Chin. Ann. Math.
26B:3(2005),457–466.

FLAT TIME-LIKE SUBMANIFOLDS IN ANTI-DE SITTER SPACE $H_1^{2n-1}(-1)$****

ZUO Dafeng*   CHEN Qing**   CHENG Yi***

Abstract

By using dressing actions of the $G^{1,1}_{n-1,n-1}$-system, the authors study geometric transformations for flat time-like n-submanifolds with flat, non-degenerate normal bundle in anti-de Sitter space $H_1^{2n-1}(-1)$, where $G^{1,1}_{n-1,n-1} = O(2n-2,2)/O(n-1,1) \times O(n-1,1)$.

Keywords Dressing action, $G^{1,1}_{n-1,n-1}$-System, Flat time-like submanifold
2000 MR Subject Classification 53A05, 35Q51

§ 1. Introduction

Recently Ferus and Pedit [1], and Terng et al [2, 3] established a beautiful relation between integrable systems and submanifold geometry. They found that submanifolds in certain symmetric space whose Gauss-Codazzi-Ricci equations are given by a nonlinear first order system, the U/K-system, which is putting the n first flows of ZS-AKNS together. This means to find submanifolds M in certain symmetric space whose Gauss-Codazzi-Ricci equations are equivalent to U/K-systems. The direct approach may provide ways to find Lax pairs for submanifolds M. Terng et al [2, 3] carried out the project for the real Grassmannian manifolds of space-like m-dimensional linear subspaces in R^{m+n} and in $R^{m+n,1}$. For instance, they proved that solutions of the $G^{0,0}_{m,n}(m \geq n)$- and the $G^{0,1}_{m,n}(m \geq n+1)$-system I correspond to local isometric immersion of $N^n(c)$ into $N^{n+m}(c)$, the $G^{0,0}_{m,n}(m > n)$- and the $G^{0,1}_{m,n}(m > n+1)$-system II give rise to n-tuple in R^m of type $O(n)$ and $(n+1)$-tuple in R^m of type $O(n,1)$ respectively. In [4–6], we obtained some space-like and time-like immersions associated with the $G^{p,q}_{m,n}$-system, where $G^{p,q}_{m,n} = O(m+n,p+q)/O(m,p) \times O(n,q)$.

The aim of this paper is to study geometric transformations for flat time-like n-submanifolds with flat, non-degenerate normal bundle in anti-de Sitter space $H_1^{2n-1}(-1)$ by using

Manuscript received October 31, 2003.
*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China; Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
E-mail: dfzuo@ustc.edu.cn
**Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
E-mail: qchen@ustc.edu.cn
***Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.
E-mail: chengy@ustc.edu.cn
****Project supported supported by the 973 Project of the Ministry of Science and Technology of China and the National Natural Science Foundation of China (No.10301030).

the $G^{1,1}_{n-1,n-1}$-system. This paper is organized as follows. In Section 2, we consider the geometry associated with the $G^{1,1}_{n-1,n-1}$-system. In Section 3, we construct a dressing action with a simple pole, and show that gives rise to Bäcklund transformations (BTs in brief) for flat time-like n-submanifolds in $H^{2n-1}_1(-1)$. In Section 4, we construct another dressing action with two simple poles, and prove that gives rise to Rabaucour transformations (RTs in brief) for flat time-like n-submanifolds in $H^{2n-1}_1(-1)$.

§ 2 . The Geometry Associated with the $G^{1,1}_{n-1,n-1}$-System

In this section, we review some known facts about the $G^{1,1}_{n-1,n-1}$-system and give a relation between flat time-like n-submanifolds in $H^{2n-1}_1(-1)$ and the $G^{1,1}_{n-1,n-1}$-system.

2.1. The general case of the $G^{1,1}_{n-1,n-1}$-system

Below we give a short review of some known facts about the $G^{1,1}_{n-1,n-1}$-system, see [2, 3, 6] for details. In this paper, we use the following notations

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

$$J = I_{n-1,1} = \mathrm{diag}(\epsilon_1, \cdots, \epsilon_n),$$

$$o(2n-2,2) = \{X \in \mathrm{gl}(2n) \mid X^t\widetilde{J} + \widetilde{J}X = 0\},$$

$$o(n-1,1) = \{X \in \mathrm{gl}(n) \mid X^tJ + JX = 0\},$$

$$G^{1,1}_{n-1,n-1} = O(2n-2,2)/O(n-1,1) \times O(n-1,1),$$

$$\mathrm{gl}(n)_* = \{(f_{ij}) \in \mathrm{gl}(n) \mid f_{ii} = 0,\ i = 1, \cdots, n\}.$$

The $G^{1,1}_{n-1,n-1}$-system, according to the terminology of [3], is the following PDE for $F = (f_{ij}) : R^n \to \mathrm{gl}(n)_*$ such that

$$\theta_\lambda = \begin{pmatrix} J\delta F^tJ - F\delta & \lambda\delta \\ -\lambda\delta & \delta F - JF^t\delta J \end{pmatrix} \tag{2.1}$$

is a family of flat connections on R^n for all $\lambda \in \mathbf{C}$, i.e.,

$$d\theta_\lambda + \theta_\lambda \wedge \theta_\lambda = 0, \tag{2.2}$$

where $\delta = \mathrm{diag}(dx_1, \cdots, dx_n)$. Hence there exists a smooth map $E : R^n \times \mathbf{C} \to O(2n-2,2)$ such that $E^{-1}dE = \theta_\lambda$ and $E(0,\lambda) = I$. Here E is often called the frame of θ_λ. The $G^{1,1}_{n-1,n-1}$-reality condition is

$$\begin{cases} \overline{g(\bar{\lambda})} = g(\lambda), \\ I_{n,n}g(\lambda)I_{n,n} = g(-\lambda), \\ g(\lambda)\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} g(\lambda)^t = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}. \end{cases} \tag{2.3}$$

Let $g = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \in O(n-1,1) \times O(n-1,1)$ be a solution of $g^{-1}dg = \theta_0$. Take the gauge transformation of θ_λ by

$$h = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}.$$

The resulting connection 1-form Ω_λ is

$$\Omega_\lambda = h\theta_\lambda h^{-1} - dhh^{-1} = \begin{pmatrix} J\delta F^t J - F\delta & J\delta A^t J\lambda \\ -A\delta\lambda & 0 \end{pmatrix}, \tag{2.4}$$

which is also a family of flat connections on R^n for all $\lambda \in \mathbf{C}$, i.e.,

$$\begin{cases} \epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \displaystyle\sum_{k=1}^n \epsilon_k f_{ki} f_{kj} = 0, & \text{if } i \neq j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj}, & \text{if } i,j,k \text{ are distinct}, \\ (a_{ij})_{x_k} = a_{ik} f_{kj}, & \text{if } j \neq k, \end{cases} \tag{2.5}$$

where $A = (a_{ij}),\ F = (f_{ij})$. Note that Eh^{-1} is the frame of Ω_λ. We call (2.5) the $G^{1,1}_{n-1,n-1}$-II system.

2.2. Flat time-like n-submanifolds in $H_1^{2n-1}(-1)$

In [13, Theorem 3.3], we have obtained

Lemma 2.1. *Let f be a flat time-like n-submanifold in $H_1^{2n-1}(-1)$ and satisfy* (i) *the second fundamental form is orthogonally diagonalizable, and* (ii) *there exists a point p of M where the principal normal curvatures are different from* 1. *Then on an open contractible region U of p, there exist a Chebyshev coordinate system $\{x_1, \cdots, x_n\}$ and $A = (a_{ij}) \in O(n-1,1)$ such that the two fundamental forms are*

$$\begin{aligned} \mathrm{I} &= \sum_{i=1}^n \epsilon_i a_{ni}^2 dx_i^2, \\ \mathrm{II} &= \sum_{i=1}^n \sum_{l=1}^{n-1} \epsilon_i a_{ni} a_{li} dx_i^2 e_{n+l}, \end{aligned} \tag{2.6}$$

where $\{e_{n+1}, \cdots, e_{2n-1}\}$ are local parallel normal frame fields.

By a direct verification, the Gauss-Codazzi-Ricci equations of M are the $G^{1,1}_{n-1,n-1}$-II system (2.5), which is gauge equivalent to the $G^{1,1}_{n-1,n-1}$-system (2.2). Hence the immersion f has a Lax pair

$$E^{-1}dE = \theta_\lambda = \begin{pmatrix} J\delta F^t J - F\delta & \delta\lambda \\ -\delta\lambda & \delta F - JF^t\delta J \end{pmatrix}. \tag{2.7}$$

Suppose $f = e_{2n},\ e_k = \frac{\mathrm{f}_{x_k}}{a_{nk}}$ for $1 \le k \le n$, and $g = (e_1, \cdots, e_{2n})$. Then we have

$$g^{-1}dg = \Omega_\lambda\big|_{\lambda=1} = \begin{pmatrix} J\delta F^t J - F\delta & J\delta A^t J \\ -A\delta & 0 \end{pmatrix}. \tag{2.8}$$

By using the fundamental theorem of pseudo-Riemannian geometry and Lemma 2.1, we get

Corollary 2.1. *Let (A,F) be a solution of the $G^{1,1}_{n-1,n-1}$-II system (2.5). Then (2.8) is solvable. Let g be a solution of (2.8) and f the $2n$-th volume of g. If all entries of the last row of A are non-zero, then f is a local isometric immersion of a flat time-like n-submanifold in $H^{2n-1}_1(-1)$ with flat, non-degenerate normal bundle such that the two fundamental forms are as in (2.6), where $A=(a_{ij})$.*

§ 3. Dressing Actions and BTs for Flat Time-Like n-Submanifolds in $H_1^{2n-1}(-1)$

In this section we shall construct a dressing action with a simple pole, and show that the dressing action on the space of solutions of the $G^{1,1}_{n-1,n-1}$-system (2.2) gives rise to BTs for flat time-like n-submanifolds in $H^{2n-1}_1(-1)$, which is constructed in [13].

Define a rational map with only a simple pole

$$K_{s,\beta}(\lambda)=\frac{1}{\lambda-is}\begin{pmatrix} s\beta & \lambda I_n \\ -\lambda I_n & sJ\beta^tJ\end{pmatrix}, \tag{3.1}$$

where $s\in R$ is a non-zero constant and $\beta\in O(n-1,1)$ is a constant matrix. It is easily verified that $g_{s,\beta}(\lambda)=\frac{\lambda-is}{\sqrt{\lambda^2+s^2}}K_{s,\beta}$ satisfies the $G^{1,1}_{n-1,n-1}$-reality condition (2.3). We call $K_{s,\beta}(\lambda)$ a simple element of the $G^{1,1}_{n-1,n-1}$-system (2.2) according to the terminology of [7, 8].

Let F be a solution of the $G^{1,1}_{n-1,n-1}$-system (2.2) and E the corresponding frame of F. It follows from the result of Terng and Uhlenbeck in [8] that $K_{s,\beta}E$ can be factored as $\widetilde{E}K_{s,\tilde\beta(x)}$ for some functions $\widetilde{E}$ and $K_{s,\tilde\beta(x)}$. Write $E(x,-is)=\begin{pmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{pmatrix}$ with $\eta_i\in \mathrm{gl}(n)$ and set

$$\tilde\beta=(i\eta_4-\beta\eta_2)^{-1}(i\beta\eta_1+\eta_3),\qquad \widetilde{E}=K_{s,\beta}E(x,\lambda)K^{-1}_{s,\tilde\beta(x)}(\lambda). \tag{3.2}$$

Note that

$$\mathrm{Res}(\widetilde{E},-is)=\frac{is}{2}\begin{pmatrix}\beta & -iI_n\\ iI_n & J\beta^tJ\end{pmatrix}\begin{pmatrix}\eta_1 & \eta_2\\ \eta_3 & \eta_4\end{pmatrix}\begin{pmatrix}J\tilde\beta^tJ & iI_n\\ -iI_n & \tilde\beta\end{pmatrix}.$$

It follows from (3.2) and $\beta\in O(n-1,1)$ that $\mathrm{Res}(\widetilde{E},-is)=0$. Since $\overline{E(x,-is)}=E(x,is)$, similarly we can show that $\mathrm{Res}(\widetilde{E},is)=0$. Hence $\widetilde{E}(x,\lambda)$ is holomorphic in $\lambda\in C$. Let $\tilde\theta_\lambda=\widetilde{E}^{-1}d\widetilde{E}$. Then $\tilde\theta_\lambda$ is holomorphic for $\lambda\in C$. By using $\theta_\lambda=E^{-1}dE$ and (3.2), we get $K_{s,\beta}\tilde\theta_\lambda=K_{s,\tilde\beta}\theta_\lambda-dK_{s,\tilde\beta}$. Comparing the coefficient of λ^j $(j=0,1,2)$, we have the following lemma.

Lemma 3.1. *Let F be a solution of the $G^{1,1}_{n-1,n-1}$-system (2.2) and E a frame of F. Then $\widetilde{F}=K_{s,\beta}\#F=JF^tJ+s\tilde\beta_*$ is a solution of the $G^{1,1}_{n-1,n-1}$-system (2.2) and $\widetilde{E}$ is a frame of $\widetilde{F}$, where $\tilde\beta_*$ is the matrix whose ij-th entry is $\tilde\beta_{ij}$ for $i\neq j$ and is 0 for $i=j$.*

Let $s_1,\ s_2$ be two unequal nonzero real constants, and $\beta_1, \beta_2 \in O(n-1,1)$ are two constant matrices. If $\phi = s_1\beta_1 - s_2\beta_2$ is non-singular, $\beta_1^t J\beta_2 \neq \frac{s_2}{s_1}J$ and $\beta_2^t J\beta_1 \neq \frac{s_1}{s_2}J$, we may set

$$\alpha_1 = (s_1 I_n - s_2 J\beta_2^t J\beta_1)\phi^{-1}, \qquad \alpha_2 = (s_1 J\beta_1^t J\beta_2 - s_2 I_n)\phi^{-1}. \tag{3.3}$$

It is easily verified that $\alpha_i \in O(n-1,1)$ for $i = 1,\ 2$ and $K_{s_1,\alpha_1} \circ K_{s_2,\beta_2} = K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}$. Moreover if $K_{s_1,\alpha_1} \circ K_{s_2,\beta_2} = K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}$, then $\beta_1,\ \beta_2$ and $\alpha_1,\ \alpha_2$ are related as in (3.3). By using this observation, we have the following permutability formula.

Lemma 3.2. *Let $s_i,\ \beta_i,\ \alpha_i$ for $i = 1,\ 2$ as above, F be a solution of the $G^{1,1}_{n-1,n-1}$-system* (2.2) *and $F_i = K_{s_i,\beta_i}\#F = JF^tJ + s_i\tilde{\beta}_{i*}$ for $i = 1,\ 2$ as given in Lemma* 3.1. *Then the permutability formula is*

$$\begin{aligned} F_3 &= (K_{s_1,\alpha_1} \circ K_{s_2,\beta_2})\#F = JF^tJ + s_1\tilde{\alpha}_{1*} + s_2\tilde{\beta}_{2*} \\ &= (K_{s_2,\alpha_2} \circ K_{s_1,\beta_1})\#F = JF^tJ + s_2\tilde{\alpha}_{2*} + s_1\tilde{\beta}_{1*}. \end{aligned} \tag{3.4}$$

According to Corollary 2.1, we know that Lemmas 3.1 and 3.2 give a method of constructing a new flat time-like n-submanifold in $H_1^{2n-1}(-1)$ from a given one. Geometrically, this gives the geometric transformation which is the BT and the analogue of the classical Bianchi theorem obtained in [13].

Theorem 3.1. *Let $F, E, K_{s,\beta}, \tilde{\beta}, \widetilde{E}, \widetilde{F}$ be as in Lemma* 3.1. *Write*

$$E(x,0) = \begin{pmatrix} B(x) & 0 \\ 0 & A(x) \end{pmatrix}, \qquad \widetilde{E}(x,0) = \begin{pmatrix} \widetilde{B}(x) & 0 \\ 0 & \widetilde{A}(x) \end{pmatrix}. \tag{3.5}$$

Let

$$\begin{aligned} N(x) &= E^{\mathrm{II}}(x,1) = E(x,1)\begin{pmatrix} I_n & 0 \\ 0 & A^{-1}(x) \end{pmatrix}, \\ \widetilde{N}(x) &= g_{s,\beta}^{-1}(1)\widetilde{E}(x,1)\begin{pmatrix} I_n & 0 \\ 0 & (A\tilde{\beta})^{-1}(x) \end{pmatrix} \\ &= N(x)\begin{pmatrix} \cos\tau J\tilde{\beta}^t J & -\sin\tau(A\tilde{\beta})^{-1} \\ \sin\tau A & \cos\tau I_n \end{pmatrix}, \end{aligned} \tag{3.6}$$

where $\tau = \arctan\dfrac{1}{s}$. Let e_i and $\tilde{e}_i$ for $1 \le i \le 2n$ denote the i-th column of $N(x)$ and $\widetilde{N}(x)$ respectively. Then $\mathcal{L} : e_{2n}(x) \to \tilde{e}_{2n}(x)$ is a BT for flat time-like n-submanifolds with constant τ in $H_1^{2n-1}(-1)$ and the line congruence is time-like.

Proof. It follows from $\beta \in O(n-1,1)$ that $\begin{pmatrix} J\beta^t J & 0 \\ 0 & \beta \end{pmatrix}\widetilde{E}(x,0)$ is also a frame of $\tilde{\theta}_\lambda$ as in the form of (2.1) for $\widetilde{F}$ at $\lambda = 0$. By using (2.4), (3.6) and Theorem 3.1, we know that (A, F) and $(\widetilde{A}, \widetilde{F})$ are solutions of the $G^{1,1}_{n-1,n-1}$-II system (2.5) and $N,\ \widetilde{N}$ are the corresponding frames at $\lambda = 1$ respectively. By using Corollary 2.1, $e_{2n}, \tilde{e}_{2n}$ are flat time-like n-submanifolds with flat, non-degenerate normal bundle in $H_1^{2n-1}(-1)$, where $\{e_{n+1},\cdots,e_{2n-1}\}$ and $\{\tilde e_{n+1},\cdots,\tilde e_{2n-1}\}$ are parallel normal frames for e_{2n} and $\tilde e_{2n}$ respectively. Let

$$N_A = N\begin{pmatrix}(A\tilde\beta)^{-1} & 0\\ 0 & I_n\end{pmatrix}, \qquad \widetilde N_A = \widetilde N\begin{pmatrix}A^{-1} & 0\\ 0 & I_n\end{pmatrix}.$$

Then we have

$$\begin{aligned}\widetilde N_A &= N\begin{pmatrix}\cos\tau J\tilde\beta^t J & -\sin\tau(A\tilde\beta)^{-1}\\ \sin\tau A & \cos\tau I_n\end{pmatrix}\begin{pmatrix}A^{-1} & 0\\ 0 & I_n\end{pmatrix}\\ &= N_A\begin{pmatrix}\cos\tau I_n & -\sin\tau I_n\\ \sin\tau I_n & \cos\tau I_n\end{pmatrix}.\end{aligned} \tag{3.7}$$

The last n column vectors of N and N_A, $\widetilde N$ and $\widetilde N_A$ are the same and they are normal frames. Geometrically, (3.7) is the BT for flat time-like n-submanifolds in $H_1^{2n-1}(-1)$.

By using Lemma 3.2 and Theorem 3.1, we get the following analogue of the classical Bianchi theorem.

Theorem 3.2. *Let $\mathcal{L}^i : M \to M_i$ be BTs for flat time-like n-submanifolds in H_1^{2n-1} corresponding to the solution of K_{s_i,β_i} for $i = 1, 2$ as in Theorem 3.1. If s_1, s_2 are two unequal nonzero real constants, $\beta_1, \beta_2 \in O(n-1,1)$ are constant matrices and $s_1\beta_1 - s_2\beta_2$ is non-singular, $\beta_1^t J\beta_2 \neq \frac{s_2}{s_1}J$ and $\beta_2^t J\beta_1 \neq \frac{s_1}{s_2}J$, then there exists a unique flat time-like n-submanifold M_3 in H_1^{2n-1} and BTs $\widetilde{\mathcal{L}}^1 : M_2 \to M_3$, $\widetilde{\mathcal{L}}^2 : M_1 \to M_3$ such that $\widetilde{\mathcal{L}}^1 \circ \mathcal{L}^2 = \widetilde{\mathcal{L}}^2 \circ \mathcal{L}^1$.*

§ 4. Dressing Actions and RTs for Flat Time-Like n-Submanifolds in $H_1^{2n-1}(-1)$

In [6], we have obtained an explicit construction of the dressing action of a rational map with two simple poles of solutions of the general $G^{p,q}_{m,n}$-system. Hence we only state our results here.

Let C^{2n} be equipped with the bi-linear form

$$\langle u,v\rangle_1 = -\sum_{i=1}^{n-1}\bar u_i v_i + \bar u_n v_n - \sum_{i=n+1}^{2n-1}\bar u_i v_i + \bar u_{2n}v_{2n}.$$

Let W and Z be two unit space-like constant vectors in $R^{1,n-1}$ respectively, and π the orthogonal projection onto the space of $\mathbf{C}\begin{pmatrix}W\\ iZ\end{pmatrix}$ with respect to $\langle\,,\,\rangle_1$. Let $0 \neq s \in R$, and define

$$g_{s,\pi} = \Big(\pi + \frac{\lambda - is}{\lambda + is}(I-\pi)\Big)\Big(\bar\pi + \frac{\lambda + is}{\lambda - is}(I-\bar\pi)\Big). \tag{4.1}$$

Lemma 4.1. *Let $F : R^n \to \mathrm{gl}_*(n)$ be a solution of the $G^{1,1}_{n-1,n-1}$-system* (2.2)*, and $E(x,\lambda)$ a frame of F such that $E(x,\lambda)$ is holomorphic for $\lambda \in \mathbf{C}$ and $E(0,\lambda) = I$. Let $g_{s,\pi}$ be as in* (4.1) *and*

$$\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}(x) = E(x,-is)^{-1}\begin{pmatrix} W \\ iZ \end{pmatrix}, \tag{4.2}$$

and $\tilde{\pi}$ be the orthogonal projection onto the space of $\mathbf{C}\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}$ with respect to $\langle\, ,\, \rangle_1$. Then $\widetilde{F} = g_{s,\pi}\sharp F = F + 2s(\widehat{W}\widehat{Z}^tJ)_*$ is a new solution of (2.2)*, and $\widetilde{E} = Eg^{-1}_{s,\tilde{\pi}}$ is a frame for $\widetilde{F}$, where $\widehat{W} = \dfrac{\widetilde{W}}{\|W\|_{R^{1,n-1}}}$ and $\widehat{Z} = \dfrac{\widetilde{Z}}{\|Z\|_{R^{1,n-1}}}$.*

Note that both E and $\widetilde{E}$ satisfy the $G^{1,1}_{n-1,n-1}$-reality condition (2.3) which implies that $E(x,0)$ and $\widetilde{E}(x,0)$ are in $O(n-1,1)\times O(n-1,1)$. Write

$$E(x,0) = \begin{pmatrix} B(x) & 0 \\ 0 & A(x) \end{pmatrix}, \qquad \widetilde{E}(x,0) = \begin{pmatrix} \widetilde{B}(x) & 0 \\ 0 & \widetilde{A}(x) \end{pmatrix}$$

for some $A,\ B,\ \widetilde{A}$ and $\widetilde{B}$. Hence we have

$$\begin{cases} \widetilde{A} = A(I + 2\widehat{Z}\widehat{Z}^tJ), \\ \widetilde{B} = B(I + 2\widehat{W}\widehat{W}^tJ). \end{cases}$$

Note that (A,F) and $(\widetilde{A},\widetilde{F})$ are solutions of the $G^{1,1}_{n-1,n-1}$-II system (2.5), the corresponding frames are $E^{\mathrm{II}}(x,\lambda)$ and $\widetilde{E}^{\mathrm{II}}(x,\lambda)$, where

$$E^{\mathrm{II}}(x,\lambda) = E(x,\lambda)\begin{pmatrix} I_m & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad \widetilde{E}^{\mathrm{II}}(x,\lambda) = \widetilde{E}(x,\lambda)\begin{pmatrix} I_m & 0 \\ 0 & \widetilde{A}^{-1} \end{pmatrix}.$$

It follows from Lemma 4.1 that

$$\widetilde{E}^{\mathrm{II}}(x,\lambda) = E^{\mathrm{II}}(x,\lambda)\left(I - \frac{2}{s^2+\lambda^2}\begin{pmatrix} s\widetilde{W} \\ -\lambda A\widetilde{Z} \end{pmatrix}(s\widetilde{W}^tJ, -\lambda\widetilde{Z}^tA^tJ)\right). \tag{4.3}$$

In the following, we use the notation

$$(\widetilde{A},\widetilde{F},\widetilde{E}^{\mathrm{II}}) = g_{s,\pi}\sharp(A,F,E^{\mathrm{II}}).$$

Analogous to the discussion of Lemma 3.2 (or see [3, 8]), we may also obtain the following permutability formula.

Lemma 4.2. *Let F be a solution of the $G^{1,1}_{n-1,n-1}$-system* (2.2)*, and E the frame of F such that $E(0,\lambda) = I$. Let W_k and Z_k be two unit space-like vectors in $R^{1,n-1}$, $v_k = \begin{pmatrix} W_k \\ iZ_k \end{pmatrix}$ and π_k the orthogonal projection onto v_k with respect to $\langle\, ,\, \rangle_1$ for $k = 1, 2$. Let $s_1, s_2 \in R$ be constants such that ${s_1}^2 \neq {s_2}^2$ and $s_1s_2 \neq 0$. Let u_k denote the unit space-like direction* *of $g_{s_j,s_k}(-is_k)(v_k)$ for $j \neq k$, and τ_k the orthogonal projection onto u_k with respect to $\langle\, ,\, \rangle_1$ for $k = 1, 2$. Then*

$$F_3 = (g_{s_2,\tau_2} g_{s_1,\pi_1})\sharp\xi = (g_{s_1,\tau_1} g_{s_2,\pi_2})\sharp\xi$$

is a new solution of the $G^{1,1}_{n-1,n-1}$-system (2.2).

RTs for surfaces in R^3 have been studied by [11, 12] and references therein. Natural generalizations of RTs for holonomic submanifolds with arbitrary dimension and codimension, that is, Riemannian submanifolds with flat normal bundle admitting a global system of principal coordinates, in (pseudo-) Riemannian space forms $N^m_p(c)$ $(p \geq 0)$ are given in [10]. In [6] we generalize it to time-like submanifolds in pseudo-Riemannian space forms $N^{m+p}_m(c)$. In the following we shall give the geometric interpretation of the dressing action of $g_{s,\pi}$ on the solution of the $G^{1,1}_{n-1,n-1}$-system (2.2).

Theorem 4.1. *Let E^{II} be a frame of the solution (A, F) of the $G^{1,1}_{n-1,n-1}$-*II *system* (2.5)*, $g_{s,\pi}$ given by* (4.1) *and $(\widetilde{A}, \widetilde{F}, \widetilde{E}^{\mathrm{II}}) = g_{s,\pi}\sharp(A, F, E^{\mathrm{II}})$. Write*

$$E^{\mathrm{II}}(x,1) = (e_1, \cdots, e_{2n-1}, X), \qquad \widetilde{E}^{\mathrm{II}}(x,1) = (\tilde{e}_1, \cdots, \tilde{e}_{2n-1}, \widetilde{X}). \tag{4.4}$$

Then

(i) *X and $\widetilde{X}$ are local isometric immersions of flat time-like n-dimensional sub-manifolds in anti-de Sitter space $H^{2n-1}_1(-1)$ with flat, non-degenerate normal bundle, $\{x_1, \cdots, x_n\}$ line of curvature coordinates, $\{e_{n+k}\}^{n-1}_{k=1}$ and $\{\tilde{e}_{n+k}\}^{n-1}_{k=1}$ are parallel normal frames for X and $\widetilde{X}$ respectively.*

(ii) *The bundle morphism $P : \vartheta(X) \to \vartheta(\widetilde{X})$ defined by $P(e_{n+k}(x)) = \tilde{e}_{n+k}(\mathcal{R}(x))$ for $1 \leq k \leq n-1$ is an RT covering the map $\mathcal{R} : X(x) \to \widetilde{X}(x)$.*

Proof. (i) follows from Corollary 2.1 and Lemma 4.1.

(ii) We first show that if (A, F) is a solution of the $G^{1,1}_{n-1,n-1}$-II system (2.5), then F is a solution of the $G^{1,1}_{n-1,n-1}$-system (2.2). Take a gauge transformation on Ω_λ by

$$h_1 = \begin{pmatrix} I_n & 0 \\ 0 & A^{-1} \end{pmatrix},$$

then the resulting 1-form is

$$h_1 * \Omega_\lambda = h_1 \Omega_\lambda h_1^{-1} - dh_1 h_1^{-1} = \begin{pmatrix} J\delta F^t J - FC_i^t & \delta\lambda \\ -\delta\lambda & A_1^{-1} dA_1 \end{pmatrix}.$$

Since (2.5), we have

$$A^{-1} A_{x_i} - (C_i^t F - JF^t C_i J) = YC_i, \qquad 1 \leq i \leq n, \tag{4.5}$$

where $Y : R^n :\to \mathrm{gl}(n)$. By using (4.5) and $A \in O(n-1, 1)$, we get

$$YC_iJ + (YC_iJ)^t = 0$$

for all $1 \leq i \leq n$ which implies that $Y = 0$. Hence F is a solution of the $G^{1,1}_{n-1,n-1}$-system (2.2).

By using Lemma 2.1, we know that (A, F) and $(\widetilde{A}, \widetilde{F})$ are solutions of the $G^{1,1}_{n-1,n-1}$-II system (2.5) corresponding to X and $\widetilde{X}$. Let $\widehat{W}, \widehat{Z}$ be given as in Lemma 4.1. Let

$$\gamma = (\gamma_1, \cdots, \gamma_{2n-1}, \gamma_{2n}) = (\cos\tau \widehat{W}^t, \sin\tau \widehat{Z}^t A^t),$$

where $\cos\tau = \frac{s}{\sqrt{1+s^2}}$ and $\sin\tau = \frac{1}{\sqrt{1+s^2}}$. Substituting $\lambda = 1$ into (4.3), we get

$$\widetilde{E}^{\mathrm{II}}(x, 1) = E^{\mathrm{II}}(x, 1)(I - 2\gamma^t\gamma)\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}. \tag{4.6}$$

Substituting (4.4) into (4.6), we have

$$\tilde{e}_k = e_k - 2\epsilon_k\gamma_k \sum_{j=1}^{2n} \gamma_j e_j, \qquad k = 1, \cdots, 2n,$$

where $\epsilon_k = \epsilon_{n+k}$ for $1 \leq k \leq n$. Since $X = e_{2n}$ and $\widetilde{X} = \tilde{e}_{2n}$, we obtain

$$\epsilon_k\gamma_k X + \gamma_{2n} e_k = \epsilon_k\gamma_k \widetilde{X} + \gamma_{2n}\tilde{e}_k, \qquad k = 1, \cdots, 2n-1. \tag{4.7}$$

Let

$$\Gamma_k = \begin{cases} \arctan\dfrac{\gamma_{2n}}{\gamma_n}, & k = n, \\ \operatorname{arctanh}\dfrac{\gamma_{2n}}{\gamma_k}, & k \neq n, \end{cases}$$

where $1 \leq k \leq 2n-1$. Then (4.7) becomes

$$\begin{aligned} \cosh\Gamma_k X + \sinh\Gamma_k e_k &= \cosh\Gamma_k \widetilde{X} + \sinh\Gamma_k \tilde{e}_k, \qquad k \neq n, \\ \cos\Gamma_n X - \sin\Gamma_n e_n &= \cos\Gamma_n \widetilde{X} - \sin\Gamma_n \tilde{e}_n. \end{aligned} \tag{4.8}$$

Geometrically, (4.8) means that the geodesic of $H_1^{2n-1}(-1)$ at $X(x)$ in the direction $e_k(x)$ intersects the geodesic of $H_1^{2n-1}(-1)$ at $\widetilde{X}(x)$ in the direction $\tilde{e}_k(x)$ at a point equidistant to $X(x)$ and $\widetilde{X}(x)$. Hence the bundle morphism $P : \vartheta(X) \to \vartheta(\widetilde{X})$ is an RT.

As a consequence of Lemma 4.2 and Theorem 4.1, we get the following permutability theorem.

Theorem 4.2. *Let $P_i : \vartheta(X) \to \vartheta(X_i)$ be RTs for flat time-like n-submanifolds in anti-de Sitter space $H_1^{2n-1}(-1)$ corresponding to the action g_{s_i,π_i} for $i = 1, 2$. If ${s_1}^2 \neq {s_2}^2$ and $s_1 s_2 \neq 0$, then there exist a unique flat time-like n-submanifold X_3 in $H_1^{2n-1}(-1)$ and RTs $\widetilde{P}_1 : \vartheta(X_2) \to \vartheta(X_3)$, $\widetilde{P}_2 : \vartheta(X_1) \to \vartheta(X_3)$ such that $\widetilde{P}_1 \circ P_2 = \widetilde{P}_2 \circ P_1$.*

Acknowledgement. The first author thanks Professor Zhang Youjin for helpful comments and constant encouragement.

References

[1] Ferus, D. & Pedit, F., Curved flats in symmetric spaces, *Manuscripta Math.*, **91**(1996), 445–454.

[2] Terng, C. L., Soliton equations and differential geometry, *J. Diff. Geom.*, **45**(1997), 407–455.

[3] Brück, M., Du, X., Park, J. S. & Terng, C. L., The submanifold geometry associated to Grassmannian system, *Mem. Amer. Math. Soc.*, **155:**735(2002), 1–95.

[4] Zuo, D., Chen, Q. & Cheng, Y., Flat time-like submanifolds in $S^{2n-1}_{2q}(1)$, *J. Phys. A*, **47**(2002), 10197–10204.

[5] Zuo, D., Chen, Q. & Cheng, Y., Darboux transformations for space-like isothermic surfaces in $R^{m,1}$, *Commun. Theor. Phys.*, **41**(2004), 816–820.

[6] Zuo, D., Chen, Q. & Cheng, Y., $G^{p,q}_{m,n}$-II systems and diagonalizable time-like immersions in $R^{p,m}$, *Inverse Problems*, **20**(2004), 319–329.

[7] Uhlenbeck, K., Harmonic maps into Lie groups (classical solutions of the chiral model), *J. Diff. Geom.*, **30**(1989), 1–50.

[8] Terng, C. L. & Uhlenbeck, K., Bäcklund transformations and loop groups, *Comm. Pure. Appl. Math.*, **53**(2000), 1–75.

[9] Zhou, Z. X., Darboux transformations for the twisted so(p,q)-system and local isometric immersions of space forms, *Inverse Problem*, **14**(1998), 1353–1370.

[10] Dajczer, M. & Tojeiro, R., An extension of the classical Ribaucour transformation, *Proc. London Math. Soc.*, **85:**3(2002), 211–232.

[11] Darboux, G., Lecon Sur La Théorie Générale Des Surfaces, 3rd edition, Chelsea, 1972.

[12] Tenenblat, K., On the Ribaucour transformations and applications to linear weigarten surfaces, *Annals of Brazilian Academy of Sciences*, **74:**4(2002), 559–575.

[13] Chen, Q., Zuo, D. & Cheng, Y., Isometric immersions of pseudo-Riemannian space forms, *J. of Geom. and Phys.*, **52:**3(2004), 241–262.