

# OSCILLATION AND GLOBAL ATTRACTIVITY OF IMPULSIVE PERIODIC DELAY RESPIRATORY DYNAMICS MODEL

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## Abstract

This paper studies the nonlinear delay impulsive respiratory dynamics model. The model describes the sudden changes of the concentration of CO<sub>2</sub> in the blood of the mammal. It is proved that the model has a unique positive periodic solution. Some sufficient conditions for oscillation of all positive solutions about the positive periodic solution are established and also some sufficient conditions for the global attractivity of the periodic solution are obtained.

**Keywords** Periodic solutions, Oscillation, Global attractivity, Respiratory dynamics model, Impulsive differential equation

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## § 1. Introduction

The nonlinear delay differential equation,

$$y'(t) = y(t) \left[ \frac{p}{q + y^n(t - \tau)} - \lambda y(t) \right], \quad (1.1)$$

was proposed by Mackey and Glass [10] for a “Dynamic Disease” involving respiratory disorders, where  $\frac{1}{y(t)}$  denotes the arterial CO<sub>2</sub> concentration of a mammal,  $p$ ,  $q$ ,  $\lambda$  and  $\tau$  are positive constants such that  $\lambda$  is the CO<sub>2</sub> production rate, and  $\tau$  is the time between oxygenation of blood in lungs and stimulation of chemoreceptors in the brainstem. The oscillation and stability of (1.1) has been studied by Kubiacyk and Saker [9].

The variation of the environment plays an important role in many biological and ecological dynamical systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the system (in a way) incorporates the periodicity of the environment (e.g., seasonal effects of weather, food supplies, mating habits, etc.).

In fact, any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this it is realistic to assume that the parameters in the models are periodic functions of period  $\omega$ .

Also, many evolution processes in nature are characterized by the fact that at certain moments of time they experience an abrupt changes of state. The phenomena exhibit

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impulsive behavior are the biological processes and the electrical circuits, for examples we refer to the articles [1, 2, 8]. The solutions of such models exhibit sudden changes or jumps which are called impulses.

Thus, the modification of (1.1) according to the environmental variation and sudden changes is the nonautonomous impulsive delay differential equation

$$y'(t) = y(t) \left[ \frac{p(t)}{q(t) + y^n(t - m\omega)} - \lambda(t)y(t) \right], \quad t \neq t_k, \quad (1.2a)$$

$$y(t_k^+) - y(t_k^-) = b_k y(t_k^-), \quad k = 1, 2, 3, \dots, \quad (1.2b)$$

where

(h<sub>1</sub>)  $0 < t_1 < t_2 < t_3 < \dots$  are fixed impulsive points with  $\lim_{k \rightarrow \infty} t_k = \infty$ ;

(h<sub>2</sub>)  $p(t), q(t), \lambda(t) \in ([0, \infty), (0, \infty))$  are locally summable functions;

(h<sub>3</sub>)  $\{b_k\}$  is a real sequence and  $b_k > -1, k = 1, 2, \dots$ ;

(h<sub>4</sub>)  $p(t), q(t), \lambda(t)$  and  $\prod_{0 < t_k < t} (1 + b_k)$  are positive periodic functions with common period

$\omega > 0$  and  $m$  is nonnegative integer.

For biological significance, we consider solutions of (1.2) with initial conditions of the form

$$y(t) = \phi(t) \quad \text{for } -m\omega \leq t \leq 0, \phi \in L([-m\omega, 0], [0, \infty)) \text{ and } \phi(0) > 0, \quad (1.3)$$

where  $L([-m\omega, 0], [0, \infty))$  denotes the set of Lebesgue measurable functions on  $[-m\omega, 0]$ .

The mathematical theory of the impulsive differential equations is much more complicated in comparison with the corresponding theory of the impulsive ordinary differential equations without delay and the theory of differential equations without impulses. In the last few years, the oscillation and global attractivity of impulsive delay differential equations have been studied by many authors, for some contributions we refer the reader to the papers [3–6, 12, 13] and the reference cited therein.

**Definition 1.1.** A function  $y \in ([-m\omega, \infty), (0, \infty))$  is said to be a solution of the equation (1.2) on  $[-m\omega, \infty)$  if

(i)  $y(t)$  is absolutely continuous on each interval  $(0, t_1]$  and  $(t_k, t_{k+1}]$ ,  $k = 1, 2, \dots$ ;

(ii) for any  $t_k, k = 1, 2, \dots$ ,  $y(t_k^+), y(t_k^-)$  exist and  $y(t_k^-) = y(t_k)$ ;

(iii)  $y(t)$  satisfies (1.2a) for almost everywhere in  $[0, \infty) \setminus \{t_k\}$  and satisfies (1.2b) for every  $t = t_k$ .

**Definition 1.2.** A function  $y(t)$  of (1.2) is said to oscillate about  $\bar{y}(t)$  if  $(y(t) - \bar{y}(t))$  has arbitrarily large zeros. Otherwise,  $y(t)$  is called nonoscillatory. When  $\bar{y}(t) = 0$ , we say  $y(t)$  oscillates about zero or simply oscillates.

**Definition 1.3.** Suppose that  $y(t)$  and  $\bar{y}(t)$  are two positive solutions of (1.2) on  $[t - m\omega, \infty)$ . The solution  $\bar{y}(t)$  is said to be asymptotically attractive to  $y(t)$  provided  $\lim_{t \rightarrow \infty} [y(t) - \bar{y}(t)] = 0$ . Further,  $\bar{y}(t)$  is called globally attractive if  $\bar{y}(t)$  is asymptotically attractive to all positive solutions of (1.2).

Our aim in this paper is to prove that (1.2) has a unique positive periodic solution  $\bar{y}(t)$  of period  $\omega$ . Next, we establish some sufficient conditions for oscillation of all positive solutions of (1.2) about  $\bar{y}(t)$ , which are the sufficient conditions for the prevalence of a dynamic disease that insure the existence of the change of the concentration of CO<sub>2</sub> in the blood around the positive periodic solution. Also, we establish some sufficient conditions for the global attractivity of  $\bar{y}(t)$ , which are the sufficient conditions for the nonexistence of dynamic diseases.

### § 2. Main Results

In this section, first we prove that the equation (1.2) has a positive periodic solution  $\bar{y}(t)$ . Next, we give an oscillation comparison theorem which enables us to establish some sufficient conditions for oscillation of all positive solutions of the equation (1.2) about  $\bar{y}(t)$  and establish some sufficient conditions for global attractivity.

Under the above hypothesis (h<sub>1</sub>)–(h<sub>4</sub>) we consider the nonlinear delay differential equation

$$z'(t) = z(t) \left[ \frac{P(t)}{Q(t) + z^n(t - m\omega)} - \Lambda(t)z(t) \right], \quad t \geq 0, \tag{2.1}$$

with the initial condition

$$z(t) = \varphi(t) \quad \text{for } -m\omega \leq t \leq 0, \quad \varphi \in L([-m\omega, 0], [0, \infty)) \text{ and } \varphi(0) > 0, \tag{2.2}$$

where

$$P(t) = \frac{p(t)}{\prod_{0 < t_k < t - m\omega} (1 + b_k)^n}, \quad Q(t) = \frac{q(t)}{\prod_{0 < t_k < t - m\omega} (1 + b_k)^n}, \quad \Lambda(t) = \lambda(t) \prod_{0 < t_k < t} (1 + b_k).$$

By a solution  $z(t)$  of (2.1) and (2.2) we mean an absolutely continuous function  $z(t)$  defined on  $[-m\omega, \infty)$  satisfies (2.1) for all  $t \geq 0$  and  $z(t) = \varphi(t)$  on  $[-m\omega, 0]$ .

Here and in the sequel we assume that a product equals unit if the number of factors is equal to zero and for a periodic function  $g$  of period  $\omega$ , we shall denote by

$$g^* = \max_{0 \leq t \leq \omega} g(t) \quad \text{and} \quad g_* = \min_{0 \leq t \leq \omega} g(t).$$

Further, each functional inequality we will write holds for all sufficiently large  $t$ .

The following lemma will be used in the proof of our main result for existence of periodic positive solution of (1.2). The proof is similar to that of Theorem 1 established by Luo [12] and will be omitted.

**Lemma 2.1.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then*

(i) *if  $z(t)$  is a solution of (2.1) on  $[-m\omega, \infty)$ , then  $y(t) = \prod_{0 < t_k < t} (1 + b_k)z(t)$  is a solution of (1.2) on  $[-m\omega, \infty)$ .*

(ii) *if  $y(t)$  is a solution of (1.2) on  $[-m\omega, \infty)$ , then  $z(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1}y(t)$  is a solution of (2.1) on  $[-m\omega, \infty)$ .*

**Lemma 2.2.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then the solutions of (1.2) and (2.1) are defined on  $[-m\omega, \infty)$  and are positive on  $[0, \infty)$ .*

**Proof.** From (2.1) and (2.2) it is clear that  $z(t)$  is defined and positive for any  $\phi \in L$ . Then, by Lemma 1.1, we see that the solution of (1.2) and (1.3) is also defined and positive on  $[-m\omega, \infty)$ . The proof is complete.

Now, we shall consider the nondelay case, i.e.,

$$y'(t) = y(t) \left[ \frac{p(t)}{q(t) + y^n(t)} - \lambda(t)y(t) \right], \quad t \neq t_k, \tag{2.3a}$$

$$y(t_k^+) - y(t_k^-) = b_k y(t_k^-), \quad k = 1, 2, 3, \dots \tag{2.3b}$$

and

$$z'(t) = z(t) \left[ \frac{P(t)}{Q(t) + z^n(t)} - \Lambda(t)z(t) \right], \quad t \geq 0, \tag{2.4}$$

where  $P(t)$ ,  $Q(t)$  and  $\Lambda(t)$  are positive periodic functions of period  $\omega$ . In the following theorem by applying the method used in [14], we prove that (2.3) has a unique positive periodic solution  $\bar{y}(t)$  and shows that  $\bar{y}(t)$  is in fact a global attractor of all other positive solutions.

**Theorem 2.1.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then*

- (a) *there exists a unique  $\omega$ -periodic positive solution  $\bar{y}(t)$  of (2.3), and*
- (b) *for every other positive solution  $y(t)$  of (2.3) the limit*

$$\lim_{t \rightarrow \infty} [y(t) - \bar{y}(t)] = 0.$$

**Proof.** To prove the theorem, we prove that (2.4) has a unique  $\omega$ -periodic positive solution  $\bar{z}(t)$ . Consider the function

$$f(z) = \frac{P}{Q + z^n} - \Lambda z, \quad z \in [0, \infty),$$

where  $\Lambda$ ,  $P$  and  $Q$  are positive constants. Clearly,  $f(0) = P/Q$ ,  $f(\infty) = -\infty$  and  $f(z)$  is monotonically decreasing. Thus, the equation  $f(z) = 0$  has a unique positive root  $z_0 \in (0, \infty)$ . Further,  $f(z) > 0$ ,  $z \in [0, z_0)$  and  $f(z) < 0$ ,  $z \in (z_0, \infty)$ .

To prove (a), we define the functions

$$f_1(z) = \frac{P^*}{Q^* + z^n} - \Lambda^* z \quad \text{and} \quad f_2(z) = \frac{P^*}{Q_* + z^n} - \Lambda_* z. \tag{2.5}$$

It is clear that  $f_1(z)$  and  $f_2(z)$  have positive zeros  $z_1$  and  $z_2$ , respectively, i.e.,  $f_1(z_1) = 0$  and  $f_2(z_2) = 0$ . Noting (2.5) we have  $z_2 > z_1 > 0$ . Now, suppose  $z(t) = z(t, 0, z_0)$  where  $z_0 > 0$  is the unique solution of (2.4) through  $(0, z_0)$ . We claim that if  $z_0 \in [z_1, z_2]$ , then  $z(t) \in [z_1, z_2]$  for all  $t \geq 0$ . Otherwise, let  $t^* = \inf\{t > 0 \mid z(t) > z_2\}$ . Then, there exists  $\bar{t} \geq t^*$  such that  $z(\bar{t}) > z_2$  and  $z'(\bar{t}) \geq 0$ . However, from (2.4) and the fact that  $z(\bar{t}) > z_2$ , we have

$$z'(\bar{t}) = z(\bar{t}) \left[ \frac{P(\bar{t})}{Q(\bar{t}) + z^n(\bar{t})} - \Lambda(\bar{t})z(\bar{t}) \right] < z(\bar{t}) \left[ \frac{P^*}{Q_* + z_2^n} - \Lambda_* z_2 \right] = 0,$$

which is a contradiction. Therefore,  $z(t) \leq z_2$ . By a similar argument, we can show that  $z(t) \geq z_1$  for all  $t \geq 0$ . Hence, in particular,  $z_\omega = z(\omega, 0, z_0) \in [z_1, z_2]$ .

Now we define a mapping  $F : [z_1, z_2] \rightarrow [z_1, z_2]$  as follows: for each  $z_0 \in [z_1, z_2]$ ,  $F(z_0) = z_\omega$ . Since the solution  $z(t, 0, z_0)$  of the equation (2.4) depends continuously on the initial value  $\bar{z}_0$ , it follows that  $F$  is continuous and maps the interval  $[z_1, z_2]$  into itself. Therefore,  $F$  has a fixed point  $\bar{z}_0$  by Brouwer's fixed point theorem. In view of the periodicity of  $P$ ,  $Q$  and  $\Lambda$  it follows that the unique solution  $\bar{z}(t) = z(t, 0, \bar{z}_0)$  of (2.4) through the initial point  $(0, z_0)$  is a positive periodic solution of period  $\omega$ . Let  $\bar{y}(t) = \prod_{0 < t_k < t} (1 + b_k) \bar{z}(t)$ . Then, by

Lemma 2.1 and (h<sub>4</sub>),  $\bar{y}(t)$  is the  $\omega$ -periodic solution of (2.3). The proof of (a) is complete.

Now we shall prove (b). Let  $\bar{y}(t)$  be a periodic positive solution of (2.3). Thus, by Lemma 1.1,  $\bar{z}(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1} \bar{y}(t)$  is the periodic solution of (2.4). Assume that  $y(t) > \bar{y}(t)$  for  $t$  sufficiently large, then  $z(t) > \bar{z}(t)$  (the proof when  $y(t) < \bar{y}(t)$  is similar and will be omitted). Set

$$z(t) = \bar{z}(t)e^{x(t)}. \tag{2.6}$$

Then,  $x(t) > 0$  for  $t$  sufficiently large, and from (2.4) we have

$$x'(t) + \Lambda(t)\bar{z}(t)(e^{x(t)} - 1) + \frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t)})} = 0.$$

However, since  $0 < (e^{x(t)} - 1)$  for  $t$  sufficiently large, it follows that

$$x'(t) + \frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t)})} \leq 0, \tag{2.7}$$

and hence

$$x'(t) \leq -\frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t)})} < 0. \tag{2.8}$$

Thus,  $x(t)$  is decreasing, and therefore  $\lim_{t \rightarrow \infty} x(t) = \alpha \in [0, \infty)$ . Now we shall prove that  $\alpha = 0$ . If  $\alpha > 0$ , then there exist  $\varepsilon > 0$  and  $T_\varepsilon > 0$  such that for  $t \geq T_\varepsilon$ ,  $0 < \alpha - \varepsilon < x(t) < \alpha + \varepsilon$ . However, then from (2.8), we find

$$x'(t) < -\frac{P_* \bar{z}_*^n}{(Q_* + (\bar{z}_*)^n)} \frac{(e^{n(\alpha - \varepsilon)} - 1)}{(Q_* + (\bar{z}_*)^n e^{n(\alpha + \varepsilon)})}, \quad t \geq T_\varepsilon. \tag{2.9}$$

But, now an integration of (2.9) from  $T_\varepsilon$  to  $\infty$  immediately gives a contradiction. Hence,  $\alpha = 0$ . Thus  $\lim_{t \rightarrow \infty} x(t) = 0$ , that is,

$$\begin{aligned} \lim_{t \rightarrow \infty} (y(t) - \bar{y}(t)) &= \lim_{t \rightarrow \infty} \prod_{0 < t_k < t} (1 + b_k)(z(t) - \bar{z}(t)) \\ &= \lim_{t \rightarrow \infty} \bar{z}(t) \prod_{0 < t_k < t} (1 + b_k)(e^{x(t)} - 1) = 0. \end{aligned}$$

Hence

$$\lim_{t \rightarrow \infty} (y(t) - \bar{y}(t)) = 0.$$

This completes the proof.

**Remark 2.1.** It is clear that in the proof of Theorem 2.1 (b) instead of (2.7) the inequality

$$x'(t) + \Lambda(t)\bar{z}(t)(e^{x(t)} - 1) \leq 0 \tag{2.10}$$

can be used to get the same conclusion that  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the following we study the existence of positive periodic solution of the delay differential equation (1.2), establish some sufficient conditions for oscillation of all positive solution about it and give sufficient condition for global attractivity.

**Theorem 2.2.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then, there exists a unique  $\omega$ -periodic positive solution  $\bar{y}(t)$  of (1.2).*

**Proof.** By Theorem 2.1, the equation (2.4) has a unique  $\omega$ -positive periodic solution  $\bar{z}(t)$ . Noting that  $\bar{z}(t) = \bar{z}(t - m\omega)$  we see that  $\bar{z}(t)$  is also an  $\omega$ -periodic positive solution of (2.1). Thus by Lemma 2.1 and (h<sub>4</sub>),  $\bar{y}(t) = \prod_{0 < t_k < t} (1 + b_k)\bar{z}(t)$  is  $\omega$ -periodic positive solution of the equation (1.2). On the other hand, if  $\bar{y}(t)$  is a periodic positive solution of (1.2), it is easy to see that  $\bar{y}(t)$  is also a positive periodic solution of (2.3). In view of Theorem 2.1, the periodic positive solution of the equation (1.2) is unique. The proof is complete.

**Remark 2.2.** From the proof of Theorem 2.1, it follows that the unique  $\omega$ -periodic positive solution  $\bar{y}(t)$  of (2.3) satisfies  $z_1 \leq \prod_{0 < t_k < t} (1 + b_k)^{-1} \bar{y}(t) \leq z_2$ . Thus, an interval for the location of  $\bar{y}(t)$  is readily available and this proves that the periodic positive solution  $\bar{y}(t)$  of (1.2) is permanent.

In the nondelay case, we have seen in Theorem 2.1(b) that every positive solution of (1.2) converges to the unique positive  $\omega$ -periodic solution  $\bar{y}(t)$ . In our final result Theorem 2.9 we shall show that the same behavior holds in the case of small delays. To show this, first we shall prove that every positive solution of (1.2) which does not oscillate about  $\bar{y}(t)$  converges to  $\bar{y}(t)$ .

**Theorem 2.3.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Let  $y(t)$  be a positive solution of (1.2) which does not oscillate about  $\bar{y}(t)$ . Then*

$$\lim_{t \rightarrow \infty} [y(t) - \bar{y}(t)] = 0. \tag{2.11}$$

**Proof.** To prove (2.11), it suffices to show that

$$\lim_{t \rightarrow \infty} [z(t) - \bar{z}(t)] = 0,$$

where  $\bar{z}(t)$  is the unique  $\omega$ -periodic positive solution of (2.1), and  $z(t)$  is any other positive solution of (2.1) which does not oscillate about  $\bar{z}(t)$ . Assume that  $z(t) > \bar{z}(t)$  for  $t$  sufficiently large (the proof for the case  $z(t) < \bar{z}(t)$  is similar and hence omitted). Using the transformation (2.6), we again have  $z(t) > 0$  for  $t$  sufficiently large, and  $z(t)$  satisfies the equation

$$x'(t) + \Lambda(t)\bar{z}(t)(e^{x(t)} - 1) + \frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t-m\omega)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t-m\omega)})} = 0. \tag{2.12}$$

Again, since  $0 < (e^{z(t)} - 1)$  for  $t$  sufficiently large, the equation (2.12) gives

$$x'(t) + \frac{Q(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t-m\omega)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t-m\omega)})} \leq 0, \tag{2.13}$$

and hence

$$x'(t) \leq -\frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} \frac{(e^{nx(t-m\omega)} - 1)}{(Q(t) + \bar{z}^n(t)e^{nx(t-m\omega)})} < 0. \tag{2.14}$$

Thus,  $z(t)$  is decreasing, and therefore  $\lim_{t \rightarrow \infty} x(t) = \alpha \in [0, \infty)$ . We need to show that  $\alpha = 0$ . If  $\alpha > 0$ , then there exist  $\varepsilon > 0$  and  $T_\varepsilon > 0$  such that for  $t \geq T_\varepsilon$ ,  $0 < \alpha - \varepsilon < x(t) < x(t - m\omega) < \alpha + \varepsilon$ . Then, from (2.14), again we obtain (2.9). The rest of the proof is the same as that of Theorem 2.1(b) and hence omitted.

**Remark 2.3.** As in Remark 2.1 in the proof of Theorem 2.3 instead of (2.13) the inequality (2.10) can be used to get the same conclusion that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the following, we prove that the oscillation of all solutions of (1.2) is equivalent to the oscillation of all solutions of the equation (2.1).

**Theorem 2.4.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then every solution of (1.2) oscillates if and only if every solution of the equation (2.1) oscillates.*

**Proof.** Suppose that  $z(t)$  is a solution of (2.1) on  $[0, \infty)$ . Let  $y(t) = \prod_{0 < t_k < t} (1 + b_k)z(t)$ . From Lemma 2.1,  $y(t)$  is a solution of (1.2) on  $[0, \infty)$ . Since  $\prod_{0 < t_k < t} (1 + b_k) > 0, t > 0, y(t)$  is oscillatory if and only if  $z(t)$  is oscillatory.

Conversely, suppose that  $y(t)$  is a solution of (1.2) on  $[0, \infty)$ . Let  $z(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1}y(t)$ . Thus, from Lemma 2.1,  $z(t)$  is a solution of (2.1) on  $[0, \infty)$  and from the fact that  $\prod_{0 < t_k < t} (1 + b_k)^{-1} > 0, t > 0, z(t)$  is oscillatory if and only if  $y(t)$  is oscillatory. The proof is complete.

From Theorem 2.4, we have the following oscillation criteria of (1.2).

**Theorem 2.5.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold and every solution of the delay differential equation*

$$W'(t) + \exp\left((1 - \varepsilon) \int_{t-m\omega}^t \Lambda(s)\bar{z}(s)ds\right) \frac{n(1 - \varepsilon)P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))^2} W(t - m\omega) = 0 \tag{2.15}$$

*oscillates, where  $\varepsilon > 0$  is arbitrarily small. Then, every solution of (1.2) oscillates about  $\bar{y}(t)$ .*

**Proof.** Set

$$z(t) = \bar{z}(t)e^{x(t)}. \tag{2.16}$$

Then it is clear that a solution  $z(t)$  of (2.1) oscillates about  $\bar{z}(t)$  if and only if  $z(t)$  oscillates about zero. Assume for the sake of contradiction that (2.1) has a solution  $z(t)$  which does not oscillate about  $\bar{z}(t)$ . Without loss of generality we assume that  $z(t) > \bar{z}(t)$ , so that  $z(t) > 0$ . (The proof for the case  $z(t) < \bar{z}(t)$ , i.e., when  $z(t) < 0$  is similar and hence omitted.) The transformation (2.16) transforms (2.1) to the equation

$$x'(t) + \Lambda(t)\bar{z}(t)f_1(x(t)) + \frac{nP(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))^2} f_2(x(t - m\omega)) = 0, \tag{2.17}$$

where

$$f_1(u) = e^u - 1 \quad \text{and} \quad f_2(u) = \frac{Q + \bar{z}^n(t)}{n} \frac{e^{nu} - 1}{Q + \bar{z}^n(t)e^{nu}}.$$

Note that

$$\begin{aligned} uf_1(u) &> 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{u \rightarrow 0} \frac{f_1(u)}{u} = 1, \\ uf_2(u) &> 0 \quad \text{for } u \neq 0 \quad \text{and} \quad \lim_{u \rightarrow 0} \frac{f_2(u)}{u} = 1. \end{aligned} \tag{2.18}$$

From (2.18) it follows that for any given arbitrarily small  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $0 < u < \delta, f_1(u) \geq (1 - \varepsilon)u$  and  $f_2(u) \geq (1 - \varepsilon)u$  (for all  $-\delta < u < 0, f_1(u) \leq (1 - \varepsilon)u$  and  $f_2(u) \leq (1 - \varepsilon)u$ ). Since in view of Theorem 2.3,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for sufficiently large  $t$  we can use these estimates in (2.17), to conclude that eventually  $x(t)$  is a positive solution of the differential inequality

$$x'(t) + \Lambda(t)\bar{z}(t)(1 - \varepsilon)x(t) + \frac{(1 - \varepsilon)nP(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))^2} x(t - m\omega) \leq 0. \tag{2.19}$$

Now, using the transformation

$$x(t) = \exp\left(- (1 - \varepsilon) \int_0^t \Lambda(s)\bar{z}(s)ds\right) X(t)$$

in (2.19) implies that  $X(t)$  is also an eventually positive solution of the differential inequality

$$X'(t) + \exp\left((1 - \varepsilon) \int_{t-m\omega}^t \Lambda(s)\bar{z}(s)ds\right) \frac{(1 - \varepsilon)nP(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))^2} X(t - m\omega) \leq 0. \tag{2.20}$$

But, then by Corollary 3.2.2 in [7] there exists an eventually positive solution of the delay differential equation (2.15) which satisfies that  $W(t) \geq X(t)$ . This contradicts our assumption that every solution of (2.15) is oscillatory. Hence, every positive solution of (2.1) oscillates about  $\bar{z}(t)$ . Then Theorem 2.4 implies that every solution of (1.2) oscillates about  $\bar{y}(t)$  if every solution of (2.15) oscillates. The proof is complete.

**Remark 2.4.** For the oscillation of the delay differential equation (2.15) several known criteria can be employed. For example, the results given in [7] when applied to (2.15) lead to the following corollary.

**Corollary 2.1.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then*

$$(1 - \varepsilon) \liminf_{t \rightarrow \infty} \int_{t-m\omega}^t \frac{nP(s)\bar{z}^n(s)}{(Q(s) + \bar{z}^n(s))^2} \exp\left((1 - \varepsilon) \int_{s-m\omega}^s \Lambda(\eta)\bar{z}(\eta)d\eta\right) ds > \frac{1}{e}, \tag{2.21}$$

or

$$(1 - \varepsilon) \limsup_{t \rightarrow \infty} \int_{t-m\omega}^t \frac{nP(s)\bar{z}^n(s)}{(Q(s) + \bar{z}^n(s))^2} \exp\left((1 - \varepsilon) \int_{s-m\omega}^s \Lambda(\eta)\bar{z}(\eta)d\eta\right) ds > 1, \tag{2.22}$$

implies that every solution of (2.15) is oscillatory.

**Remark 2.5.** Clearly, if the strict inequalities hold in (2.21) and (2.22) for  $\varepsilon = 0$ , then the same must be true for all sufficiently small  $\varepsilon > 0$  also. Thus, we can restate Corollary 2.1 as follows:

**Corollary 2.2.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then*

$$\liminf_{t \rightarrow \infty} \int_{t-m\omega}^t \frac{nP(s)\bar{z}^n(s)}{(Q(s) + \bar{z}^n(s))^2} \exp\left(\int_{s-m\omega}^s \Lambda(\eta)\bar{z}(\eta)d\eta\right) ds > \frac{1}{e}, \tag{2.23}$$

or

$$\Gamma_0 = \limsup_{t \rightarrow \infty} \int_{t-m\omega}^t \frac{nP(s)\bar{z}^n(s)}{(Q(s) + \bar{z}^n(s))^2} \exp\left((1 - \varepsilon) \int_{s-m\omega}^s \Lambda(\eta)\bar{z}(\eta)d\eta\right) ds > 1, \tag{2.24}$$

implies that every solution of (2.15) is oscillatory.

From Theorem 2.4, Theorem 2.5 and Corollary 2.2 the following oscillation criterion for (1.2) is immediate.

**Theorem 2.6.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold. Then*

$$\liminf_{t \rightarrow \infty} \int_{t-m\omega}^t A(s) \exp\left(\int_{s-m\omega}^s \Lambda(\eta) \left(\prod_{0 < t_k < \eta} (1 + b_k)^{-1} \bar{y}(\eta)\right) d\eta\right) ds > \frac{1}{e}, \tag{2.25}$$

or

$$\limsup_{t \rightarrow \infty} \int_{t-m\omega}^t A(s) \exp\left(\int_{s-m\omega}^s \Lambda(\eta) \left(\prod_{0 < t_k < \eta} (1 + b_k)^{-1} \bar{y}(\eta)\right) d\eta\right) ds > 1, \tag{2.26}$$

where

$$A(s) = \frac{nP(s) \left(\prod_{0 < t_k < s} (1 + b_k)^{-1} \bar{y}(s)\right)^n}{\left(Q(s) + \left(\prod_{0 < t_k < s} (1 + b_k)^{-1} \bar{y}(s)\right)^n\right)^2}, \tag{2.27}$$

implies that every solution of (1.2) oscillates about  $\bar{y}(t)$ .

To show that  $\bar{y}(t)$  is a global attractor of (1.2), we need to prove that  $\bar{z}(t)$  is a global attractor of the equation (2.1). To prove this we need to find some upper and lower bounds for solutions of (2.1) which oscillate about  $\bar{z}(t)$ .

**Theorem 2.7.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold, and let  $z(t)$  be a positive solution of (2.1) which oscillates about  $\bar{z}(t)$ . Then, there exists a  $T$  such that for all  $t \geq T$ ,*

$$Z_1 = z_1 \exp \left( \left[ \frac{P^*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] m\omega \right) \leq z(t) \leq z_2 \exp \left( \frac{P^* m\omega}{Q^*} \right) = Z_2. \tag{2.28}$$

**Proof.** First we shall show the upper bound in (2.28). For this, let  $m\omega \leq t_1 < t_2 < \dots < t_l < \dots$  be a sequence of zeros of  $z(t) - \bar{z}(t)$  with  $\lim_{l \rightarrow \infty} t_l = \infty$ . Our strategy is to show that the upper bound holds in each interval  $(t_l, t_{l+1})$ . For this, let  $\zeta_l \in (t_l, t_{l+1})$  be a point where  $z(t)$  attains its maximum in  $(t_l, t_{l+1})$ . Then, it suffices to show that

$$z(\zeta_l) \leq z_2 \exp \left( \frac{P^* m\omega}{Q^*} \right) = Z_2. \tag{2.29}$$

We can assume that there exists a  $\zeta_l$  where  $z(\zeta_l) > z_2$ , otherwise there is nothing to prove. Since  $z'(\zeta_l) = 0$ , it follows that

$$0 = z'(\zeta_l) < z(\zeta_l) \left[ \frac{P^*}{Q^* + (y(\zeta_l - m\omega))^n} - \Lambda_* z(\zeta_l) \right],$$

and hence

$$0 < \frac{P^*}{Q^* + (z(\zeta_l - m\omega))^n} - \Lambda_* z_2.$$

Thus, if  $z(t)$  attains its maximum at  $\zeta_l$ , then it follows (cf. see the proof of Theorem 2.1) that  $z(\zeta_l - m\omega) < z_2$ . Now, since  $z(\zeta_l) > z_2$  and  $z(\zeta_l - m\omega) < z_2$ , we can let  $\bar{\zeta}_l$  to be the first zero of  $z(t) - z_2$  in  $(\zeta_l - m\omega, \zeta_l)$ , i.e.,  $z(\bar{\zeta}_l) = z_2$ . Integrating (2.1) from  $\bar{\zeta}_l$  to  $\zeta_l$ , we get

$$\begin{aligned} \ln \frac{z(\zeta_l)}{z(\bar{\zeta}_l)} &= \int_{\bar{\zeta}_l}^{\zeta_l} \left( \frac{P(t)}{Q(t) + (z(t - m\omega))^n} - \Lambda(t)z(t) \right) dt \\ &< \int_{\bar{\zeta}_l}^{\zeta_l} \frac{P(t)}{Q(t)} dt < \int_{\zeta_l - m\omega}^{\zeta_l} \frac{p^*}{Q^*} dt < \frac{P^* m\omega}{Q^*}. \end{aligned}$$

Hence, there exists a  $T_1$  such that  $z(t) \leq z_2 \exp \left( \frac{P^* m\omega}{Q^*} \right)$  for all  $t \geq T_1$ .

Now, we shall show the lower bound in (2.28) for  $t \geq T_1 + m\omega$ . For this, following as above let  $\mu_l \in (t_l, t_{l+1})$  be a point where  $z(t)$  attains its minimum in  $(t_l, t_{l+1})$ . Then, it suffices to show that

$$Z_1 = z_1 \exp \left( \left[ \frac{P^*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] m\omega \right) \leq z(\mu_l). \tag{2.30}$$

Since,  $Z_2 > z_2 > z_1$  it follows that

$$\left[ \frac{P^*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] < 0. \tag{2.31}$$

Thus,  $Z_1 < z_1$ . Now, assume that there exists a  $\mu_l \geq T_1 + m\omega$  where  $z(\mu_l) < z_1$ , otherwise there is nothing to prove. Since  $z'(\mu_l) = 0$ , we have

$$0 = z'(\mu_l) > z(\mu_l) \left[ \frac{P^*}{Q^* + (z(\mu_l - m\omega))^n} - \Lambda^* z(\mu_l) \right],$$

and hence

$$0 > \frac{P_*}{Q^* + (z(\mu_l - m\omega))^n} - \Lambda^* z_1.$$

Thus, it is necessary that  $z(\mu_l - m\omega) > z_1$ . Hence, there exists a  $\bar{\mu}_l \in (\mu_l - m\omega, \mu_l)$  where  $z(\bar{\mu}_l) = z_1$ . Integrating (2.1) from  $\bar{\mu}_l$  to  $\mu_l$ , and using  $z(t) \leq Z_2$  and (2.31), we get

$$\begin{aligned} \ln \frac{z(\mu_l)}{z(\bar{\mu}_l)} &= \int_{\bar{\mu}_l}^{\mu_l} \left( \frac{P(t)}{Q(t) + (z(t - m\omega))^n} - \Lambda(t)z(t) \right) dt \\ &> \int_{\bar{\mu}_l}^{\mu_l} \left[ \frac{P_*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] dt > \int_{\mu_l - m\omega}^{\mu_l} \left[ \frac{P_*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] dt, \end{aligned}$$

which immediately leads to (2.30).

From Theorem 2.7 and Lemma 2.1, we have the following permanent theorem for every solution of (1.2) about  $\bar{y}(t)$ .

**Theorem 2.8.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold, and let  $y(t)$  be a positive solution of (1.1) which oscillates about  $\bar{y}(t)$ . Then, there exists a  $T$  such that for all  $t \geq T$ ,*

$$z_1 \prod_{0 < t_k < t} (1 + b_k) \exp \left( \left[ \frac{P_*}{Q^* + Z_2^n} - \Lambda^* Z_2 \right] m\omega \right) \leq y(t) \leq z_2 \prod_{0 < t_k < t} (1 + b_k) \exp \left( \frac{P_* m\omega}{Q_*} \right).$$

where  $Z_2$  be as defined in Theorem 2.7.

The following result provides sufficient conditions for the global attractivity of  $\bar{y}(t)$ .

**Theorem 2.9.** *Assume that (h<sub>1</sub>)–(h<sub>4</sub>) hold, and*

$$\lim_{t \rightarrow \infty} \int_{t - m\omega}^t \frac{nP(s)(Z_2)^n}{(Q(s) + (Z_1)^n)^2} \exp \left( Z_2 \int_{s - m\omega}^s \Lambda(u) du \right) ds < \frac{\pi}{2}, \tag{2.32}$$

where  $Z_1$  and  $Z_2$  are as in (2.28). Then (2.11) holds for any positive solution  $y(t)$  of (1.2).

**Proof.** Clearly from Lemma 2.1, it suffices to prove that

$$\lim_{t \rightarrow \infty} [z(t) - \bar{z}(t)] = 0. \tag{2.33}$$

where  $\bar{z}(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1} \bar{z}(t)$  is the unique  $\omega$ -periodic positive solution of (2.1), and  $z(t) = \prod_{0 < t_k < t} (1 + b_k)^{-1} y(t)$  is any other positive solution of (2.1). In the nondelay case, we have established (2.33) in Theorem 2.1(b), and for the positive solutions of (2.1) which are nonoscillatory about  $\bar{z}(t)$  we have shown (2.33) in Theorem 2.3. Thus, it remains to prove (2.33) for the positive solutions of (2.1) which oscillate about  $\bar{z}(t)$ . The transformation (2.6) transforms (2.1) to the equation (2.12), which can be written as

$$x'(t) + \Lambda(t)\bar{z}(t)f_1(x(t)) + \frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} f_2(x(t - m\omega)) = 0, \tag{2.34}$$

where

$$f_1(u) = e^u - 1 \quad \text{and} \quad f_2(u) = \frac{e^{nu} - 1}{Q(t) + \bar{z}^n(t)e^{nu}}.$$

Let

$$G_1(t, u) = \Lambda(t)\bar{z}(t)f_1(u) \quad \text{and} \quad G_2(t, u) = \frac{P(t)\bar{z}^n(t)}{(Q(t) + \bar{z}^n(t))} f_2(u).$$

Then, we have

$$\frac{\partial G_1(t, u)}{\partial u} = \Lambda(t)\bar{z}(t)e^u \quad \text{and} \quad \frac{\partial G_2(t, u)}{\partial u} = \frac{nP(t)\bar{z}^n(t)e^{nu}}{(Q(t) + \bar{z}^n(t)e^{nu})^2}.$$

The equation (2.34) is the same as

$$x'(t) + G_1(t, x(t)) - G_1(t, 0) + G_2(t, x(t - m\omega)) - G_2(t, 0) = 0. \tag{2.35}$$

Clearly, by the mean value theorem (2.35) can be written as

$$x'(t) + F_1(t)x(t) + F_2(t)x(t - m\omega) = 0, \tag{2.36}$$

where

$$F_1(t) = \left. \frac{\partial G_1(t, u)}{\partial u} \right|_{u=\zeta_1(t)} = \Lambda(t)\bar{z}(t)e^{\zeta_1(t)} = \Lambda(t)\eta_1(t),$$

$$F_2(t) = \left. \frac{\partial G_2(t, u)}{\partial u} \right|_{u=\zeta_2(t)} = \frac{nP(t)\bar{z}^n(t)e^{n\zeta_2(t)}}{(Q(t) + \bar{z}^n(t)e^{n\zeta_2(t)})^2} = \frac{nP(t)(\eta_2(t))^n}{(Q(t) + (\eta_2(t))^n)^2},$$

where  $\eta_1(t)$  lies between  $\bar{z}(t)$  and  $z(t)$ , and  $\eta_2(t)$  lies between  $\bar{z}(t)$  and  $z(t - m\omega)$ . Let

$$x(t) = X(t) \exp\left(-\int_0^t F_1(s)ds\right).$$

Then, (2.36) can be written as

$$X'(t) + F_2(t) \exp\left(\int_{t-m\omega}^t F_1(s)ds\right)X(t - m\omega) = 0. \tag{2.37}$$

From Theorem 2.8, we find

$$\int_{t-m\omega}^t F_2(t) \exp\left(\int_{t-m\omega}^t F_1(s)ds\right) \leq \int_{t-m\omega}^t \frac{nP(s)(Z_2)^n}{(Q(s) + (Z_1)^n)^2} \exp\left(Z_2 \int_{s-m\omega}^s \Lambda(u)du\right) ds$$

and hence in view of (2.32), we have

$$\lim_{t \rightarrow \infty} \int_{t-m\omega}^t F_2(t) \exp\left(\int_{t-m\omega}^t F_1(s)ds\right) < \frac{\pi}{2}.$$

But, now by the well-known result in [11] every solution of (2.37) satisfies  $\lim_{t \rightarrow \infty} X(t) = 0$ , and hence  $\lim_{t \rightarrow \infty} [z(t) - \bar{z}(t)] = 0$ . The proof is complete.

### § 3. Conclusion

We conclude with some remarks related to the literature on mathematical model systems with delays in production component of models and dynamical diseases. Many human diseases are characterized by changes in the qualitative behavior of physiological control mechanism. Systems which normally and regularly oscillate can stop oscillating or systems which do not normally oscillate can start oscillating. Such changes in the qualitative dynamics can be sudden and one of the mechanisms used to describe such onset of changes is by means of bifurcations. One of the necessary conditions for dynamical systems to have

potential for complicated behavior, is that the system is oscillatory. A major part of the literature deals with either a linear analysis of systems or computer simulations, we have however carried out a full nonlinear analysis and our conclusions are valid beyond the local approximations based on linearizations. While the existence of periodic solutions in delay differential equations has been established by means of mechanism of a Hopf bifurcation, the stability of the bifurcating periodic solution has been rarely investigated.

Our emphasis in this paper has been on a different aspect dealing with the dynamics of the respiratory dynamics model with periodic coefficients and impulsive effect. In Theorem 2.6, we have derived some sufficient conditions for the prevalence of a dynamic disease, which insure the existence of the change of the concentration of  $\text{CO}_2$  in the blood around the positive periodic solution which is equivalent to the minimum capacity in the autonomous case. In Theorem 2.8, we have established the maximum and the minimum concentration of  $\text{CO}_2$  in the blood. In Theorem 2.9, we have derived sufficient conditions for the nonexistence of dynamic diseases, which are the conditions for the global attractivity of the positive periodic solution for all the solutions.

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