots,K_{n-1}))^{n-1}\le V(I(K_1))\cdots V(I(K_{n-1})),$$

and the equality holds if and only if $K_1,\cdots,K_{n-1}$ are dilatates.

Taking $K_1=\cdots=K_{n-2-j}=K$, $K_{n-1-j}=L$, $K_{n-j}=\cdots=K_{n-1}=B$ in (4.1), where $1\le j\le n-2$, we obtain

Corollary 4.2. *If $K, L \in \mathcal{L}_o^n$, $1 \le j < n-2$, then*

$$V(I(\underbrace{K,\cdots,K}_{n-2-j},L,\underbrace{B,\cdots,B}_{j}))^{n-1-j} \le V(I_jK)^{n-2-j}V(I_jL),$$

and the equality holds if and only if K and L are dilatates.

Remark 4.1. Take $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = K'$, $i = n-1$ in (4.1), and Theorem 3.1 follows immediately.

§ 5 . Dual Brunn-Minkowski Inequalities for Mixed Intersection Bodies

First, in this section, we will consider the dual Brunn-Minkowski inequality for the mixed intersection bodies and the radial Blaschke linear combination.

Let K and K' be star bodies in $\mathbb{E}^n$ with $o \in K, K'$, $\alpha_1, \alpha_2 \ge 0$, and define the radial Blaschke linear combination $\alpha_1 \cdot K \hat{+} \alpha_2 \cdot K'$ by

$$\rho_{\alpha_1\cdot K\hat{+}\alpha_2\cdot K'}(u)^{n-1} = \alpha_1\rho_K(u)^{n-1} + \alpha_2\rho_{K'}(u)^{n-1}. \tag{5.1}$$

Theorem 5.1. *If K, K', $L \in \mathcal{L}_o^n$, $\alpha_1, \alpha_2 \ge 0$, then*

$$V(I_1(\alpha_1\cdot K\hat{+}\alpha_2\cdot K', L))^{1/n} \le \alpha_1 V(I_1(K,L))^{1/n} + \alpha_2 V(I_1(K',L))^{1/n},$$

and the equality holds if and only if $I_1(K,L), I_1(K',L)$ are dilatates.

Proof. Let $u \in S^{n-1}$. Since $K\cap u^\perp, K'\cap u^\perp$ are star bodies containing the origin in $u^\perp$, by (5.1), we have

$$\rho_{\alpha_1\cdot(K\cap u^\perp)\hat{+}\alpha_2\cdot(K'\cap u^\perp)}(v)^{n-2} = \alpha_1\rho_{K\cap u^\perp}(v)^{n-2} + \alpha_2\rho_{K'\cap u^\perp}(v)^{n-2}.$$

Then, from (2.2) and the above equality, we get

$$\begin{aligned}
&\rho_{I_1(\alpha_1\cdot K\hat{+}\alpha_2\cdot K', L)}(u)\\
&= \frac{1}{n-1}\int_{S^{n-1}\cap S}\rho_{(\alpha_1\cdot K\hat{+}\alpha_2\cdot K')\cap u^\perp}(v)^{n-2}\rho_{L\cap u^\perp}(v)dv\\
&= \frac{1}{n-1}\int_{S^{n-1}\cap S}\rho_{(\alpha_1\cdot(K\cap u^\perp)\hat{+}\alpha_2\cdot(K'\cap u^\perp))}(v)^{n-2}\rho_{L\cap u^\perp}(v)dv\\
&= \frac{\alpha_1}{n-1}\int_{S^{n-1}\cap S}\rho_{K\cap u^\perp}(v)^{n-2}\rho_{L\cap u^\perp}(v)dv + \frac{\alpha_2}{n-1}\int_{S^{n-1}\cap S}\rho_{K'\cap u^\perp}(v)^{n-2}\rho_{L\cap u^\perp}(v)dv\\
&= \frac{\alpha_1}{n-1}\int_{S^{n-1}\cap u^\perp}\rho_K(v)^{n-2}\rho_L(v)dv + \frac{\alpha_2}{n-1}\int_{S^{n-1}\cap u^\perp}\rho_{K'}(v)^{n-2}\rho_L(v)dv\\
&= \alpha_1\rho_{I_1(K,L)}(u) + \alpha_2\rho_{I_1(K',L)}(u).
\end{aligned}$$

Using (1.4), the above result and Minkowski integral inequalities, we have

$$\begin{aligned}
&V(I_1(\alpha_1\cdot K\hat{+}\alpha_2\cdot K', L))\\
&= \frac{1}{n}\int_{S^{n-1}}(\alpha_1\rho_{I_1(K,L)}(u) + \alpha_2\rho_{I_1(K',L)}(u))^n du
\end{aligned}$$

$$\leq \Big(\Big(\frac{1}{n}\int_{S^{n-1}} \alpha_1^n \rho_{I_1(K,L)}(u)^n du\Big)^{1/n} + \Big(\frac{1}{n}\int_{S^{n-1}} \alpha_2^n \rho_{I_1(K',L)}(u)^n du\Big)^{1/n}\Big)^n$$
$$= (\alpha_1 V(I_1(K,L))^{1/n} + \alpha_2 V(I_1(K',L))^{1/n})^n.$$

From the equality condition of Minkowski integral inequality, it shows that $\rho_{I_1(K,L)}(u) = a\rho_{I_1(K',L)}(u)$, where $a \geq 0$. Thus, Theorem 5.1 holds with equality if and only if $I_1(K,L)$, $I_1(K',L)$ are dilatates.

Second, the dual Brunn-Minkowski inequalities for the radial linear combination and intersection bodies, which will be established is: If $K, L \in \mathcal{L}_o^n$, then

$$V(I(K\widetilde{+}L))^{1/(n(n-1))} \leq V(IK)^{1/(n(n-1))} + V(IL)^{1/(n(n-1))},$$

and the equality holds if and only if K and L are dilatates.

This is the special case $j = 0, i = 0$ of Theorem 5.2.

Theorem 5.2. *If $0 \leq i < n, 0 \leq j < n-1$, K, L, $M_1, \cdots, M_i$, $M_1', \cdots, M_j' \in \mathcal{L}_o^n$, $C = (M_1, \cdots, M_i)$, and $D = (M_1', \cdots, M_j')$, then*

$$\widetilde{V}_i(I_j(K\widetilde{+}L, D), C)^{1/((n-i)(n-j-1))}$$
$$\leq \widetilde{V}_i(I_j(K,D),C)^{1/((n-i)(n-j-1))} + \widetilde{V}_i(I_j(L,D),C)^{1/((n-i)(n-j-1))},$$

and the equality holds if and only if K, L are dilatates.

In order to prove Theorem 5.2, we need the following two lemmas.

Lemma 5.1. *If $K_1, \cdots, K_{n-1}, L_1, \cdots, L_{n-1} \in \mathcal{L}_o^n$, then*

$$\widetilde{V}(K_1, \cdots, K_{n-1}, I(L_1, \cdots, L_{n-1})) = \widetilde{V}(L_1, \cdots, L_{n-1}, I(K_1, \cdots, K_{n-1})).$$

Proof. Let $u \in S^{n-1}$. Then

$$\widetilde{V}(K_1, \cdots, K_{n-1}, I(L_1, \cdots, L_{n-1}))$$
$$= \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\cdots\rho_{K_{n-1}}(u)\rho_{I(L_1,\cdots,L_{n-1})}(u)du$$
$$= \frac{1}{n}\cdot\frac{1}{n-1}\int_{S^{n-1}} \rho_{K_1}(u)\cdots\rho_{K_{n-1}}(u)\int_{S^{n-1}\cap u^\perp} \rho_{L_1}(v)\cdots\rho_{L_{n-1}}(v)dvdu.$$

Let $f = \rho_{K_1}\cdots\rho_{K_{n-1}}$, $g = \rho_{L_1}\cdots\rho_{L_{n-1}}$. Then

$$\widetilde{V}(K_1, \cdots, K_{n-1}, I(L_1, \cdots, L_{n-1})) = \frac{1}{n}\cdot\frac{1}{n-1}\int_{S^{n-1}} f(u)Rg(u)du.$$

In the same way,

$$\widetilde{V}(L_1, \cdots, L_{n-1}, I(K_1, \cdots, K_{n-1})) = \frac{1}{n}\cdot\frac{1}{n-1}\int_{S^{n-1}} g(u)Rf(u)du.$$

From the property (1.9) of the spherical Radon transformation, we obtain Lemma 5.1.

Lemma 5.2. *If $0 \leq i < n-1$, $K, L, K_1, \cdots, K_i \in \mathcal{L}_o^n$, and $C = (K_1, \cdots, K_i)$, then*

$$\widetilde{V}_i(K\widetilde{+}L, C)^{1/(n-i)} \leq \widetilde{V}_i(K,C)^{1/(n-i)} + \widetilde{V}_i(L,C)^{1/(n-i)}, \tag{5.2}$$

and the equality holds if and only if K and L are dilatates.

Proof. Let $\rho_C = \rho_{K_1} \cdots \rho_{K_i}$.

By (1.3), (1.1) and Minkowski integral inequality, we obtain

$$\begin{aligned}
&\widetilde{V}_i(K\widetilde{+}L, C)^{1/(n-i)} \\
&= \Big(\frac{1}{n}\int_{S^{n-1}} \rho_{K\widetilde{+}L}(u)^{n-i}\rho_C(u)du\Big)^{1/(n-i)} \\
&= \Big(\frac{1}{n}\int_{S^{n-1}} (\rho_K(u)\rho_C(u)^{1/(n-i)} + \rho_L(u)\rho_C(u)^{1/(n-i)})^{n-i}du\Big)^{1/(n-i)} \\
&\le \Big(\frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i}\rho_C(u)du\Big)^{1/(n-i)} + \Big(\frac{1}{n}\int_{S^{n-1}} \rho_L(u)^{n-i}\rho_C(u)du\Big)^{1/(n-i)} \\
&= \widetilde{V}_i(K, C)^{1/(n-i)} + \widetilde{V}_i(L, C)^{1/(n-i)}.
\end{aligned}$$

From the equality condition of Minkowski integral inequality, we get that (5.2) holds with equality if and only if $\rho_K(u)\rho_C(u)^{1/(n-i)} = a\rho_L(u)\rho_C(u)^{1/(n-i)}$, where $a \ge 0$. This implies that $\rho_K(u) = \rho_{aL}(u)$, i.e., $K = aL$. The proof is completed.

Proof of Theorem 5.2. If $j = n-2$, then from Proposition 2.2, it follows that

$$I_{n-2}(K\widetilde{+}L, D) = I_{n-2}(K, D)\widetilde{+}I_{n-2}(L, D).$$

Hence, for $j = n-2$, the inequality of Theorem 5.2 reduces to the one of Lemma 5.2.

For $i = n-1$, using Lemma 5.1 and Lemma 5.2, we obtain

$$\begin{aligned}
&\widetilde{V}_{n-1}(I_j(K\widetilde{+}L, D), C)^{1/(n-j-1)} \\
&= \widetilde{V}(\underbrace{K\widetilde{+}L, \cdots, K\widetilde{+}L}_{n-j-1}, \underbrace{D, \cdots, D}_{j}, I(C, \cdots, C))^{1/(n-j-1)} \\
&\le \widetilde{V}(\underbrace{K, \cdots, K}_{n-j-1}, \underbrace{D, \cdots, D}_{j}, I(C, \cdots, C))^{1/(n-j-1)} \\
&\quad + \widetilde{V}(\underbrace{L, \cdots, L}_{n-j-1}, \underbrace{D, \cdots, D}_{j}, I(C, \cdots, C))^{1/(n-j-1)} \\
&= \widetilde{V}_{n-1}(I_j(K, D), C)^{1/(n-j-1)} + \widetilde{V}_{n-1}(I_j(L, D), C)^{1/(n-j-1)}.
\end{aligned}$$

Thus, only the cases where $j < n-2$, and $i < n-1$ need to be treated.

Suppose $Q \in \mathcal{L}_o^n$. Using Lemma 5.1 and Lemma 5.2, we get

$$\begin{aligned}
&\widetilde{V}(Q, n-i-1; C; I_j(K\widetilde{+}L, D))^{1/(n-j-1)} \\
&= \widetilde{V}(K\widetilde{+}L, n-j-1; D; I_i(Q, C))^{1/(n-j-1)} &\quad (5.3)\\
&\le \widetilde{V}(K, n-j-1; D; I_i(Q, C))^{1/(n-j-1)} \\
&\quad + \widetilde{V}(L, n-j-1; D; I_i(Q, C))^{1/(n-j-1)}. &\quad (5.4)
\end{aligned}$$

Using Lemma 5.1 again and the dual Aleksandrov-Fenchel inequality, we have

$$\begin{aligned}
&\widetilde{V}(K, n-j-1; D; I_i(Q,C))^{1/(n-j-1)} \\
&= \widetilde{V}(Q, n-i-1; C; I_j(K,D))^{1/(n-j-1)} \\
&= \widetilde{V}(Q, n-i-1; I_j(K,D); C)^{1/(n-j-1)} \\
&\leq \widetilde{V}_i(Q,C)^{(n-1-i)/((n-j-1)(n-i))} \widetilde{V}_i(I_j(K,D),C)^{1/((n-j-1)(n-i))}.
\end{aligned} \tag{5.5}$$

In exactly the same way, it can be seen that

$$\begin{aligned}
&\widetilde{V}(L, n-j-1; D; I_i(Q,C))^{1/(n-j-1)} \\
&\leq \widetilde{V}_i(Q,C)^{(n-1-i)/((n-j-1)(n-i))} \widetilde{V}_i(I_j(L,D),C)^{1/((n-j-1)(n-i))}.
\end{aligned} \tag{5.6}$$

Combining (5.3)–(5.6), we obtain

$$\begin{aligned}
&\widetilde{V}(Q, n-i-1; C; I_j(K \widetilde{+} L, D))^{1/(n-j-1)} \widetilde{V}_i(Q,C)^{-(n-1-i)/((n-j-1)(n-i))} \\
&\leq \widetilde{V}_i(I_j(K,D),C)^{1/((n-j-1)(n-i))} + \widetilde{V}_i(I_j(L,D),C)^{1/((n-j-1)(n-i))}.
\end{aligned}$$

Take $Q = I_j(K \widetilde{+} L, D)$, and the inequality of Theorem 5.2 is obtained.

Consider the equality condition of Theorem 5.2. From the equality condition of (5.4), it follows that K and L are dilatates. There exists $a \geq 0$, such that $K = aL$.

Let $\rho_D(u) = \rho_{M'_1}(u) \cdots \rho_{M'_j}(u)$. When $K = aL$,

$$\begin{aligned}
\rho_{I_j(K \widetilde{+} L, D)}(u) &= \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_{K \widetilde{+} L}(v)^{n-j-1} \rho_D(v) dv \\
&= \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_{aL \widetilde{+} L}(v)^{n-j-1} \rho_D(v) dv \\
&= \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_{(a+1)L}(v)^{n-j-1} \rho_D(v) dv \\
&= (a+1)^{n-j-1} \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_L(v)^{n-j-1} \rho_D(v) dv \\
&= (a+1)^{n-j-1} \rho_{I_j(L,D)}(u).
\end{aligned}$$

In the same way, when $K = aL$, $\rho_{I_j(K \widetilde{+} L, D)}(u) = (1 + \frac{1}{a})^{n-j-1} \rho_{I_j(K,D)}(u)$.

Since (5.5) holds with equality if and only if $I_j(K \widetilde{+} L, D)$ and $I_j(K,D)$ are dilatates, and (5.6) holds with equality if and only if $I_j(K \widetilde{+} L, D)$ and $I_j(L,D)$ are dilatates, this shows that Theorem 5.2 holds.

Corollary 5.1. *If $K, L, M, M'_1, \cdots, M'_{n-2} \in \mathcal{L}_o^n$, $D = (M'_1, \cdots, M'_{n-2})$, then*

$$\widetilde{V}(M, n-1; I(D, K \widetilde{+} L)) \leq V(M)^{(n-1)/n} (V(I(D,K))^{1/n} + V(I(D,L))^{1/n}),$$

and the equality holds if and only if $M, I(D,L), I(D,L)$ are dilatates.

Proof. Taking $i = n-1,\ j = n-2,\ M_1 = \cdots = M_i = M$ in Theorem 5.2, we get

$$\widetilde{V}(M, n-1; I(D, K \widetilde{+} L)) \leq \widetilde{V}(M, n-1; I(D,K)) + \widetilde{V}(M, n-1; I(D,L)).$$

According to (1.7), Corollary 5.1 follows immediately.

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