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GLOBAL STABILITY OF SOLUTIONS WITH DISCONTINUOUS INITIAL DATA CONTAINING VACUUM STATES FOR THE RELATIVISTIC EULER EQUATIONS***

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Abstract

The global stability of Lipschitz continuous solutions with discontinuous initial data for the relativistic Euler equations is established in a broad class of entropy solutions in L^∞ containing vacuum states. As a corollary, the uniqueness of Lipschitz solutions with discontinuous initial data is obtained in the broad class of entropy solutions in L^∞.

Keywords Relativistic Euler equations, Entropy solutions, Vacuum, Uniqueness, Global stability
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§ 1 . Introduction

We are concerned with the global stability of entropy solutions in L^∞ containing vacuum states for the relativistic Euler equations (cf. e.g. [10, 27–30])

$$\begin{cases} \partial_t\Big((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)=0, \\ \partial_t\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\Big)=0, \end{cases} \tag{1.1}$$

where ρ, p, and v represent the proper energy density, the pressure, and the particle speed respectively, and are in the physical region

$$\mathcal{V}=\{U=(\rho,v):0\le\rho<\rho_{\max},\ |v|<c\}, \tag{1.2}$$

where the constant c is the speed of light,

$$\rho_{\max}=\sup\{\rho:p'(\rho)\le c^2\},$$

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which means that the sound speed $\sqrt{p'(\rho)}$ is less than the light speed c.

In this paper, we study System (1.1) and establish the stability of Lipschitz continuous solutions with discontinuous initial data in a broad class of entropy solutions in L^∞ containing the vacuum states for (1.1). This broad class of entropy solutions requires only that the solutions are weak solutions and satisfy one physical entropy inequality.

For the classical (nonrelativistic) Euler equations, similar problems have been studied. Chen in [3] first introduced an effective method to handle with such entropy solutions in L^∞ and showed the stability of rarefaction waves. Li in [20] further developed Chen's method and solved the stability problem for Lipschitz continuous solutions with discontinuous initial data in a broad class of entropy solutions in L^∞ containing the vacuum states. One of the main motivations for the stability problem for the Euler equations is the instability of solutions of the corresponding classical Navier-Stokes equations containing vacuum as discussed in [3]. Also see [13].

In this paper we study the relativistic Euler equations and extend the results for the classical Euler equations to the relativistic case. As we will see below, the relativistic Euler equations are much more complicated and have more rich phenomena, while the classical Euler equations are just the limit system of the relativistic Euler equations as the light speed tends to infinity. One of the main difficulties is that strict hyperbolicity of System (1.1) fails at the vacuum, which yields additional singularity.

One of the main new ingredients in this paper is to analyze the singularity and other behaviors of solutions in detail in the relativistic regime, that is, in the regime when the light speed is finite. In particular, for the general case, we identify the invariant regions for the Riemann solutions to the system. Another new ingredient is to identify a global Lyapunov functional for the stability problem for the relativisitic Euler equations so that any solution in the family of Lipschitz continuous solutions with discontinuous initial data is globally stable under the Lyapunov functional norm, a weighted L^2-norm, and thus is unique in the broad class of entropy solutions in L^∞. To achieve this, we require the generalized Gauss-Green theorem for divergence-measure fields recently established in [4] since the solutions are not in BV.

The organization of this paper is the following. In Section 2, we review and discuss some basic properties of System (1.1) of the relativistic Euler equations, analyze the Riemann problems with vacuum for subsequent development. In Section 3, the existence of a global Lipschitz continuous solution with vacuum and discontinuous initial data to the Cauchy problem is shown under some monotone conditions on the data which exclude the propagation of the discontinuity. Finally, in Section 4, we prove the stability and uniqueness of global Lipschitz continuous solutions with discontinuous initial data in a broad class of entropy solutions in L^∞ containing vacuum states by the use of an appropriate Lyapunov functional and the monotonicity of this functional.

About the uniqueness and stability of Riemann solutions staying away from vacuum, see [10, 11]. For the classical non-relativistic case, see [4, 6–8, 13–15, 17]. Other related results and discussions regarding vacuum problems can be found in [1–3, 5, 9, 13, 16, 18, 19, 21–25].

§ 2. The System of Relativistic Euler Equations and the Riemann Problem with Vacuum

In this section, let us first review some basic and important properties of the system of relativistic Euler equations (1.1). Then we analyze the Riemann problem of System (1.1) containing vacuum states.

2.1. Relativistic Euler equations

System (1.1) fits into the following general form of conservation laws

$$\partial_t U + \partial_x F(U) = 0, \tag{2.1}$$

by setting

$$U = \Big((p+\rho c^2)\frac{v^2}{c^2(c^2-v^2)} + \rho, (p+\rho c^2)\frac{v}{c^2-v^2}\Big)^\top \tag{2.2}$$

and

$$F(U) = \Big((p+\rho c^2)\frac{v}{c^2-v^2}, (p+\rho c^2)\frac{v^2}{c^2-v^2} + p\Big)^\top. \tag{2.3}$$

The equation of state is

$$p = p(\rho),$$

where $p(\rho)$ is a smooth function of ρ and satisfies that, for $\rho = 0$ (vacuum states),

$$p(0) = 0, \quad p'(0) = 0, \quad \lim_{\rho\to 0}\frac{p'(\rho)}{\rho^2} = c_1 > 0, \tag{2.4}$$

and, for $\rho > 0$ (non-vacuum states),

$$p(\rho) > 0, \tag{2.5}$$

$$p'(\rho) > 0, \tag{2.6}$$

$$p''(\rho) > 0. \tag{2.7}$$

The condition (2.6) means the strict hyperbolicity, and the conditions (2.5)–(2.7) imply the following genuine nonlinearity of System (1.1):

$$\rho p''(\rho) + 2p'(\rho) + \frac{p(\rho)p''(\rho) - 2p'(\rho)^2}{c^2} > 0, \tag{2.8}$$

when $\rho > 0$ and the sound speed $\sqrt{p'(\rho)}$ is less than the light speed c.

For polytropic γ-law fluids with

$$p(\rho) = \kappa\rho^\gamma, \qquad \gamma > 1,\ \kappa > 0,$$

it is easy to verify that $p = p(\rho)$ clearly satisfies the conditions (2.4)–(2.7).

If we rewrite (1.1) as

$$\partial_t u + A\partial_x u = 0, \tag{2.9}$$

where

$$\begin{aligned}
&u = (\rho, v)^\top, \quad A = (dU)^{-1} dF(U),\\
&dU := \frac{\partial(U_1, U_2)}{\partial(\rho, v)} = \begin{pmatrix} U_{1\rho} & U_{1v} \\ U_{2\rho} & U_{2v} \end{pmatrix},\\
&dF(U) := \frac{\partial(F_1(U), F_2(U))}{\partial(\rho, v)} = \begin{pmatrix} F_{1\rho} & F_{1v} \\ F_{2\rho} & F_{2v} \end{pmatrix},
\end{aligned}$$

then it is not difficult to calculate the two eigenvalues of System (1.1):

$$\lambda_1(u) = \frac{c^2(v - \sqrt{p'(\rho)}\,)}{c^2 - v\sqrt{p'(\rho)}}, \qquad \lambda_2(u) = \frac{c^2(v + \sqrt{p'(\rho)}\,)}{c^2 + v\sqrt{p'(\rho)}}.$$

Since

$$\lambda_2 - \lambda_1 = \frac{2c^2(v^2 - c^2)\sqrt{p'(\rho)}}{c^4 - v^2 p'(\rho)},$$

we know that System (1.1) is strictly hyperbolic in non-vacuum states in $\mathcal{V} \cap \{\rho > 0\}$. But, from the condition (2.4), we have

$$\lim_{\rho \to 0} (\lambda_2 - \lambda_1) = 0,$$

which means that the strict hyperbolicity fails in vacuum states. We can also calculate the two eigenvectiors corresponding to λ_j :

$$r_j(u) = \alpha_j(\rho, v)\Big(\frac{(-1)^j}{c^2 - v^2}, \frac{\sqrt{p'(\rho)}}{p + \rho c^2}\Big)^\top, \qquad j = 1, 2.$$

By choosing

$$\alpha_j(\rho, v) = \frac{2(c^2 + (-1)^{j+1} v\sqrt{p'(\rho)}\,)^2 (p + \rho c^2)\sqrt{p'(\rho)}}{c^2(\rho p''(\rho) + 2p'(\rho)) + p(\rho)p''(\rho) - 2(p'(\rho))^2} > 0, \qquad j = 1, 2, \tag{2.10}$$

we have $\nabla_u \lambda_j(u) \cdot r_j = 1,\ j = 1, 2$, so both families of System (1.1) are genuinely nonlinear.

On the other hand, we can also rewrite System (1.1) as

$$\partial_t U + \nabla F(U)\, \partial_x U = 0, \tag{2.11}$$

where

$$\begin{aligned}
\nabla F(U) &:= \frac{\partial(F_1(U), F_2(U))}{\partial(U_1, U_2)} = \begin{pmatrix} F_{1U_1} & F_{1U_2} \\ F_{2U_1} & F_{2U_2} \end{pmatrix} = dF(dU)^{-1}\\
&= \begin{pmatrix} 0 & 1 \\ \frac{c^4(p' - v^2)}{c^4 - p'v^2} & \frac{-2c^2 v(p' - c^2)}{c^4 - p'v^2} \end{pmatrix}.
\end{aligned} \tag{2.12}$$

In this case, the eigenvalues of the system keep unchanged, but the corresponding eigenvectors are changed to

$$\begin{aligned}\widetilde{r}_j &:= dU \cdot r_j \\ &= \frac{\alpha_j(\rho, v)}{c^2(c^2-v^2)^2}\begin{pmatrix}(-1)^j(c^4+p'(\rho)v^2)+2vc^2\sqrt{p'(\rho)} \\ (-1)^j(p'(\rho)+c^2)vc^2+c^2(c^2+v^2)\sqrt{p'(\rho)}\end{pmatrix}, \qquad j=1,2, \quad (2.13)\end{aligned}$$

with

$$\nabla\lambda_j(U)\cdot\widetilde{r}_j = \nabla_u\lambda_j(u)(dU)^{-1}dU\cdot r_j = \nabla_u\lambda_j(u)\cdot r_j = 1.$$

We recall that an entropy-entropy flux pair for (1.1) is a pair of C^1 functions $(\eta(U), q(U))$ satisfying

$$\nabla\eta(U)\nabla F(U) = \nabla q(U).$$

In particular, the physical entropy-entropy flux pair $(\eta_*(U), q_*(U))$ of (1.1) is

$$\eta_*(U) = -\frac{c^3}{\sqrt{c^2-v^2}}\exp\Big(c^2\int_0^\rho \frac{ds}{p(s)+c^2 s}\Big)+c^2U_1, \quad q_*(U) = \eta_*(U)v + c^2F_1(U), \quad (2.14)$$

and we notice that the Newton limit of this pair is

$$\Big(\frac{1}{2}\rho v^2 + \rho\int_0^\rho \frac{p(r)}{r^2}dx,\ \frac{1}{2}\rho v^3 + \rho v\int_0^\rho \frac{p'(r)}{r}dr\Big),$$

which is exactly the physical entropy-entropy flux pair of the following classical non-rela -tivistic Euler equations

$$\begin{cases}\partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho)) = 0.\end{cases}$$

A direct calculation yields

$$\nabla^2\eta_*(U) = \alpha_0(\rho, v)\begin{pmatrix} c^2(p'c^2+v^2c^2+2p'v^2) & -(c^4+2p'c^2+p'v^2)v \\ -(c^4+2p'c^2+p'v^2)v & c^4+3p'v^2\end{pmatrix}, \quad (2.15)$$

with

$$\alpha_0(\rho, v) = \frac{c^5\exp\big(c^2\int_1^\rho \frac{ds}{p(s)+c^2s}\big)}{\sqrt{c^2-v^2}(c^4-p'v^2)(p+\rho c^2)^2} > 0. \quad (2.16)$$

Thus $\eta_*(U)$ is strictly convex in U in any compact domain of $\mathcal{V}\cap\{\rho>0\}$.

2.2. Riemann problem containing vacuum states

We consider a special Cauchy problem–Riemann problem, which is the initial value problem with initial data

$$U|_{t=0} = R_0(x) \equiv \begin{cases} U_-, & x<0, \\ U_+, & x>0,\end{cases} \quad (2.17)$$

$(\rho_\pm, v_\pm) \in \mathcal{V}$, $U_\pm = U(\rho_\pm, v_\pm)$, and $U_+ \neq U_-$.

Given a state U_l, we consider all the possible states U that can be connected to state U_l on the right by a centered rarefaction wave in the j-families, $j = 1, 2$. Consider the self-similar solutions $U(\xi)$, $\xi = \frac{x}{t}$, of the Riemann problem (1.1) and (2.17). We have the ordinary differential equations

$$\begin{cases} \xi = \lambda_j(U)(\xi), \\ (\xi I - \nabla F(U(\xi)))U'(\xi) = 0, \end{cases}$$

with boundary conditions

$$U(\lambda_j(U_l)) = U_l,$$

and, on the j-family centered rarefaction waves,

$$\frac{\partial U}{\partial x} = \frac{1}{t}\frac{dU}{d\xi} = \frac{1}{t}\widetilde{r}_j\Big(U\Big(\frac{x}{t}\Big)\Big), \qquad j = 1, 2. \tag{2.18}$$

For a rarefaction wave $R(\frac{x}{t})$ with right state U_r, it holds that

$$\begin{cases} r(U_l) - s(U_r) > 0, \\ r(U_l) \leq r(R(\frac{x}{t})) \leq r(U_r), \quad s(U_l) \leq s(R(\frac{x}{t})) \leq s(U_r), \\ r(R(\frac{x}{t})) - s(R(\frac{x}{t})) > 0, \end{cases}$$

where

$$\begin{cases} r(\rho, v) = \dfrac{c}{2}\ln\Big(\dfrac{c+v}{c-v}\Big) + c^2\displaystyle\int_0^\rho \frac{\sqrt{p'(s)}}{p(s)+c^2 s}ds, \\ s(\rho, v) = \dfrac{c}{2}\ln\Big(\dfrac{c+v}{c-v}\Big) - c^2\displaystyle\int_0^\rho \frac{\sqrt{p'(s)}}{p(s)+c^2 s}ds \end{cases} \tag{2.19}$$

are the Riemann invariants. So the two families of rarefaction wave curves corresponding to the j-th characteristic families, $j = 1, 2$, can be given respectively by

$$\begin{cases} R_1(\rho, v) : \dfrac{c}{2}\ln\Big(\dfrac{c+v}{c-v}\Big) + c^2\displaystyle\int_{\rho_l}^\rho \frac{\sqrt{p'(s)}}{p(s)+c^2 s}ds = \text{constant}, \qquad 0 \leq \rho < \rho_l, \\ R_2(\rho, v) : \dfrac{c}{2}\ln\Big(\dfrac{c+v}{c-v}\Big) + c^2\displaystyle\int_{\rho_l}^\rho \frac{\sqrt{p'(s)}}{p(s)+c^2 s}ds = \text{constant}, \qquad \rho > \rho_l. \end{cases} \tag{2.20}$$

Standardly, the Riemann solutions can be constructed as follows.

A. If $\rho_- > 0$ and $\rho_+ = 0$, then there exists a unique v_c such that

$$R\Big(\frac{x}{t}\Big) = \begin{cases} U_-, & \dfrac{x}{t} < \lambda_1(U_-), \\ R_1\Big(\dfrac{x}{t}\Big), & \lambda_1(U_-) \leq \dfrac{x}{t} \leq v_c, \\ \text{vacuum}, & \dfrac{x}{t} > v_c, \end{cases} \tag{2.21}$$

where $R_1(\xi)$ is the solution of the boundary value problem:

$$R_1'(\xi) = r_1(R_1(\xi)), \quad \xi > \lambda_1(U_-); \qquad R_1(\lambda_1(U_-)) = U_-. \tag{2.22}$$

B. If $\rho_- = 0$ and $\rho_+ > 0$, then there exists a unique $\tilde{v}_c$ such that

$$R\Big(\frac{x}{t}\Big) = \begin{cases} \text{vacuum}, & \dfrac{x}{t} < \tilde{v}_c, \\ R_2\Big(\dfrac{x}{t}\Big), & \tilde{v}_c \le \dfrac{x}{t} \le \lambda_2(U_+), \\ U_+, & \dfrac{x}{t} > \lambda_2(U_+), \end{cases} \tag{2.23}$$

where $R_2(\xi)$ is the solution of the boundary value problem:

$$R_2'(\xi) = r_2(R_2(\xi)), \quad \xi < \lambda_2(U_+); \qquad R_2(\lambda_2(U_+)) = U_+. \tag{2.24}$$

C. If $\rho_\pm > 0$, there are two cases:

(C1) There exist unique $v_{c_1}, v_{c_2}, v_{c_1} < v_{c_2}$, such that

$$R\Big(\frac{x}{t}\Big) = \begin{cases} U_-, & \dfrac{x}{t} < \lambda_1(U_-), \\ R_1\Big(\dfrac{x}{t}\Big), & \lambda_1(U_-) \le \dfrac{x}{t} \le v_{c_1}, \\ \text{vacuum}, & v_{c_1} < \dfrac{x}{t} < v_{c_2}, \\ R_2\Big(\dfrac{x}{t}\Big), & v_{c_2} \le \dfrac{x}{t} \le \lambda_2(U_+), \\ U_+, & \dfrac{x}{t} > \lambda_2(U_+), \end{cases} \tag{2.25}$$

where $R_1(\xi)$ and $R_2(\xi)$ are the solutions of the boundary value problems (2.22) and (2.24), respectively.

(C2) There exists a unique $U_M = U(\rho_M, v_M), \rho_M > 0$, such that

$$R\Big(\frac{x}{t}\Big) \equiv \begin{cases} U_-, & \dfrac{x}{t} < \lambda_1(U_-), \\ R_1\Big(\dfrac{x}{t}\Big), & \lambda_1(U_-) \le \dfrac{x}{t} \le \lambda_1(U_M), \\ U_c, & \lambda_1(U_M) < \dfrac{x}{t} < \lambda_2(U_M), \\ R_2\Big(\dfrac{x}{t}\Big), & \lambda_2(U_M) \le \dfrac{x}{t} \le \lambda_2(U_+), \\ U_+, & \dfrac{x}{t} > \lambda_2(U_+), \end{cases} \tag{2.26}$$

where $R_1(\xi)$ and $R_2(\xi)$ are the solutions of the boundary value problems (2.22) and (2.24), respectively.

Lemma 2.1. *The regions*

$$\sum(\widetilde{r}_0, \tilde{s}_0) = \{(\rho, v) : r \le \widetilde{r}_0, s \ge \tilde{s}_0, r - s \ge 0\}$$

are invariant regions of the Riemann problem (1.1) *and* (2.17). *That is, if the Riemann data lies in* $\sum(\widetilde{r}_0, \tilde{s}_0)$*, then the corresponding solution of the Riemann problem also lies in* $\sum(\widetilde{r}_0, \tilde{s}_0)$.

The proof of Lemma 2.1 needs the following two lemmas.

Lemma 2.2. *As long as the genuine nonlinearity condition* (2.8) *is satisfied, on the* $U_1 - U_2$ *plane the* 1*-rarefaction wave curve* R_1 *is concave and the* 2*-rarefaction wave curve* R_2 *is convex, where* R_1 *and* R_2 *are defined by* (2.20).

Proof. (1) First we show the concavity of R_1.

It is easy to know from (2.20) that, along R_1,

$$v_\rho = -(c^2 - v^2)\frac{\sqrt{p'(\rho)}}{p(\rho) + \rho c^2}, \tag{2.27}$$

and

$$\frac{dU_2}{dU_1} = \frac{\partial U_2/\partial\rho + (\partial U_2/\partial v)(\partial v/\partial\rho)}{\partial U_1/\partial\rho + (\partial U_1/\partial v)(\partial v/\partial\rho)}.$$

Noticing that

$$\frac{\partial U_2}{\partial\rho} = \frac{(p'(\rho) + c^2)v}{(c^2 - v^2)}, \qquad \frac{\partial U_1}{\partial\rho} = \frac{(p'(\rho) + c^2)v^2}{c^2(c^2 - v^2)} + 1,$$

and

$$\frac{\partial U_2}{\partial v} = \frac{(p(\rho) + \rho c^2)(c^2 + v^2)}{(c^2 - v^2)^2}, \qquad \frac{\partial U_1}{\partial v} = \frac{2v(p(\rho) + \rho c^2)}{(c^2 - v^2)^2},$$

we can obtain

$$\frac{dU_2}{dU_1} = \frac{c^2(v - \sqrt{p'(\rho)}\,)}{c^2 - v\sqrt{p'(\rho)}}. \tag{2.28}$$

From

$$\frac{d^2U_2}{dU_1^2} = \frac{d(\frac{dU_2}{dU_1})}{d\rho}\Big/\frac{dU_1}{d\rho},$$

and a detailed calculation, we arrive at

$$\frac{d^2U_2}{dU_1^2} = -\frac{c^6(c^2 - v^2)^2[\rho p''(\rho) + 2p'(\rho) + (p(\rho)p''(\rho) - 2p'(\rho)^2)/c^2]}{2\sqrt{p'(\rho)}(p(\rho) + \rho c^2)(c^2 - v\sqrt{p'(\rho)}\,)^4}, \tag{2.29}$$

whose sign is negative from (2.8), thus R_1 is concave on the $U_1 - U_2$ plane.

(2) Next we show the convexity of R_2.

Similar computation shows that, along R_2,

$$v_\rho = (c^2 - v^2)\frac{\sqrt{p'(\rho)}}{p(\rho) + \rho c^2}, \tag{2.30}$$

$$\frac{dU_2}{dU_1} = \frac{c^2(v + \sqrt{p'(\rho)}\,)}{c^2 + v\sqrt{p'(\rho)}}, \tag{2.31}$$

and then

$$\frac{d^2U_2}{dU_1^2} = \frac{c^6(c^2 - v^2)^2[\rho p''(\rho) + 2p'(\rho) + (p(\rho)p''(\rho) - 2p'(\rho)^2)/c^2]}{2\sqrt{p'(\rho)}\,(p(\rho) + \rho c^2)(c^2 - v\sqrt{p'(\rho)}\,)^4}, \tag{2.32}$$

which is positive from (2.8), thus R_2 is convex on the $U_1 - U_2$ plane.

Lemma 2.3. *The mapping* $(\rho, v) \longrightarrow (U_1, U_2)$ *is one-to-one.*

This is because the Jacobian of the mapping is nonsingular in the region $\mathcal{V}$:

$$\begin{aligned} J &= \det \frac{\partial(U_1,U_2)}{\partial(\rho,v)} = \det \begin{pmatrix} \frac{c^4+v^2p'(\rho)}{c^2(c^2-v^2)} & \frac{2v(p(\rho)+\rho c^2)}{(c^2-v^2)^2} \\ \frac{(p'(\rho)+c^2)v}{(c^2-v^2)} & \frac{(p(\rho)+\rho c^2)(c^2+v^2)}{(c^2-v^2)^2} \end{pmatrix} \\ &= \frac{(p(\rho)+\rho c^2)(c^4-v^2p'(\rho))}{c^2(c^2-v^2)^2} \neq 0. \end{aligned} \tag{2.33}$$

Then Lemma 2.1 follows immediately from Lemma 2.2 and Lemma 2.3 (cf. [13, 16]).

§ 3. The Global Existence of Lipschitz Solutions

Let us first identify a class of monotone initial data which can allow discontinuity and may generate vacuum.

The given initial data is

$$U|_{t=0} = V_0(x) = (U_1(\rho_0(x), v_0(x)), U_2(\rho_0(x), v_0(x)))^\top, \tag{3.1}$$

where $(\rho_0(x), v_0(x)) \in \mathcal{V}$, such that

$$r(\rho_0(x), v_0(x)) = \begin{cases} r_0, & x \le 0, \\ r_0(x), & x > 0, \end{cases} \tag{3.2}$$

$$s(\rho_0(x), v_0(x)) = \begin{cases} s_0(x), & x \le 0, \\ s_0, & x > 0, \end{cases} \tag{3.3}$$

where $r(\rho, v)$ and $s(\rho, v)$ are the Riemann invariants, $r_0(x)$ and $s_0(x)$ are the nondecreasing piecewise continuous functions of x, r_0 and s_0 are constants, and it holds that

$$r_0 \le r_0(x) \le r_{\sup}, \qquad s_{\inf} \le s_0(x) \le s_0, \tag{3.4}$$

$$r_0(0) \neq r_0, \qquad s_0(0) \neq s_0. \tag{3.5}$$

In this section, we first construct the approximate solutions, and then obtain the existence of Lipschitz solutions via the uniform boundedness estimates of the approximate solutions.

3.1. The construction of approximate solutions

In order to obtain the approximate solution sequences of (1.1) and (3.1), we consider the Cauchy problems with initial data

$$U|_{t=0} \equiv V_0\Big(\Big(j - \frac{1}{2}\Big)h\Big) \stackrel{\text{def}}{=} V^0_{j-\frac{1}{2}}, \qquad (j-1)h < x \le jh. \tag{3.6}$$

Then the self-similar solutions of (1.1) and (3.6) can be described as follows.

A. (Fig. 1) There exists unique v_{c_1} and v_{c_2}, such that

$$V^h(x,t)=\begin{cases} R^1_{-j+\frac12}\big(\frac{x+jh}{t}\big), & -jh+\lambda_1\big(V^0_{-j-\frac12}\big)t\le x<-jh+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ V^0_{-j+\frac12}, & -jh+\lambda_1\big(V^0_{-j+\frac12}\big)t\le x<-(j-1)h+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ R^1_0\big(\frac{x}{t}\big), & \lambda_1\big(V^0_{-\frac12}\big)t\le x<v_{c_1}t,\\ \text{vacuum}, & v_{c_1}t\le x<v_{c_2}t,\\ R^2_0\big(\frac{x}{t}\big), & v_{c_2}t\le x<\lambda_2\big(V^0_{\frac12}\big)t,\\ V^0_{j-\frac12}, & (j-1)h+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j-\frac12}\big)t,\\ & j=1,2,\cdots,\\ R^2_{j-\frac12}\big(\frac{x-jh}{t}\big), & jh+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j+\frac12}\big)t,\\ & j=1,2,\cdots. \end{cases} \tag{3.7}$$

B. (Fig. 2) There exists unique v_{c_1} and v_{c_2}, such that

$$V^h(x,t)=\begin{cases} R^1_{-j+\frac12}\big(\frac{x+jh}{t}\big), & -jh+\lambda_1\big(V^0_{-j-\frac12}\big)t\le x<-jh+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ V^0_{-j+\frac12}, & -jh+\lambda_1\big(V^0_{-j+\frac12}\big)t\le x<-(j-1)h+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ R^1_0\big(\frac{x}{t}\big), & \lambda_1\big(V^0_{-\frac12}\big)t\le x<v_{c_1}t,\\ \text{vacuum}, & v_{c_1}t\le x<j_0h+v_{c_2}t,\\ R^2_{j_0-\frac12}\big(\frac{x-j_0h}{t}\big), & j_0h+v_{c_2}t\le x<j_0h+\lambda_2\big(V^0_{j_0+\frac12}\big)t,\\ V^0_{j-\frac12}, & jh+\lambda_2\big(V^0_{j+\frac12}\big)t\le x<(j+1)h+\lambda_2\big(V^0_{j+\frac12}\big)t,\\ & j=j_0,j_0+1,\cdots,\\ R^2_{j-\frac12}\big(\frac{x-jh}{t}\big), & (j+1)h+\lambda_2\big(V^0_{j+\frac12}\big)t\le x<(j+1)h+\lambda_2\big(V^0_{j+\frac32}\big)t,\\ & j=j_0,j_0+1,\cdots. \end{cases} \tag{3.8}$$

C. (Fig. 3) There exists unique v_{c_1} and v_{c_2}, such that

$$V^h(x,t)=\begin{cases} R^1_{-j-\frac12}\big(\frac{x+jh}{t}\big), & -(j+1)h+\lambda_1\big(V^0_{-j-\frac32}\big)t\le x<-(j+1)h+\lambda_1\big(V^0_{-j-\frac12}\big)t,\\ & j=j_0,j_0+1,\cdots,\\ V^0_{-j-\frac12}, & -(j+1)h+\lambda_1\big(V^0_{-j-\frac12}\big)t\le x<-jh+\lambda_1\big(V^0_{-j-\frac12}\big)t,\\ & j=j_0,j_0+1,\cdots,\\ R^1_{-j_0+\frac12}\big(\frac{x+j_0h}{t}\big), & -j_0h+\lambda_1\big(V^0_{-j_0-\frac12}\big)t\le x<-j_0h+v_{c_1}t,\\ \text{vacuum}, & -j_0h+v_{c_1}t\le x<v_{c_2}t,\\ R^2_0\big(\frac{x}{t}\big), & v_{c_2}t\le x<\lambda_2\big(V^0_{\frac12}\big)t,\\ V^0_{j-\frac12}, & (j-1)h+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j\frac12 2}\big)t,\\ & j=1,2,\cdots,\\ R^2_{j-\frac12}\big(\frac{x-jh}{t}\big), & jh+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j+\frac12}\big)t,\\ & j=1,2,\cdots. \end{cases} \tag{3.9}$$

D. (Fig. 4) There exists unique v_{c_1} and v_{c_2}, such that

$$V^h(x,t)=\begin{cases} R^1_{-j+\frac12}\big(\frac{x+jh}{t}\big), & -jh+\lambda_1\big(V^0_{-j-\frac12}\big)t\le x<-jh+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ V^0_{-j+\frac12}, & -jh+\lambda_1\big(V^0_{-j+\frac12}\big)t\le x<-(j-1)h+\lambda_1\big(V^0_{-j+\frac12}\big)t,\\ & j=1,2,\cdots,\\ R^1_0\big(\frac{x}{t}\big), & \lambda_1\big(V^0_{-\frac12}\big)t\le x<v_{c_1}t,\\ \rho>0, & v_{c_1}t\le x<v_{c_2}t,\\ R^2_0\big(\frac{x}{t}\big), & v_{c_2}t\le x<\lambda_2\big(V^0_{\frac12}\big)t,\\ V^0_{j-\frac12}, & (j-1)h+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j-\frac12}\big)t,\\ & j=1,2,\cdots,\\ R^2_{j-\frac12}\big(\frac{x-jh}{t}\big), & jh+\lambda_2\big(V^0_{j-\frac12}\big)t\le x<jh+\lambda_2\big(V^0_{j+\frac12}\big)t,\\ & j=1,2,\cdots. \end{cases} \tag{3.10}$$

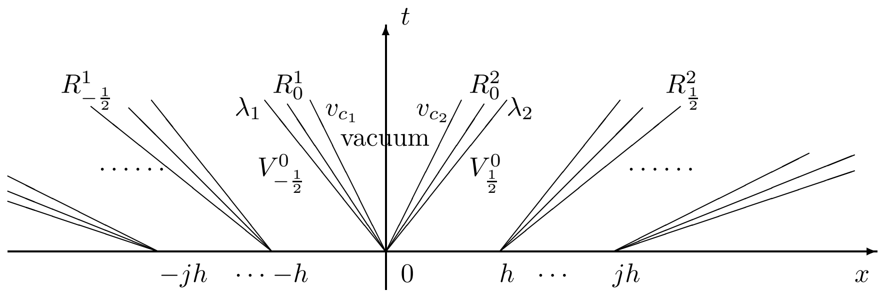

Fig. 1

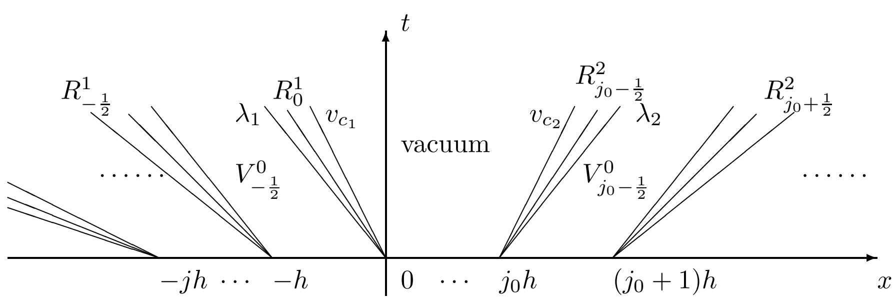

Fig. 2

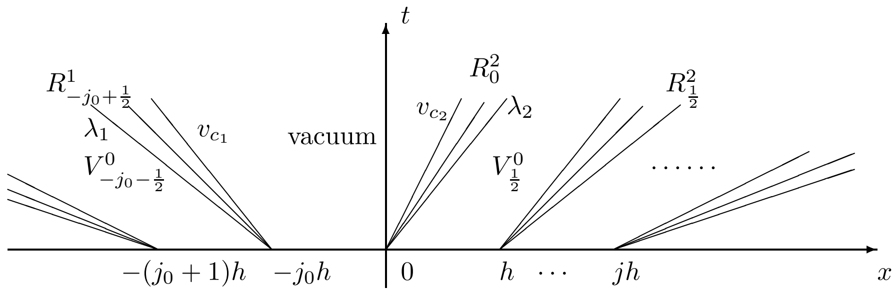

Fig. 3

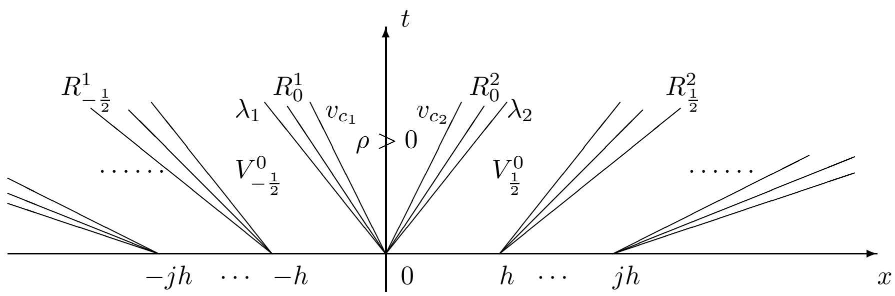

Fig. 4

3.2. Uniform bounds for the approximate solutions and the existence of global Lipschitz solutions

In order to obtain the global Lipschitz solution, we need to estimate the uniform bounds of the approximate solution sequences and its derivatives. Let us first give the uniform bounds of the approximate solution sequence $V^h(x,t)$ itself.

Lemma 3.1. *Given initial data* $(\rho_0(x), v_0(x))$ *satisfying* (3.1)–(3.5)*, there exists a constant* $C > 0$*, such that*

$$|V^h(x,t)| \leq C. \tag{3.11}$$

Proof. Noticing that $r_{\sup} \geq r > s \geq s_{\inf}$ (where r, s are the Riemann invatiants), from Lemma 2.1 we have

$$s_{\inf} \leq \frac{c}{2} \ln \frac{c+v}{c-v} \leq r_{\sup},$$

thus

$$\frac{2}{c} s_{\inf} \leq \ln \frac{c+v}{c-v} \leq \frac{2}{c} r_{\sup}.$$

Then we can solve

$$-c < \underline{c} = c\,\frac{e^{\frac{2}{c}s_{\inf}}-1}{e^{\frac{2}{c}s_{\inf}}+1} \le v \le c\,\frac{e^{\frac{2}{c}r_{\sup}}-1}{e^{\frac{2}{c}r_{\sup}}+1} = \overline{c} < c, \tag{3.12}$$

that is, v is strictly away from the light speed c. Therefore there exists a constant $C_1 > 0$, such that

$$\frac{1}{c^2 - v^2} \le C_1. \tag{3.13}$$

From the construction of $V^h(x,t)$, we obtain the uniform boundedness of $V^h(x,t)$ immediately.

Lemma 3.2. *Given initial data* $(\rho_0(x), v_0(x))$ *satisfying* (3.1)–(3.5)*, there exists a constant* $C > 0$*, such that*

$$|\bigtriangledown V^h(x,t)\,| \le \frac{C}{t}. \tag{3.14}$$

Proof. We need to prove that $\left|\frac{\partial V^h(x,t)}{\partial x}\right| \le \frac{C}{t}$ and $\left|\frac{\partial V^h(x,t)}{\partial t}\right| \le \frac{C}{t}$.

From (2.18), we first show the uniform boundedness of $\widetilde{r}_j$. Noticing the expression (2.13) of $\widetilde{r}_j(V^h(x,t))$, we see that the boundedness of

$$c^2(c^2-v^2)^2,$$

$$(-1)^j(c^4 + p'(\rho)v^2) + 2vc^2\sqrt{p'(\rho)}$$

and

$$(-1)^j(p'(\rho)+c^2)vc^2 + c^2(c^2+v^2)\sqrt{p'(\rho)}$$

is obvious from (3.13) and the fact $\sqrt{p'(\rho)} < c$ and $|v| < c$, even near $\rho = 0$ (vacuum states).

From the formula (2.11) of $\alpha_j(\rho, v)$, we have

$$\alpha_j(\rho,v) = \frac{2(c^2 + (-1)^{j+1}v\sqrt{p'(\rho)}\,)^2(p(\rho)+\rho c^2)\sqrt{p'(\rho)}}{(p(\rho)+\rho c^2)p''(\rho) + 2p'(\rho)(c^2 - p'(\rho)^2)}.$$

Then it is easy to obtain its uniform boundedness for $\rho > 0$. But near $\rho = 0$, because

$$\alpha_j(\rho,v) \le \frac{2c^4\sqrt{p'(\rho)}}{p''(\rho)},$$

from (2.4) we know that

$$\lim_{\rho\to 0}\alpha_j(\rho,v) \le \frac{c^4}{\sqrt{c_1}}.$$

Therefore we can have the uniform boundedness of $\widetilde{r}_j$. Then $\left|\frac{\partial V^h(x,t)}{\partial x}\right| \le \frac{C}{t}$. From (1.1) and (3.12), it follows that $\left|\frac{\partial V^h(x,t)}{\partial t}\right| \le \frac{C}{t}$.

From Lemma 3.1 and Lemma 3.2, we have the following theorem immediately.

Theorem 3.1. *Consider the problem* (1.1) *and* (3.1)*. For any given initial data* $(\rho_0(x), v_0(x))$ *satisfying* (3.1)–(3.5)*, there exists a global Lipschitz continuous solution* $V(x,t) = U(\rho(x,t), v(x,t))$ *of* (1.1) *and* (3.1) *satisfying*

$$|V(x,t)| \le C, \qquad |\bigtriangledown V(x,t)\,| \le C.$$

§ 4. Global Stability of Lipschitz Solution with Discontinuous Initial Data Containing Vacuum

In this section, we show the global stability of Lipschitz continuous solutions with discontinuous initial data in a broad class of entropy solutions in L^∞ containing vacuum states.

4.1. Gauss-Green formula and normal traces for divergence-measure fields in L^∞

Since the solutions are not in BV space, we require the generalized Gauss-Green theorem for divergence-measure fields recently established in [4]. For completeness, let us first review divergence-measure fields in L^∞, and the corresponding generalized Gauss-Green formula and normal traces.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open. For $F \in L^\infty(\Omega;\mathbb{R}^N)$, set*

$$|\mathrm{div}\, F|(\Omega) := \sup\{\, \langle F, \nabla\varphi\rangle : \varphi \in C_0^1(\Omega),\ |\varphi(x)| \le 1,\ x \in \Omega\,\}.$$

We say that F is an L^∞ divergence-measure field over Ω, i.e., $F \in \mathcal{DM}^\infty(\Omega)$, if

$$\|F\|_{\mathcal{DM}^\infty(\Omega)} := \|F\|_{L^\infty(\Omega;\mathbb{R}^N)} + |\mathrm{div}\, F|(\Omega) < \infty, \tag{4.1}$$

which means that $\mathrm{div}\, F$ is a Radon measure over Ω.

If $F \in \mathcal{DM}^\infty(\Omega)$ for any open set Ω with $\Omega \Subset D \subset \mathbb{R}^N$, then we say $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(D)$. Here, for open sets $A, B \subset \mathbb{R}^N$, the relation $A \Subset B$ means that the closure of A, $\overline{A}$, is a compact subset of B.

Definition 4.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset. We say that $\partial\Omega$ is a deformable Lipschitz boundary, provided that*

(i) $\forall x \in \partial\Omega$, $\exists r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ such that, after rotating and relabelling coordinates if necessary,

$$\Omega \cap Q(x,r) = \{y \in \mathbb{R}^N : \gamma(y_1, \cdots, y_{N-1}) < y_N\,\} \cap Q(x,r),$$

where $Q(x,r) = \{y \in \mathbb{R}^N : |x_i - y_i| \le r,\ i = 1, \cdots, N\,\}$;

(ii) $\exists\, \Psi : \partial\Omega \times [0,1] \to \overline{\Omega}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(\omega, 0) = \omega$ for all $\omega \in \partial\Omega$. The map Ψ is called a Lipschitz deformation of the boundary $\partial\Omega$.

Denote $\partial\Omega_s \equiv \Psi(\partial\Omega \times \{s\})$, $s \in [0,1]$, and denote Ω_s the open subset of Ω whose boundary is $\partial\Omega_s$. We call Ψ a Lipschitz deformation of $\partial\Omega$.

Definition 4.3. *We say that the Lipschitz deformation is regular if*

$$\lim_{s\to 0+} D\Psi_s \circ \widetilde{\gamma} = D\widetilde{\gamma}, \qquad in \ \ L^1_{\mathrm{loc}}(B), \tag{4.2}$$

where $\widetilde{\gamma}$ is a map as in Condition (i) of Definition 4.2, and Ψ_s denotes the map of $\partial\Omega$ into Ω, given by $\Psi_s(x) = \Psi(x,s)$. Here B denotes the greatest open set such that $\widetilde{\gamma}(B) \subset \partial\Omega$.

Then we have the following results on generalized Gauss-Green formula and normal traces.

Theorem 4.1. (see [4]) *Let $F \in \mathcal{DM}^\infty(\Omega)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz deformable boundary. Then there exists a function $F\cdot\nu|_{\partial\Omega} \in L^\infty(\partial\Omega)$ such that, for any $\phi \in \mathrm{Lip}(\mathbb{R}^N)$,*

$$\int_{\partial\Omega} F\cdot\nu|_{\partial\Omega}\,\phi\,d\,\mathcal{H}^{N-1} = \langle \operatorname{div} F,\,\phi\rangle_\Omega + \int_\Omega \nabla\phi\cdot F\,dx. \tag{4.3}$$

Moreover, let $\nu : \Psi(\partial\Omega\times[0,1]) \to \mathbb{R}^N$ be such that $\nu(x)$ is the unit outer normal to $\partial\Omega_s$ at $x \in \partial\Omega_s$, defined for a.e. $x \in \Psi(\partial\Omega\times[0,1])$. Let $h : \mathbb{R}^N \to \mathbb{R}$ be the level set function of $\partial\Omega_s$, that is,

$$h(x) := \begin{cases} 0 & \text{for } x \in \mathbb{R}^N - \overline{\Omega}, \\ 1 & \text{for } x \in \Omega - \Psi(\partial\Omega\times[0,1]), \\ s & \text{for } x \in \partial\Omega_s, 0 \le s \le 1. \end{cases}$$

Then, for any $\psi \in \mathrm{Lip}(\partial\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle = -\lim_{s\to 0}\frac{1}{s}\int_{\Psi(\partial\Omega\times(0,s))} \mathcal{E}(\psi)\,\nabla h\cdot F\,dx, \tag{4.4}$$

where $\mathcal{E}(\psi)$ is any Lipschitz extension of ψ to all $\mathbb{R}^N$. Furthermore, the normal trace $F\cdot\nu|_{\partial\Omega}$ is a function in $L^\infty(\partial\Omega)$ satisfying $\|F\cdot\nu\|_{L^\infty(\partial\Omega)} \le C\|F\|_{L^\infty(\Omega)}$, for some constant C independent of F; if $\partial\Omega$ admits a regular Lipschitz deformation, then $C = 1$. Furthermore, for any field $F \in \mathcal{DM}^\infty(\Omega)$,

$$\langle F\cdot\nu|_{\partial\Omega}, \psi\rangle = \operatorname*{ess\,lim}_{s\to 0}\int_{\partial\Omega_s} \psi\circ\Psi_s^{-1}\,F\cdot\nu\,d\mathcal{H}^{N-1} \qquad \text{for any } \psi \in L^1(\Omega). \tag{4.5}$$

4.2. The main stability theorem and the proof

We consider the Cauchy problem which is an initial $L^\infty \bigcap L^1(\mathbb{R})$ perturbation of (3.1):

$$U_0(x) \equiv V_0(x) + P_0(x), \qquad P_0(x) \in L^\infty \cap L^1(\mathbb{R}). \tag{4.6}$$

Definition 4.4. *A bounded measurable function $U(x,t)$ is an entropy solution of* (1.1) *and* (4.6) *in $\mathbb{R}^2_+$, if $U(t,x) \in \mathcal{V}$ and satisfies the following:*

(i) *The equations* (1.1) *and initial data* (4.6) *are satisfied in the weak sense in $\mathbb{R}^2_+$, i.e., for all $\phi \in C^1_0(\mathbb{R}^2_+)$,*

$$\int_0^\infty\int_{-\infty}^\infty (U\partial_t\phi + F(U)\partial_x\phi)\;dx\,dt + \int_{-\infty}^\infty U_0(x)\phi(x,0)dx = 0 \tag{4.7}$$

with U and $F(U)$ defined by (2.2) *and* (2.3).

(ii) *One physical entropy inequality holds in the sense of distributions in $\mathbb{R}^2_+$, i.e., for any nonnegative function $\phi \in C^1_0(\mathbb{R}^2_+)$,*

$$\int_0^\infty\int_{-\infty}^\infty (\eta_*(U)\partial_t\phi + q_*(U)\partial_x\phi)\,dx\,dt + \int_{-\infty}^\infty \eta_*(U_0)(x)\phi(x,0)dx \ge 0, \tag{4.8}$$

where (η_*, q_*) *is the mechanical energy-energy flux pair defined by* (2.14).

Then we have the following stability theorem.

Theorem 4.2. *Let* $V(x,t)$ *be the Lipschitz continuous solution of* (1.1) *and* (3.1) *containing vacuum states, as constructed in Section* 3. *Let* $U(x,t)$ *be any entropy solution of* (1.1) *and* (4.6) *in the sense of Definition* 4.4. *Then, for any* $L > 0$,

$$\int_{|x|\le L} \alpha(U,V)(x,t)\,dx \le \int_{|x|\le L+Kt} \alpha(U_0,V_0)(x)\,dx, \tag{4.9}$$

where $K > 0$ *is independent of* t, *and*

$$\alpha(U,R) \equiv (U-V)^\top \Big(\int_0^1 \nabla^2\eta_*(R+\tau(U-R))\,d\tau\Big)(U-V) > 0,$$

if $U \neq V$ *and both are away from the vacuum.*

In particular, if $U_0(x) = V_0(x)$ *a.e., then* $U(x,t) = V(x,t)$ *a.e.*

Proof. The proof is based on the normal traces and Gauss-Green formula in Subsection 4.1 for divergence-measure vector fields in L^∞. Without loss of generality, we suppose that the approximate solution $V^h(x,t)$ has the form of Case A.

Step 1. First we renormalize the mechanical energy-energy flux pair (η_*, q_*) through the following relative entropy pair:

$$\begin{aligned}
\alpha(U,V) &= \eta_*(U) - \eta_*(V) - \nabla\eta_*(V)(U-V),\\
\beta(U,V) &= q_*(U) - q_*(V) - \nabla\eta_*(V)(F(U)-F(V)),
\end{aligned}$$

and consider

$$\begin{aligned}
\mu &= \partial_t\alpha(U(x,t),V^h(x,t)) + \partial_x\beta(U(x,t),V^h(x,t)),\\
\nu &= \partial_t\eta_*(U(x,t)) + \partial_x q_*(U(x,t)).
\end{aligned}$$

Since $U(x,t)$ is an entropy solution, $\nu \le 0$, and $\mu \le 0$ in any region in which $V^h(x,t)$ is constant, in the sense of distributions. Then by using the Schwartz lemma (see [26]) and the product rule for divergence-measure fields in [4], we see that μ and ν are Radon measures, and $(\beta(U(x,t),V^h(x,t)),\alpha(U(x,t),V^h(x,t)))$ and $(q_*(U(x,t)),\eta_*(U(x,t)))$ are divergence-measure vector fields on $\mathbb{R}^2_+$.

For any $L > 0$, let

$$\Pi^\delta_{L,t} = \{\,(x,s) : |x| < L + K(t-s),\ 0 < \delta < s < t\},$$

where

$$K \ge K_0 \equiv \sup_h \|\beta(U,V^h)/\alpha(U,V^h)\|_{L^\infty(\mathbb{R}^2_+)}.$$

First, from the entropy inequality (4.8), the normal traces and Gauss-Green formula for divergence-measure vector fields, and the convexity of $\eta_*(U)$ in U, we notice that any entropy solution defined in Definition 4.4 assumes its initial data $U_0(x)$ strongly in $L^1_{\rm loc}$ (cf. [12]):

$$\lim_{t\to 0}\int_{|x|\le L}|U(x,t)-U_0(x)|dx=0 \qquad \text{for any } L>0. \tag{4.10}$$

Furthermore, we use Theorem 4.1 again to conclude that

$$\begin{aligned}\mu\{\Pi^\delta_{t,L}\} &= \int_{|x|\le L}\alpha(U(x,t),V^h(x,t))\,dx-\int_{|x|\le L+K(t-\delta)}\alpha(U(x,\delta),V^h(x,\delta))\,dx\\ &\quad+\int_{\partial\Pi^\delta_{t,L}}(\beta,\alpha)\cdot\nu\,d\sigma,\end{aligned}$$

where $\partial\Pi^\delta_{t,L}=\{(x,s):|x|=L+K(t-s),0<\delta<s<t\}$, ν is the unit outward normal field, and σ is the boundary measure. By choosing $K\ge K_0$ such that

$$\int_{\partial\Pi^\delta_{t,L}}(\beta,\alpha)\cdot\nu\,d\sigma\ge 0,$$

we have

$$\mu\{\Pi^\delta_{t,L}\}\ge\int_{|x|\le L}\alpha(U(x,t),V^h(x,t))\,dx-\int_{|x|\le L+K(t-\delta)}\alpha(U(x,\delta),V^h(x,\delta))\,dx. \tag{4.11}$$

Step 2. Set

$$\begin{aligned}&\Omega^1_j=\{\,(x,t)\,|\,-jh+\lambda_1(V^0_{-j-\frac12})t<x<-jh+\lambda_1(V^0_{-j+\frac12})t,\ t>0\},\\ &\Omega^1_0=\{\,(x,t)\,|\,\lambda_1(V^0_{-\frac12})<x<v_{c_1}t,\ t>0\},\\ &\Omega^2_0=\{\,(x,t)\,|\,v_{c_2}t<x<\lambda_2(V^0_{\frac12})t,\ t>0\},\\ &\Omega^2_j=\{\,(x,t)\,|\,jh+\lambda_2(V^0_{j-\frac12})t<x<jh+\lambda_2(V^0_{j+\frac12})t,\ t>0\}\qquad (j=1,2,\cdots)\end{aligned}$$

the rarefaction wave regions of $V^h(x,t)$, and

$$\Omega_0:=\Big\{(x,t):v_{c_1}<\frac{x}{t}<v_{c_2},\,t>0\Big\} \tag{4.12}$$

the vacuum region.

Then we have

$$\mu\{\Pi^\delta_{t,L}\}=\mu\{\text{constant states regions}\}+\mu\{\text{rarefaction wave regions}\}+\mu\{\text{vacuum region}\}. \tag{4.13}$$

Let

$$\begin{aligned}&\Omega^k_{j,\delta}(t)=\Omega^k_j\cap\Pi^\delta_{L,t},\quad \Omega^k_j(t)=\Omega^k_j\cap\{(x,s)\,|\,0<s<t\},\qquad k=1,2,\\ &\Omega_{0,\delta}(t)=\Omega_0\cap\Pi^\delta_{L,t}.\end{aligned}$$

Then we have

$$\mu\{\text{rarefaction wave regions}\}=\mu\Big\{\bigcup_{j,k}\Omega^k_{j,\delta}(t)\Big\}, \tag{4.14}$$

$$\mu\{\text{vacuum region}\}=\mu\{\Omega_{0,\delta}(t)\}. \tag{4.15}$$

Over the rarefaction wave regions,

$$\mu = \partial_t \alpha(U, V^h) + \partial_x \beta(U, V^h) = \nu - (\partial_x V^h)^\top \nabla^2 \eta_*(V^h) QF(U, V^h), \tag{4.16}$$

where $QF(U, V^h) = F(U) - F(V^h) - \nabla F(V^h)(U - V^h)$, and we used the fact that $\nabla^2 \eta_* \nabla F$ is symmetric. Recall that, for $(x, t) \in \Omega_j^k, j = 0, 1, \cdots, k = 1, 2$,

$$\frac{\partial V^h(x,t)}{\partial x} = \frac{1}{t} \widetilde{r}_k(V^h(x,t)), \qquad k = 1, 2. \tag{4.17}$$

Then, for any Borel set $E \subset \Omega_j^k$, $j = 0, 1, \cdots, k = 1, 2$, we have

$$\mu(E) = \nu(E) - \int_E \frac{1}{t} \widetilde{r}_k(V^h)^\top \nabla^2 \eta_*(V^h) QF(U, V^h)(x,t)\, dxdt. \tag{4.18}$$

Hence we have

$$\begin{aligned} &\mu\Big\{\bigcup_{j,k} \Omega_{j,\delta}^k(t)\Big\} \\ &= \nu\Big\{\bigcup_{j,k} \Omega_{j,\delta}^k(t)\Big\} - \sum_{j,k} \int_{\Omega_{j,\delta}^k(t)} \frac{1}{s} \widetilde{r}_j(V^h)^\top \nabla^2 \eta_*(V^h) QF(U, V^h)(x,s)\, dxds. \end{aligned} \tag{4.19}$$

Over the vacuum region Ω_0, $\rho^h(x,t) = 0 := \bar{\rho}(x,t)$, we may choose the velocity

$$v^h(x,t) = \frac{x}{t} := \bar{v}(x,t), \qquad v_{c_1} < \frac{x}{t} < v_{c_2}.$$

Then a careful calculation as before yields

$$\mu \le \nu - \frac{1}{t} \frac{\alpha_3}{(c^2 - v^2)^2 (c^2 - \bar{v}^2)^{3/2}} \big((c^2 - v^2)(c^2 - \bar{v}^2) p(\rho) + c^2 (v - \bar{v})^2 (p + c^2 \rho)\big),$$

where

$$\alpha_3 = 2c^5 \liminf_{\rho \to 0} \frac{p'(\rho)}{\rho p''(\rho) + 2p'(\rho)} \ge 0.$$

This implies that, for any Borel set $E \subset \Omega_0$,

$$\begin{aligned} \mu(E) \le \nu(E) &- \int_E \frac{1}{t} \frac{\alpha_3}{(c^2 - v^2)^2 (c^3 - \bar{v}^2)^{3/2}} \\ &\cdot \big((c^2 - v^2)(c^2 - \bar{v}^2) p(\rho) + c^2 (v - \bar{v})^2 (p + c^2 \rho)\big) dxdt \le 0. \end{aligned} \tag{4.20}$$

Since $V^h(x,t)$ is constant in each component of $\Pi_{t,L}^\delta - \Big\{\bigcup_{j,k} \Omega_{j,\delta}^k(t) \bigcup \Omega_{0,\delta}(t)\Big\}$ and $\nu \le 0$, we have

$$\mu\{\Pi_{t,L}^\delta\} \le -\sum_{j,k} \int_{\Omega_{j,\delta}^k(t)} \frac{1}{s} \widetilde{r}_j(V^h)^\top \nabla^2 \eta_*(V^h) QF(U, V^h)(x,s) dxds \tag{4.21}$$

from (4.18) and (4.20).

Step 3. Now we are going to show that

$$\mu\{\Pi_{t,L}^\delta\} \le 0. \tag{4.22}$$

In fact, a careful direct calculation from (2.13) and (2.15) yields

$$\widetilde{r}_2(V^h)^\top \nabla^2\eta_*(V^h) = \alpha_0(\bar\rho,\bar v)\alpha_2(\bar\rho,\bar v)\frac{\sqrt{\bar p'(c^4-\bar p'\bar v^2)}}{c^2(c^2-\bar v^2)}\big(c^2(\sqrt{\bar p'}-\bar v),\, c^2-\bar v\sqrt{\bar p'}\,\big) \tag{4.23}$$

and

$$QF(U,V^h) = \begin{pmatrix}0\\ \Theta\end{pmatrix}, \tag{4.24}$$

where

$$\Theta = \frac{c^2}{(c^4-\bar p'\bar v^2)(c^2-v^2)}\cdot\big((c^2-v^2)(c^2-\bar v^2)(p-\bar p-\bar p'(\rho-\bar\rho)) + (v-\bar v)^2(c^2-\bar p')(p+\rho c^2)\big).$$

Here we denote $\bar\rho=\rho^h, \bar v = v^h$, and $\bar p = p(\rho^h)$. Therefore, we have

$$h_j(s,x) := \widetilde{r}_j(V^h)^\top\nabla^2\eta_*(V^h)QF(U,V^h)(s,x)\ge 0. \tag{4.25}$$

Then (4.22) follows immediately.

Step 4. Combining (4.11) and (4.22), we arrive at

$$\int_{|x|\le L}\alpha(U(x,t),V^h(x,t))\,dx \le \int_{|x|\le L+K(t-\delta)}\alpha(U(x,\delta),V^h(x,\delta))\,dx. \tag{4.26}$$

Letting $\delta\to 0$, we see that (4.10) and (4.26) imply

$$\int_{|x|\le L}\alpha(U(x,t),V^h(x,t))\,dx \le \int_{|x|\le L+Kt}\alpha(U_0(x),V^h(x,0))\,dx. \tag{4.27}$$

Let $h\to 0$ in (4.27). Using the strong convergence of $V^h(x,t)$ to $V(x,t)$ as $h\to 0$, we have

$$\int_{|x|\le L}\alpha(U(x,t),V(x,t))\,dx \le \int_{|x|\le L+Kt}\alpha(U_0(x),V_0(x))\,dx. \tag{4.28}$$

This completes the proof.

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