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MONOTONIZATION IN GLOBAL OPTIMIZATION****

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Abstract

A general monotonization method is proposed for converting a constrained programming problem with non-monotone objective function and monotone constraint functions into a monotone programming problem. An equivalent monotone programming problem with only inequality constraints is obtained via this monotonization method. Then the existing convexification and concavefication methods can be used to convert the monotone programming problem into an equivalent better-structured optimization problem.

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§ 1 . Introduction

We consider a global optimization problem of the following form:

$$\begin{aligned} \min\ & f(x), \\ \text{s.t.}\ & g_i(x) = 0, \qquad i = 1, \cdots, m, \\ & g_i(x) \le 0, \qquad i = m+1, \cdots, m_0, \\ & x \in R^n, \end{aligned} \tag{1.1}$$

where $f : R^n \to R$, $g_i : R^n \to R$, $i = 1, 2, \cdots, m_0$.

It is well known that when both the objective function and the constraint set are convex, then any local minimizer of the problem (1.1) is the global minimizer. Many efficient algorithms can be used to obtain a local minimizer (see [1, 3]). When the objective function or the constraint set fails to be convex, a local minimizer may not be a global one. Up to now, the global optimization techniques for general nonconvex programming problem are not well developed (see [9]). Fortunately, many nonconvex programming problems encountered in real life possess some kind of convexity, and can be formulated into concave minimization problem or reverse convex programming problem, or more general, D. C. programming problem. Some prominent features in these nonconvex programming problems lead to the

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development of various implementable global optimal algorithms for solving these programming problems (see, e.g., [2, 4–6, 8, 11]). Thus, if a programming problem can be converted into an equivalent concave minimization problem or reverse convex programming problem or D. C. programming problem, then its global optimal solution can be obtained by using the existing algorithms.

Recently, some convexification and concavification transformations have been proposed to convert a strictly monotone programming problem into an equivalent concave minimization problem or reverse convex programming problem or canonical D. C. programming problem (see, e.g., [7, 10, 12, 13]). Thus the global optimal solution of a strictly monotone programming problem can be obtained by solving the converted better structured programming problem via the existing algorithms. In [7], the authors established a special monotonization transformation to convert a non-monotone objective function with a single linear constraint into a strictly monotone objective function and showed that a non-monotone programming problem with a single linear constraint can be converted into an equivalent strictly monotone programming problem. In [13], the authors gave another special monotonization transformation for a programming problem with multiple linear constraints, but they did not give the proof for the equivalence between the primal problem and the converted monotone programming problem.

In this paper, we propose a monotonization transformation to convert a non-monotone programming problem with general monotone constraints into an equivalent monotone programming problem. The rigorous proof for the equivalence between the primal problem and the converted monotone programming problem is given.

The rest of this paper is organized as follows. In Section 2, we present a general monotonization transformation to convert a non-monotone objective function into a monotone one. The equivalence between the primal problem and the converted monotone programming problem is established in Section 3.

§ 2. Monotonization Transformation

To begin with, we give the following definitions.

Definition 2.1. *A function $h : R^n \to R$ is called an increasing (decreasing) function in a set $D \subset R^n$ if for any $x, y \in D$ with $x_i \le y_i$, $i = 1, \cdots, n$, we have $h(x) \le (\ge) h(y)$.*

Definition 2.2. *A function $h : R^n \to R$ is called a strictly increasing (decreasing) function in a set $D \subset R^n$ if for any $x, y \in D$ with $x_i \le y_i$, $i = 1, \cdots, n$, and $x \ne y$, we have $h(x) < (>) h(y)$.*

Definition 2.3. *A programming problem is called a monotone programming problem if each of the objective function and the constraint functions is monotone (increasing or decreasing) on the feasible set. A programming problem is called a strictly monotone programming problem if each of the objective function and the constraint functions is strictly monotone (strictly increasing or strictly decreasing) on the feasible set.*

Let

$$S_0 = \{x \in R^n \mid g_i(x) = 0,\ i = 1, \cdots, m;\ g_i(x) \le 0,\ i = m+1, \cdots, m_0\}.$$

We make the following assumptions.

Assumption 2.1. *S_0 is a compact set or $f(x)$ satisfies the following coercive condition:*

$$\lim_{||x||\to\infty} f(x) = +\infty.$$

Then there exists a box

$$X = \{x \in R^n \mid c_i \le x_i \le d_i,\ i = 1, \cdots, n\}, \tag{2.1}$$

such that

$$G(1.1) \subset \mathrm{int}(X), \tag{2.2}$$

where $c_i, d_i \in R,\ c_i \le d_i,\ i = 1, \cdots, n$, $G(1.1)$ is the set of global minima of the problem (1.1) and $\mathrm{int}(X)$ denotes the interior of X. Then the problem (1.1) is equivalent to the following problem as far as the global minima are concerned:

$$\begin{aligned} \min\ \ & f(x), \\ \text{s.t.}\ \ & g_i(x) = 0, \qquad i = 1, \cdots, m \\ & g_i(x) \le 0, \qquad i = m+1, \cdots, m_0, \\ & x \in X, \end{aligned} \tag{2.3}$$

i.e., $G(1.1) = G(2.3)$, where $G(2.3)$ is the set of global minima of the problem (2.3).

$$S = \{x \in X \mid g_i(x) = 0,\ i = 1, \cdots, m;\ g_i(x) \le 0,\ i = m+1, \cdots, m_0\}. \tag{2.4}$$

Assumption 2.2. *Each of the equality constraints $g_i(x),\ i = 1, \cdots, m$, is monotone on X. Furthermore, there exists a positive number $\delta_0 > 0$, such that*

$$\sum_{i\in I} \frac{\partial g_i(x)}{\partial x_k} - \sum_{j\in \bar{I}} \frac{\partial g_j(x)}{\partial x_k} \ge \delta_0, \qquad \forall\, x \in X,\ k = 1, \cdots, n, \tag{2.5}$$

where

$$\begin{aligned} & I = \{i \in \{1, \cdots, m\} \mid g_i(x) \text{ is increasing}\}, \\ & \bar{I} = \{1, \cdots, m\} \setminus I = \{i \in \{1, \cdots, m\} \mid g_i(x) \text{ is decreasing}\}. \end{aligned} \tag{2.6}$$

Note that here we just require that the equality constraints are monotone and satisfy (2.5). In some cases, in order to assure that the condition (2.5) holds, we should introduce some relaxation variables to convert some monotone inequality constraints into equality constraints. Throughout the rest of this paper, we suppose that $f(x) > 0$ for all $x \in X$. Otherwise, we can add a very large positive number M to $f(x)$ to make $f(x)+M$ be positive on X.

Let

$$\phi_q(x) = T(r_{1,q}(g_1(x)), r_{2,q}(g_2(x)), \cdots, r_{m,q}(g_m(x)), f(x)), \tag{2.7}$$

$$\widetilde{X} = \{x \in X \mid g_i(x) \ge 0,\ i \in I;\ g_i(x) \le 0,\ i \in \bar{I}\} \tag{2.8}$$

$$0 < m_0 \le \min_{x\in X} f(x) \tag{2.9}$$

$$\widehat{R}^{m+1}_{+} = \{x \in R^{m+1} \mid x_i \ge 0,\ i = 1, \cdots, m;\ x_{m+1} \ge m_0\}, \tag{2.10}$$

where $T : A \to R,\ r_{i,q} : B_i \to R,\ A \subset R^{m+1},\ B_i \subset R,\ i = 1, 2, \cdots, m$, and q is a parameter.

The following theorem shows that $\phi_q(x)$ is strictly increasing on $\widetilde{X}$ under some conditions.

Theorem 2.1. *Suppose that*

(i) *$f,\ g_i,\ i = 1, 2, \cdots, m$, are continuously differentiable on X;*

(ii) *Assumption 2.2 holds;*

(iii) *T is continuously differentiable and strictly increasing on $\widehat{R}_+^{m+1}$, furthermore, there exists an $\varepsilon_0 > 0$, such that*

$$\frac{\partial T(z)}{\partial z_i} \geq \varepsilon_0 \qquad \text{for any } z \in \widehat{R}_+^{m+1} \text{ and any } i = 1, \cdots, m,$$

and an $M_0 > 0$, such that

$$0 < \frac{\partial T(z)}{\partial z_{m+1}} \leq M_0 \qquad \text{for any } z \in \widehat{R}_+^{m+1},$$

where $\widehat{R}_+^{m+1}$ is given in (2.10);

(iv) *for any $i \in I$, $r_{i,q}$ is continuously differentiable and strictly increasing on R_+ and satisfies*

$$\begin{aligned} &r_{i,q}(0) \geq 0, \\ &r'_{i,q}(t) \rightrightarrows +\infty \ (q \to +\infty) \qquad \text{for all } \ t \in R_+; \end{aligned} \tag{2.11}$$

for any $i \in \bar{I}$, $r_{i,q}$ is continuously differentiable and strictly decreasing on R_- and satisfies

$$\begin{aligned} &r_{i,q}(0) \geq 0, \\ &r'_{i,q}(t) \rightrightarrows -\infty \ (q \to +\infty) \qquad \text{for all } \ t \in R_-, \end{aligned} \tag{2.12}$$

where $R_+ = \{t \mid t \geq 0,\ t \in R\}, R_- = \{t \mid t \leq 0,\ t \in R\}$, "$\rightrightarrows$" represents the uniform convergence.

Then, for any given $\varepsilon_1 > 0$, there exists a $q_0 > 0$, such that for all $q > q_0$,

$$\frac{\partial \phi_q(x)}{\partial x_k} \geq \varepsilon_1, \qquad \forall\, x \in \widetilde{X},\ k = 1, 2, \cdots, n,$$

where $\widetilde{X}$ is defined by (2.8).

Proof. Let $z_i = r_{i,q}(g_i(x)),\ i = 1, \cdots, m,\ z_{m+1} = f(x),\ z = (z_1, \cdots, z_{m+1})$. Thus, for any $x \in \widetilde{X}$, we have $z \in \widehat{R}_+^{m+1}$. By (2.7), for any $x \in \widetilde{X},\ k = 1, \cdots, n$, we have

$$\begin{aligned} \frac{\partial \phi_q(x)}{\partial x_k} &= \sum_{i=1}^{m} \frac{\partial T(z)}{\partial z_i} r'_{i,q}(g_i(x)) \frac{\partial g_i(x)}{\partial x_k} + \frac{\partial T(z)}{\partial z_{m+1}} \frac{\partial f(x)}{\partial x_k} \\ &= \sum_{i \in I} \frac{\partial T(z)}{\partial z_i} r'_{i,q}(g_i(x)) \frac{\partial g_i(x)}{\partial x_k} + \sum_{i \in \bar{I}} \frac{\partial T(z)}{\partial z_i} r'_{i,q}(g_i(x)) \frac{\partial g_i(x)}{\partial x_k} + \frac{\partial T(z)}{\partial z_{m+1}} \frac{\partial f(x)}{\partial x_k}. \end{aligned} \tag{2.13}$$

Let $\lambda_0 \leq \min\limits_{k=1,\cdots,n} \min\limits_{x \in X} \frac{\partial f(x)}{\partial x_k}$.

Then

$$\frac{\partial f(x)}{\partial x_k} \geq \lambda_0 \geq -|\lambda_0|, \qquad \forall\, x \in X,\ k = 1, \cdots, n.$$

For a given positive number $\varepsilon_1 > 0$, for $\varepsilon_2 = \frac{\varepsilon_1 + |\lambda_0| M_0}{\varepsilon_0 \delta_0} > 0$, by the condition (2.11), there exists a $q_0' > 0$, such that, when $q > q_0'$,

$$r_{i,q}'(t) > \varepsilon_2, \qquad \forall\, t \in R_+,\ \forall i \in I;$$

by the condition (2.12), there exists a $q_0'' > 0$, such that, when $q > q_0''$,

$$r_{i,q}'(t) < -\varepsilon_2, \qquad \forall t \in R_-,\ \forall i \in \bar{I},$$

where ε_0 and M_0 are given in the condition (iii). Let $q_0 = \max\{q_0', q_0''\}$. By (2.13), we have

$$\begin{aligned} \frac{\partial \phi_q(x)}{\partial x_k} &> \sum_{i \in I} \varepsilon_0 \varepsilon_2 \frac{\partial g_i(x)}{\partial x_k} + \sum_{i \in \bar{I}} \varepsilon_0(-\varepsilon_2) \frac{\partial g_i(x)}{\partial x_k} - M_0|\lambda_0| \\ &= \varepsilon_0 \varepsilon_2 \Big[\sum_{i \in I} \frac{\partial g_i(x)}{\partial x_k} - \sum_{i \in \bar{I}} \frac{\partial g_i(x)}{\partial x_k} \Big] - |\lambda_0| M_0 \\ &\geq \varepsilon_0 \delta_0 \frac{\varepsilon_1 + |\lambda_0| M_0}{\varepsilon_0 \delta_0} - |\lambda_0| M_0 = \varepsilon_1 \end{aligned}$$

for any $x \in \widetilde{X}$, any $k = 1, \cdots, n$, and any $q > q_0$.

Let

$$\widehat{X} = \{x \in R^n \mid g_i(x) \leq 0 \ \text{ for all } \ i \in I;\ g_i(x) \geq 0 \ \text{ for all } \ i \in \bar{I}\}. \tag{2.14}$$

$\phi_p(x)$ can also be strictly decreasing on $\widehat{X}$ if we replace the condition (iv) of Theorem 2.1 by the corresponding one.

Theorem 2.2. *Suppose that the conditions* (i), (ii) *and* (iii) *of Theorem* 2.1 *hold. Moreover, suppose that* (iv) *for any $i \in I$, $r_{i,q}$ is continuously differentiable and strictly decreasing on R_- and satisfies*

$$\begin{aligned} &r_{i,q}(0) \geq 0, \\ &r_{i,q}'(t) \rightrightarrows -\infty \ (q \to +\infty) \qquad \textit{for all} \ \ t \in R_-; \end{aligned}$$

for any $i \in \bar{I}$, $r_{i,q}$ is continuously differentiable and strictly increasing on R_+ and satisfies

$$\begin{aligned} &r_{i,q}(0) \geq 0, \\ &r_{i,q}'(t) \rightrightarrows +\infty \ (q \to +\infty) \qquad \textit{for all} \ \ t \in R_+, \end{aligned}$$

where $R_+ = \{t \mid t \geq 0, t \in R\}$, $R_- = \{t \mid t \leq 0, t \in R\}$.

Then, for any given $\varepsilon_1 > 0$, there exists a $q_0 > 0$, such that for all $q > q_0$,

$$\frac{\partial \phi_q(x)}{\partial x_k} \leq -\varepsilon_1, \qquad \forall x \in \widehat{X},\ k = 1, 2, \cdots, n,$$

where $\widehat{X}$ is defined by (2.14).

Proof. The proof can be readily obtained from the proof of Theorem 2.1.

Note that $f(x) \geq m_0$ for any $x \in X$, where m_0 is given in (2.9). Let

$$\widetilde{R}_{m+1} = \{z \in R^{m+1} \mid z_i \in R \text{ for all } i = 1, \cdots, m;\ z_{m+1} \geq m_0\}. \tag{2.15}$$

If we use stronger conditions than those in Theorem 2.1 or Theorem 2.2, then $\phi_q(x)$ can be strictly monotone on X as the following corollary shows.

Corollary 2.1. *Suppose that the conditions* (i) *and* (ii) *of Theorem* 2.1 *hold, and*

(iii)$'$ *the condition* (iii) *of Theorem* 2.1 *holds on* $\widetilde{R}_{m+1}$ *(i.e.,* $\widehat{R}_{+}^{m+1}$ *is replaced by* $\widetilde{R}_{m+1}$*). Furthermore, if*

(iv)$'$(a) *the condition* (iv) *of Theorem* 2.1 *holds on* R *(i.e.,* R_+ *and* R_- *are replaced by* R*), then for any given* $\varepsilon_1 > 0$*, there exists a positive number* $q_0 > 0$*, such that when* $q > q_0$*,*

$$\frac{\partial \phi_q(x)}{\partial x_k} \geq \varepsilon_1, \qquad \forall x \in X,\ k = 1, 2, \cdots, n.$$

Otherwise, if

(iv)$'$(b) *the condition* (iv) *of Theorem* 2.2 *holds on* R *(i.e.,* R_+ *and* R_- *are replaced by* R*), then for any given* $\varepsilon_1 > 0$*, there exists a positive number* $q_0 > 0$*, such that when* $q > q_0$*,*

$$\frac{\partial \phi_q(x)}{\partial x_k} \leq -\varepsilon_1, \qquad \forall x \in X,\ k = 1, 2, \cdots, n.$$

Proof. The proof can be readily obtained from the proof of Theorem 2.1.

For any $z \in \widetilde{R}_{m+1}$, let

$$T_1(z) = \sum_{i=1}^{m+1} z_i, \qquad T_2(z) = \sum_{i=1}^{m} z_i + \ln(1 + z_{m+1}).$$

Then $T_1(z)$ and $T_2(z)$ satisfy the condition (iii)$'$ of Corollary 2.1. Let

$$r_{i,q}^1(t) = \begin{cases} qt & \text{for any } i \in I, \\ -qt & \text{for any } i \in \bar{I}. \end{cases}$$

Then $r_{i,q}^1(t)$ satisfies the condition (iv)$'$(a) of Corollary 2.1. On the other hand, let

$$r_{i,q}^2(t) = \begin{cases} -qt & \text{for any } \ i \in I, \\ qt & \text{for any } \ i \in \bar{I}. \end{cases}$$

Then $r_{i,q}^2(t)$ satisfies the condition (iv)$'$(b) of Corollary 2.1.

For any $z \in \widetilde{R}_{m+1}$, let

$$T_3(z) = \exp\Big(\sum_{i=1}^{m} z_i\Big) + z_{m+1}, \quad T_4(z) = \sum_{i=1}^{m}(z_i + z_i^2) + z_{m+1}, \quad T_5(z) = \sum_{i=1}^{m} \exp(z_i) + z_{m+1}.$$

Then $T_i(z), i = 3, 4, 5$, satisfy the condition (iii) of Theorem 2.1, but they do not satisfy the condition (iii)$'$ of Corollary 2.1.

Let

$$r^3_{i,q}(t)=\begin{cases} t^2+qt & \text{for any } \ i\in I,\\ t^2-qt & \text{for any } \ i\in \bar{I},\end{cases}$$

$$r^4_{i,q}(t)=\begin{cases} \exp(qt) & \text{for any } \ i\in I,\\ \exp(-qt) & \text{for any } \ i\in \bar{I},\end{cases} \qquad r^{\widehat{4}}_{i,q}(t)=\begin{cases} \exp(qt) & \text{for any } \ i\in I,\\ \exp(-qt) & \text{for any } \ i\in \bar{I},\end{cases}$$

$$r^5_{i,q}(t)=\begin{cases} \ln(1+\exp(qt)) & \text{for any } \ i\in I,\\ \ln(1+\exp(-qt)) & \text{for any } \ i\in \bar{I}.\end{cases} \qquad r^{\widehat{5}}_{i,q}(t)=\begin{cases} \ln(1+\exp(-qt)) & \text{for any } \ i\in I,\\ \ln(1+\exp(qt)) & \text{for any } \ i\in \bar{I}.\end{cases}$$

Then $r^j_{i,q}(t),\ j=3,4,5$, satisfy the condition (iv) of Theorem 2.1, but they do not satisfy the condition $(\mathrm{iv})'$(a) of Corollary 2.1, and $r^{\widehat{j}}_{i,q}(t),\ j=4,5$, satisfy the condition (iv) of Theorem 2.2, but they do not satisfy the condition $(\mathrm{iv})'$(b) of Corollary 2.1.

§ 3 . Equivalence

In this section, the equivalence between the problem (1.1) and its transformed monotone programming problems is established in Theorem 3.3 (Theorem 3.4). It is shown that under some conditions, the primal problem (1.1) can be transformed into an equivalent monotone programming problem with only inequality constraints.

Let

$$\varphi_q(x)=T(r_{1,q}(b_1(x)),r_{2,q}(b_2(x)),\cdots,r_{m,q}(b_m(x)),f(x))+q\sum_{i=m+1}^{m_0}b_i(x), \tag{3.1}$$

where

$$b_i(x)=\begin{cases} |g_i(x)|, & i\in I,\\ -|g_i(x)|, & i\in\bar{I},\\ \max\{0,g_i(x)\}, & i\in\{m+1,\cdots,m_0\}.\end{cases} \tag{3.2}$$

Note the definitions of $T, r_{i,q}, i=1,\cdots,m$, and we can regard $\varphi_q(x)$ as a modified penalty function.

Throughout the paper, the pair (x^*,λ^*) is said to satisfy the second order sufficiency condition (see [1, p.169]) if

$$\begin{aligned} \nabla_x L(x^*,\lambda^*)&=0,\\ g_i(x^*)&=0, && i=1,\cdots,m,\\ g_i(x^*)&\le 0, && i=m+1,\cdots,m_0,\\ \lambda_i^*&\ge 0, && i=m+1,\cdots,m_0,\\ \lambda_i^* g_i(x^*)&=0, && i=m+1,\cdots,m_0,\\ y^T\nabla^2_{xx}L(x^*,\lambda^*)y&>0, && \forall y\in V(x^*), \end{aligned} \tag{3.3}$$

where $L(x,\lambda) = f(x) + \sum_{i=1}^{m_0} \lambda_i g_i(x)$ and

$$V(x^*) = \left\{ y \in R^n \;\middle|\; \begin{array}{ll} \nabla^T g_i(x^*)y = 0, & i = 1, \cdots, m \\ \nabla^T g_i(x^*)y = 0, & i \in A(x^*) \\ \nabla^T g_i(x^*)y \leq 0, & i \in B(x^*) \end{array} \right\},$$

$$A(x^*) = \{i \in \{m+1, \cdots, m_0\} \mid g_i(x^*) = 0,\ \lambda_i^* > 0\},$$

$$B(x^*) = \{i \in \{m+1, \cdots, m_0\} \mid g_i(x^*) = 0,\ \lambda_i^* = 0\}.$$

It is well known that the second order sufficiency condition implies strictly local exact penalization of the l_1 penalty function. The following theorem shows that under some conditions, the second order sufficiency condition also implies strictly local exact penalization of the function $\varphi_q(x)$.

Theorem 3.1. *Suppose that*

(i) $f(x)$ *and* $g_i(x)$, $i = 1, \cdots, m_0$, *are twice continuously differentiable on* X, $T(z)$ *satisfies the condition* (iii) *of Theorem* 2.1 *and* $r_{i,q}$, $i = 1, \cdots, m$, *satisfy the condition* (iv) *of Theorem* 2.1*;*

(ii) *the pair* (x^*, λ^*) *with* $x^* \in \mathrm{int}(X)$ *satisfies the second order sufficiency condition* (3.3).

Then, there exists a positive number q_0*, such that when* $q > q_0$*,* x^* *is a strict local minimizer of* $\varphi_q(x)$.

Proof. Since $r_{i,q}(t)$, $i = 1, \cdots, m$, satisfy the condition (iv) of Theorem 2.1, for

$$\bar{q} = \frac{M_0}{\varepsilon_0} \max_{1 \leq i \leq m_0} (|\lambda_i^*| + 1),$$

there exists a $q_0 > M_0 \max_{1 \leq i \leq m_0} (|\lambda_i^*| + 1) > 0$, such that

$$\begin{aligned} & r'_{i,q}(t) > \bar{q} \qquad \text{for any } \ t \in R_+,\ i \in I, \\ & r'_{i,q}(t) < -\bar{q} \qquad \text{for any } \ t \in R_-,\ i \in \bar{I}, \end{aligned} \tag{3.4}$$

for all $q > q_0$, where ε_0 and M_0 are given in the condition (iii) of Theorem 2.1. Then, we conclude that when $q > q_0$, x^* is a strict local minimizer of $\varphi_q(x)$ on R^n.

In fact, by contradiction, suppose that there exists a $q > q_0$, such that x^* is not a strict local minimizer of $\varphi_q(x)$ on R^n. Then there exists a sequence $\{x_n\}$ converging to x^*, such that $x_n \neq x^*$ for any $n = 1, 2, \cdots$, and

$$\varphi_q(x_n) \leq \varphi_q(x^*).$$

Since for any $i \in I$, $n = 1, 2, \cdots$, we have $b_i(x_n) \geq 0$; for any $i \in \bar{I}$, $n = 1, 2, \cdots$, we have $b_i(x_n) \leq 0$ and since $r_{i,q}$, $i = 1, \cdots, m$, satisfy the condition (iv) of Theorem 2.1, for any $n = 1, 2, \cdots$ we have

$$r_{i,q}(b_i(x_n)) \geq 0, \qquad i = 1, \cdots, m.$$

Since $r_{i,q}(b_i(x^*)) = 0,\ i = 1, \cdots, m$, for any $n = 1, 2, \cdots$ we have $r_{i,q}(b_i(x_n)) \geq r_{i,q}(b_i(x^*)),\ i = 1, \cdots, m$. Since T is strictly increasing, and for any $i \in \{m+1, \cdots, m_0\}$, $b_i(x_n) \geq 0 = b_i(x^*)$; for any $n = 1, 2, \cdots$, $\varphi_q(x_n) \leq \varphi_q(x^*)$, we have

$$f(x_n) \leq f(x^*).$$

Since $\left\{\frac{x_n - x^*}{\|x_n - x^*\|}\right\}$ is a bounded sequence, there exists a subsequence $\{n_k\}$ of $\{n\}$, such that $\left\{\frac{x_{n_k} - x^*}{\|x_{n_k} - x^*\|}\right\}$ converges to a vector s with unit norm, i.e., $\|s\| = 1$.

Without loss of generality, we suppose that the subsequence $\{n_k\}$ is just $\{n\}$, i.e.,

$$s = \lim_{n \to +\infty} \frac{x_n - x^*}{\|x_n - x^*\|}.$$

Let

$$\begin{aligned} &y_n^i = r_{i,q}(b_i(x_n)), \quad i = 1, \cdots, m, && y_n^{m+1} = f(x_n), \\ &Z_n = (y_n^1, \cdots, y_n^m, y_n^{m+1}), && Z^* = (r_{1,q}(0), \cdots, r_{m,q}(0), f(x^*)). \end{aligned}$$

Then, there exist $0 < \theta < 1,\ 0 < \theta_i < 1,\ i = 1, \cdots, m$, such that

$$\begin{aligned} &\varphi_q(x_n) - \varphi_q(x^*) \\ =\ & T(Z_n) - T(Z^*) + q \sum_{i=m+1}^{m_0} b_i(x_n) \\ =\ & \sum_{i=1}^{m} \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_i} r'_{i,q}(\theta_i b_i(x_n)) b_i(x_n) \\ & + \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_{m+1}} (f(x_n) - f(x^*)) + q \sum_{i=m+1}^{m_0} b_i(x_n) \\ =\ & \sum_{i \in I} \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_i} r'_{i,q}(\theta_i b_i(x_n)) |\nabla^T g_i(x^*)(x_n - x^*) + o(\|x_n - x^*\|)| \\ & - \sum_{i \in \bar{I}} \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_i} r'_{i,q}(\theta_i b_i(x_n)) |\nabla^T g_i(x^*)(x_n - x^*) + o(\|x_n - x^*\|)| \\ & + \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_{m+1}} (\nabla^T f(x^*)(x_n - x^*) + o(\|x_n - x^*\|)) \\ & + q \sum_{i=m+1}^{m_0} \max\{0, g_i(x^*) + \nabla^T g_i(x^*)(x_n - x^*) + o(\|x_n - x^*\|)\}. \end{aligned} \tag{3.5}$$

Since for any $n = 1, 2, \cdots$, $\varphi_q(x_n) \leq \varphi_q(x^*)$, we have that for any $n = 1, 2, \cdots$,

$$\begin{aligned} & \sum_{i \in I} \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_i} r'_{i,q}(\theta_i b_i(x_n)) |\nabla^T g_i(x^*)(x_n - x^*) + o(\|x_n - x^*\|)| \\ & - \sum_{i \in \bar{I}} \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_i} r'_{i,q}(\theta_i b_i(x_n)) |\nabla^T g_i(x^*)(x_n - x^*) + o(\|x_n - x^*\|)| \\ & + \frac{\partial T(\theta Z_n + (1-\theta)Z^*)}{\partial y_{m+1}} (\nabla^T f(x^*)(x_n - x^*) + o(\|x_n - x^*\|)) \end{aligned}$$

$$+\,q\sum_{i=m+1}^{m_0}\max\{0,\nabla^T g_i(x^*)(x_n-x^*)+o(\|x_n-x^*\|)\}\le 0.$$

Therefore

$$\begin{aligned}&\sum_{i\in I}\frac{\partial T(\theta Z_n+(1-\theta)Z^*)}{\partial y_i}r'_{i,q}(\theta_i b_i(x_n))\Big|\nabla^T g_i(x^*)\frac{(x_n-x^*)}{\|x_n-x^*\|}+\frac{o(\|x_n-x^*\|)}{\|x_n-x^*\|}\Big|\\&-\sum_{i\in\bar I}\frac{\partial T(\theta Z_n+(1-\theta)Z^*)}{\partial y_i}r'_{i,q}(\theta_i b_i(x_n))\Big|\nabla^T g_i(x^*)\frac{(x_n-x^*)}{\|x_n-x^*\|}+\frac{o(\|x_n-x^*\|)}{\|x_n-x^*\|}\Big|\\&+\frac{\partial T(\theta Z_n+(1-\theta)Z^*)}{\partial y_{m+1}}\Big(\nabla^T f(x^*)\frac{(x_n-x^*)}{\|x_n-x^*\|}+\frac{o(\|x_n-x^*\|)}{\|x_n-x^*\|}\Big)\\&+q\sum_{i=m+1}^{m_0}\max\Big\{0,\nabla^T g_i(x^*)\frac{(x_n-x^*)}{\|x_n-x^*\|}+\frac{o(\|x_n-x^*\|)}{\|x_n-x^*\|}\Big\}\le 0.\end{aligned}$$

By taking limit on the both sides, we have

$$\begin{aligned}&\sum_{i\in I}\frac{\partial T(Z^*)}{\partial y_i}r'_{i,q}(0)|\nabla^T g_i(x^*)s|-\sum_{i\in\bar I}\frac{\partial T(Z^*)}{\partial y_i}r'_{i,q}(0)|\nabla^T g_i(x^*)s|\\&+\frac{\partial T(Z^*)}{\partial y_{m+1}}\nabla^T f(x^*)s+q\sum_{i=m+1}^{m_0}\max\{0,\nabla^T g_i(x^*)s\}\le 0.\end{aligned}\tag{3.6}$$

Thus, it implies that $\nabla^T f(x^*)s\le 0$. By (3.4) and the condition (iii) of Theorem 2.1 and since $q>q_0$, we have

$$\begin{aligned}&\sum_{i\in I}\frac{\partial T(Z^*)}{\partial y_i}r'_{i,q}(0)|\nabla^T g_i(x^*)s|-\sum_{i\in\bar I}\frac{\partial T(Z^*)}{\partial y_i}r'_{i,q}(0)|\nabla^T g_i(x^*)s|\\&\quad+\frac{\partial T(Z^*)}{\partial y_{m+1}}\nabla^T f(x^*)s+q\sum_{i=m+1}^{m_0}\max\{0,\nabla^T g_i(x^*)s\}\\&\ge M_0\Big\{\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\sum_{i=1}^{m}|\nabla^T g_i(x^*)s|\\&\quad+\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\sum_{i=m+1}^{m_0}\max\{0,\nabla^T g_i(x^*)s\}+\nabla^T f(x^*)s\Big\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\sum_{1\le i\le m}|\nabla^T g_i(x^*)s|\\&+\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\sum_{i=m+1}^{m_0}\max\{0,\nabla^T g_i(x^*)s\}+\nabla^T f(x^*)s\le 0.\end{aligned}$$

By the condition (3.3), we have

$$\nabla^T f(x^*)s+\sum_{i=1}^{m_0}\lambda_i^*\nabla^T g_i(x^*)s=0.$$

Therefore, we have

$$\sum_{i=m+1}^{m_0}\Big[\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\max\{0,\nabla^T g_i(x^*)s\}-\lambda_i^*\nabla^T g_i(x^*)s\Big]$$
$$+\sum_{i=1}^{m}\Big[\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]|\nabla^T g_i(x^*)s|-\lambda_i^*\nabla^T g_i(x^*)s\Big]\le 0.$$

Since for any $i=m+1,\cdots,m_0$, $\max\limits_{1\le i\le m_0}(|\lambda_i^*|+1)\max\{0,\nabla^T g_i(x^*)s\}-\lambda_i^*\nabla^T g_i(x^*)s\ge 0$; and for any $i=1,\cdots,m$, $\max\limits_{1\le i\le m_0}(|\lambda_i^*|+1)|\nabla^T g_i(x^*)s|-\lambda_i^*\nabla^T g_i(x^*)s\ge 0$, we have

$$\max_{1\le i\le m_0}(|\lambda_i^*|+1)\max\{0,\nabla^T g_i(x^*)s\}-\lambda_i^*\nabla^T g_i(x^*)s=0,\qquad i=m+1,\cdots,m_0,$$
$$\max_{1\le i\le m_0}(|\lambda_i^*|+1)|\nabla^T g_i(x^*)s|-\lambda_i^*\nabla^T g_i(x^*)s=0,\qquad i=1,\cdots,m.$$

Thus we have

$$\nabla^T g_i(x^*)s\begin{cases}=0, & i=1,\cdots,m,\\ =0, & i\in A(x^*),\\ \le 0, & i\in B(x^*),\end{cases}$$

i.e., $s\in V(x^*)$. By (3.3), we have

$$s^T\nabla_{xx}^2 L(x^*,\lambda^*)s>0.$$

Therefore, when n is large enough, we have

$$f(x_n)+\sum_{i=1}^{m_0}\lambda_i^* g_i(x_n)>f(x^*).$$

By (3.4), (3.5) and the condition (iii) of Theorem 2.1, since $q>q_0$, we have

$$\begin{aligned}
&\varphi_q(x_n)-\varphi_q(x^*)\\
\ge\ & M_0\Big[\sum_{i\in I}\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]b_i(x_n)-\sum_{i\in\bar I}\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]b_i(x_n)\\
&+\sum_{i=m+1}^{m_0}\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]b_i(x_n)+f(x_n)-f(x^*)\Big]\\
=\ & M_0\Big[\sum_{i=1}^{m}\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]|g_i(x_n)|\\
&+\sum_{i=m+1}^{m_0}\Big[\max_{1\le i\le m_0}(|\lambda_i^*|+1)\Big]\max\{0,g_i(x_n)\}+f(x_n)-f(x^*)\Big]\\
\ge\ & M_0\Big(\sum_{i=1}^{m}|\lambda_i^*||g_i(x_n)|+\sum_{i=m+1}^{m_0}\lambda_i^*\max\{0,g_i(x_n)\}+f(x_n)-f(x^*)\Big)\\
\ge\ & M_0\Big(\sum_{i=1}^{m_0}\lambda_i^* g_i(x_n)+f(x_n)-f(x^*)\Big).
\end{aligned}$$

Thus, when n is large enough, we have

$$\varphi_q(x_n) - \varphi_q(x^*) \geq M_0\Big(\sum_{i=1}^{m_0} \lambda_i^* g_i(x_n) + f(x_n) - f(x^*)\Big) > 0,$$

which contradicts $\varphi_q(x_n) \leq \varphi_q(x^*)$ for all $n = 1, 2, \cdots$. We complete the proof.

Consider the following simply constrained programming problem:

$$\min_{x \in X} \varphi_q(x). \tag{3.7}$$

The set of global minima of the problem (3.7) is denoted by $G(3.7)$.

In order to obtain the relationship for global minima between the original problem (1.1) and the simply constrained problem (3.7), we need the following assumptions.

Assumption 3.1. *The set of global minima of the problem* (1.1) *is a finite set.*

Assumption 3.2. *For any $x^* \in G(1.1)$, there exists a vector λ^* such that the pair (x^*, λ^*) satisfies the second order sufficiency condition* (3.3)*, where $G(1.1)$ is the set of global minima of the problem* (1.1).

The following theorem shows the global exact penalization of $\varphi_q(x)$.

Theorem 3.2. *Suppose that*

(i) *$f(x)$ and $g_i(x)$, $i = 1, \cdots, m_0$, are twice continuously differentiable functions, T satisfies the condition* (iii) *of Theorem* 2.1 *and $r_{i,q}$, $i = 1, \cdots, m$, satisfy the condition* (iv) *of Theorem* 2.1*;*

(ii) *Assumption* 2.1*, Assumption* 3.1 *and Assumption* 3.2 *hold.*

Then there exists a $q^* > 0$, such that when $q > q^*$, $G(3.7) = G(1.1)$.

Proof. By Assumption 3.1, we know that $G(1.1)$ is a finite set. Let

$$G(1.1) = \{x_1^*, \cdots, x_{k_0}^*\}. \tag{3.8}$$

By Theorem 3.1, for any $i = 1, \cdots, k_0$, there exists a positive number $q_{i,0}$, such that when $q > q_{i,0}$, x_i^* is a strict local minimizer of function $\varphi_q(x)$. Let $q_0 = \max\limits_{1 \leq i \leq k_0} q_{i,0}$. Thus, when $q > q_0$, for any $i = 1, \cdots, k_0$, there exists a positive number $\delta_{x_i^*}$, such that for any $x \in N(x_i^*, \delta_{x_i^*}) \setminus \{x_i^*\}$, $i = 1, \cdots, k_0$, we have

$$\varphi_q(x) > \varphi_q(x_i^*),$$

where $N(x_i^*, \delta_{x_i^*}) = \{x \in R^n \mid \|x - x_i^*\| < \delta_{x_i^*}\}$. For any $x \in S \setminus G(1.1)$, we have $f(x) > f(x_i^*)$, where S is defined in (2.4). Thus, for any $x \in S \setminus G(1.1)$, there exists a positive number δ_x, such that for any $y \in N(x, \delta_x)$ and for any $i = 1, \cdots, k_0$, we have $f(y) > f(x_i^*)$. Thus, for any $y \in \bigcup\limits_{x \in S \setminus G(1.1)} N(x, \delta_x)$, for any $i = 1, \cdots, k_0$, we have $f(y) > f(x_i^*)$. Thus, for any $y \in \bigcup\limits_{x \in S \setminus G(1.1)} N(x, \delta_x)$, $i = 1, \cdots, k_0$, since $r_{i,q}(b_i(y)) \geq 0$, we have

$$\begin{aligned}\varphi_q(y) &= T(r_{1,q}(b_1(y)), \cdots, r_{m,q}(b_m(y)), f(y)) + q \sum_{i=m+1}^{m_0} b_i(y) \\ &\geq T(0, \cdots, 0, f(y)) > T(0, \cdots, 0, f(x_i^*)) = \varphi_q(x_i^*).\end{aligned}$$

Therefore, for any $y \in \Big(\bigcup_{x\in S} N(x,\delta_x)\Big)\Big\backslash G(1.1),\ i=1,\cdots,k_0$, we have

$$\varphi_q(y) > \varphi_q(x_i^*).$$

Since S is a compact set, there exists a positive number δ_0, such that $(S+\delta_0 N(0,1)) \subset \bigcup_{x\in S} N(x,\delta_x)$, thus, for any $x \in (S+\delta_0 N(0,1)) \setminus G(1.1),\ i=1,\cdots,k_0$, we have $\varphi_q(x) > \varphi_q(x_i^*)$. Since $X \setminus (S+\delta_0 N(0,1))$ is a compact set, and for any $x \in X \setminus (S+\delta_0 N(0,1))$, $\sum_{i\in I} b_i(x) - \sum_{i\in \bar I} b_i(x) + \sum_{i=m+1}^{m_0} b_i(x) > 0$, there exists a positive number θ_0, such that for any $x \in X \setminus (S+\delta_0 N(0,1))$ we have

$$\sum_{i\in I} b_i(x) - \sum_{i\in \bar I} b_i(x) + \sum_{i=m+1}^{m_0} b_i(x) \geq \theta_0 > 0.$$

Let $\bar q_0' = \frac{M_0(\overline{f}-\underline{f})}{\varepsilon_0\theta_0}$. By the condition (iv) of Theorem 2.1, there exists a positive number $q_0' > \frac{M_0(\overline{f}-\underline{f})}{\theta_0}$, such that when $q > q_0'$, we have

$$\begin{aligned} &r_{i,q}'(t) > \bar q_0' && \text{for any } \ t \in R_+,\ i \in I,\\ &r_{i,q}'(t) < -\bar q_0' && \text{for any } \ t \in R_-,\ i \in \bar I, \end{aligned}$$

where $\overline{f} \geq \max_{x\in X} f(x)$, $\underline{f} \leq \min_{x\in X} f(x)$, ε_0 and M_0 are given in the condition (iii) of Theorem 2.1. Let

$$\begin{aligned} &y^i = r_{i,q}(b_i(x)), \quad i=1,\cdots,m, && y^{m+1} = f(x),\\ &Z = (y^1,\cdots,y^m,y^{m+1}), && Z_i^* = (r_{1,q}(0),\cdots,r_{m,q}(0), f(x_i^*)). \end{aligned}$$

Then, for any $x \in X \setminus (S+\delta_0 N(0,1)),\ i=1,\cdots,k_0,\ q > q_0'$, we have

$$\begin{aligned} \varphi_q(x) &= \varphi_q(x_i^*) + \sum_{k=1}^{m} \frac{\partial T(\theta_i Z + (1-\theta_i)Z_i^*)}{\partial y_k} r_{k,q}'(\alpha_{i,k} b_k(x)) b_k(x)\\ &\quad + q\sum_{i=m+1}^{m_0} b_i(x) + \frac{\partial T(\theta_i Z + (1-\theta_i)Z_i^*)}{\partial y_{m+1}} (f(x) - f(x_i^*))\\ &> \varphi_q(x_i^*) + \frac{M_0(\overline{f}-\underline{f})}{\theta_0}\Big(\sum_{i\in I} b_i(x) - \sum_{i\in \bar I} b_i(x) + \sum_{i=m+1}^{m_0} b_i(x)\Big) + M_0(\underline{f} - f(x_i^*))\\ &\geq \varphi_q(x_i^*), \end{aligned}$$

where $0<\theta_i<1,\ i=1,\cdots,k_0;\ 0<\alpha_{i,k}<1,\ k=1,\cdots,m,\ i=1,\cdots,k_0$.

Let $q^* = \max\{q_0, q_0'\}$. When $q > q^*$, for any $x \in X \setminus G(1.1),\ i,j=1,\cdots,k_0$, we have $\varphi_q(x) > \varphi_q(x_i^*) = \varphi_q(x_j^*)$. Thus, we have $G(3.7) = G(1.1)$.

Note that $\varphi_q(x)$ is not necessarily monotone. Moreover, it is not differentiable. Consider the following programming problem

$$\begin{aligned} \min\ & \phi_q(x) = T(r_{1,q}(g_1(x)),\cdots,r_{m,q}(g_m(x)), f(x)),\\ \text{s.t.}\ & g_i(x) \geq 0, \qquad i \in I,\\ & g_i(x) \leq 0, \qquad i \in \bar I, \end{aligned}$$

$$
\begin{aligned}
& g_i(x) \le 0, \qquad i \in \{m+1, \cdots, m_0\}, \\
& x \in X.
\end{aligned} \tag{3.9}
$$

Denote the set of global minima of the problem (3.9) by $G(3.9)$. The objective function $\phi_q(x)$ is continuously differentiable under the conditions of Theorem 3.2. The following theorem shows that under some conditions the primal problem (1.1) is equivalent to the problem (3.9) and the problem (3.9) is a monotone programming problem with strictly increasing objective function.

Theorem 3.3. *If the conditions of Theorem* 3.2 *hold, furthermore, if Assumption* 2.2 *holds, and for any* $i = m+1, \cdots, m_0$, $g_i(x)$ *is monotone, then there exists a* $q_1 > 0$, *such that when* $q > q_1$, $G(3.9) = G(1.1)$, *and the problem* (3.9) *is a monotone programming problem with strictly increasing objective function.*

Proof. Let

$$
S_2 = \left\{ x \in X \;\middle|\; \begin{array}{ll} g_i(x) \ge 0, & i \in I, \\ g_i(x) \le 0, & i \in \bar{I}, \\ g_i(x) \le 0, & i \in \{m+1, \cdots, m_0\} \end{array} \right\}.
$$

For any $x \in S_2$, we have

$$
b_i(x) = \begin{cases} |g_i(x)| = g_i(x), & i \in I, \\ -|g_i(x)| = g_i(x), & i \in \bar{I}, \\ \max\{0, g_i(x)\} = 0, & i \in \{m+1, \cdots, m_0\}. \end{cases}
$$

Thus, for any $x \in S_2$, we have

$$
\begin{aligned}
\varphi_q(x) &= T(r_{1,q}(b_1(x)), \cdots, r_{m,q}(b_m(x)), f(x)) + q \sum_{i=m+1}^{m_0} b_i(x) \\
&= T(r_{1,q}(g_1(x)), \cdots, r_{m,q}(g_m(x)), f(x)) = \phi_q(x).
\end{aligned}
$$

By $G(1.1) \subset S_2$ and Theorem 3.2, when $q > q^*$, for any $x^* \in G(1.1)$ and $x \in S_2 \setminus G(1.1)$, we have $\phi_q(x) = \varphi_q(x) > \varphi_q(x^*) = \phi_q(x^*)$, where q^* is given in Theorem 3.2. Thus, when $q > q^*$, we have $G(1.1) = G(3.9)$. Furthermore, by Theorem 2.1, we know that there exists a positive number q_0, such that when $q > q_0$, function $\phi_q(x)$ is strictly increasing on $\widetilde{X}$, where $\widetilde{X}$ is defined in (2.8). Let $q_1 = \max\{q^*, q_0\}$. Thus, when $q > q_1$, $G(1.1) = G(3.9)$ and the problem (3.9) is a monotone problem.

Consider another programming problem

$$
\begin{aligned}
\min\ & \phi_q(x) = T(r_{1,q}(g_1(x)), \cdots, r_{m,q}(g_m(x)), f(x)), \\
\text{s.t.}\ & g_i(x) \le 0, \qquad i \in I, \\
& g_i(x) \ge 0, \qquad i \in \bar{I}, \\
& g_i(x) \le 0, \qquad i \in \{m+1, \cdots, m_0\}, \\
& x \in X.
\end{aligned} \tag{3.10}
$$

Similarly, we can obtain that under some conditions the original problem (1.1) is equivalent to the problem (3.10) and the problem (3.10) is a monotone programming problem with strictly decreasing objective function.

Theorem 3.4. *Suppose that*

(i) $f(x)$ *and* $g_i(x),\ i = 1, \cdots, m_0$ *are twice continuously differentiable functions,* T *and* $r_{i,q},\ i = 1, \cdots, m$ *satisfy the conditions* (iii) *and* (iv) *of Theorem* 2.2 *respectively;*

(ii) *Assumptions* 2.1*,* 2.2*,* 3.1 *and* 3.2 *hold;*

(iii) *for any* $i = m + 1, \cdots, m_0$*,* $g_i(x)$ *is monotone.*

Then there exists a positive number q_2*, such that when* $q > q_2$*,* $G(3.10) = G(1.1)$ *and the problem* (3.10) *is a monotone programming problem with strictly decreasing objective function.*

Proof. The proof can be readily obtained from Theorem 2.2 and Theorems 3.1–3.3.

We know that if the conditions of Theorem 2.1 hold, then $\phi_q(x)$ is an increasing function when q is sufficiently large; if the conditions of Theorem 2.2 hold, then $\phi_q(x)$ is a decreasing function when q is sufficiently large. In addition, in order to assure the equivalence of the original problem and the converted monotone problem, we need Assumption 3.1 and Assumption 3.2. The following theorem shows that when $m = m_0 = 1$, we can assure the equivalence without Assumption 3.1 and Assumption 3.2.

Theorem 3.5. *Suppose that*

(i) $m = m_0 = 1$*;*

(ii) *the conditions of Theorem* 2.1 (*Theorem* 2.2) *hold;*

(iii) *the problem* (2.4) *has at least one feasible solution.*

Then there exists a positive number q_0*, such that when* $q > q_0$*, the problem* (3.9) (*problem* (3.10)) *is a monotone programming problem and* $G(1.1) = G(3.9)$ ($G(1.1) = G(3.10)$).

Proof. Assume that $g_1(x)$ is strictly increasing on X. Let $\overline{X} = \{x \in X \mid g_1(x) \geq 0\}$, $\overline{S} = \{x \in X \mid g_1(x) = 0\}$. By Theorem 2.1, there exists a positive number q_0, such that when $q > q_0$, $\phi_q(x)$ is a strictly increasing function on $\overline{X}$. Thus, the problem (3.9) is a monotone programming problem on $\overline{X}$.

By the condition (iii), $\overline{S}$ is not empty. We firstly prove that when $q > q_0$, $G(3.9) \subset \overline{S}$.

By contradiction, suppose that there exist a $q > q_0$ and a $x_q^* \in G(3.9)$, such that $x_q^* \notin \overline{S}$. Then we have

$$g_1(x_q^*) > 0. \tag{3.11}$$

By the continuity of $g_1(x)$, there exists a positive number $\varepsilon > 0$, such that for any $x \in X$ with $|x_i - x_{q,i}^*| \leq \varepsilon,\ i = 1, 2, \cdots, n$, it holds that $g_1(x) > 0$, where $x = (x_1, \cdots, x_n),\ x_q^* = (x_{q,1}^*, \cdots, x_{q,n}^*)$.

If for any i, $x_{q,i}^* = c_i$, then we have $x_{q,i}^* \leq x_i^\circ$, $i = 1, \cdots, n$, where $x^\circ = (x_1^\circ, \cdots, x_n^\circ) \in S$. Since $g_1(x)$ is strictly increasing on $\overline{X}$, we have $g_1(x_q^*) \leq g_1(x^\circ) = 0$. This contradicts (3.11). Thus, there exists an i_0 $(1 \leq i_0 \leq n)$, such that $x_{q,i_0}^* > c_{i_0}$. Let $\varepsilon_0 = \min\{\varepsilon, x_{q,i_0}^* - c_{i_0}\}$, $u^* = (x_{q,1}^*, x_{q,2}^*, \cdots, x_{q,i_0-1}^*, x_{q,i_0}^* - \varepsilon_0, x_{q,i_0+1}^*, \cdots, x_{q,n}^*)$. Then, $u^* \in X$ and $g_1(u^*) > 0$.

Thus, $u^* \in \overline{X}$ and

$$\phi_q(u^*) \geq \phi_q(x_q^*). \tag{3.12}$$

On the other hand, since $\phi_q(x)$ is strictly increasing on $\overline{X}$ and $u^* < x_q^*$, we have $\phi_q(u^*) < \phi_q(x_q^*)$, which contradicts (3.12). Thus, we must have $G(3.9) \subset \overline{S}$.

Thus, for any $x^* \in G(3.9)$ and $y^* \in G(1.1)$, we have

$$\phi_q(y^*) \geq \phi_q(x^*) = T(r_{1,q}(0), f(x^*)) \geq T(r_{1,q}(0), f(y^*)) = \phi_q(y^*).$$

Thus, we have $\phi_q(y^*) = \phi_q(x^*)$ and $f(x^*) = f(y^*) = \phi_q(y^*)$, which imply $x^* \in G(1.1)$ and $y^* \in G(3.9)$. Therefore, it holds that $G(3.9) = G(1.1)$.

Similarly, we can prove the corresponding result in other cases.

A research topic that needs further pursuing in the future is to identify the lower bound of q which guarantees the success of the monotonization and the equivalence.

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