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ON THE PRIMARY DECOMPOSITION THEOREM OF MODULAR LIE SUPERALGEBRAS***

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Abstract

This gives some identities of associative Lie superalgebras and some properties of modular Lie superalgebras. Furthermore, the primary decomposition theorem of modular Lie superalgebras is shown. It is well known that the primary decomposition theorem of modular Lie algebras has played an important role in the classification of the finite-dimensional simple modular Lie algebras (see [5, 6]). Analogously, the primary decomposition theorem of modular Lie superalgebras may play an important role in the open classification of the finite dimensional simple modular Lie superalgebras.

Keywords Modular Lie superalgebras, Fitting decomposition, The primary decomposition theorem
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§ 1 . Introduction

$L = L_{\bar 0} \bigoplus L_{\bar 1}$ is called a superalgebra by physicists if $[a, b] \in L_{\alpha+\beta}$ for any $a \in L_\alpha$, $b \in L_\beta$, $\alpha, \beta \in Z_2 = \{\bar 0, \bar 1\}$. L is called a Lie superalgebra if L is a superalgebra with the operation $[\ ,\ ]$ satisfying the following axioms:

(1) $[a, b] = -(-1)^{\alpha\beta}[b, a]$ (graded skew-symmetry),

(2) $[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]]$ (graded Jacobi identity),

for all $a \in L_\alpha,\ b \in L_\beta,\ c \in L,\ \alpha, \beta \in Z_2 = \{\bar 0, \bar 1\}$.

Lie superalgebras are frequently called Z_2-graded Lie algebras by physicists (see [3, 11]). $L_{\bar 1}$ is an $L_{\bar 0}$-module. $L_{\bar 0}$ is an ordinary Lie algebra. Generally speaking, Lie superalgebras are not Lie algebras. Moreover, many important features of Lie algebras are not necessarily true for Lie superalgebras (see [3]). For instance, Lie's theorem and Levi's theorem of Lie algebras are not true, in general, for Lie superalgebras. In addition, it is well known that a semisimple Lie algebra is a direct sum of simple ones, but this is by no means true for Lie superalgebras.

In the 1950s A. Nijenhuis gave an example of Lie superalgebras. As a natural generalization of Lie algebras, Lie superalgebras become an efficient tools for analyzing the properties

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of physical systems (see [13, 15]), shortly after they are defined in an abstract way. Concrete applications comprise the formulation of supersymmetries of Hamiltonian systems, the description of atomic, molecular and nuclear spectra, particle physics, unified field theory and many others. Lie superalgebras are also interesting from a purely mathmatical point of view, and their enveloping algebras provide various rich classes of associative algebras.

During the last fifty years, the theory of Lie superalgebras has undergone a remarkable evolution in mathmatics. The study of Lie superalgebras mainly contains classifications, structures and representations. At present the most important results in the theory seem to be the classification by V. G. Kac and S. J. Cheng of finite-dimensional and infinite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero (see [2, 3, 9, 10]), the Classification by V. G. Kac and M. Wakimoto of modular invariant representations of affine superalgebras (see [4]) and the classification by V. G. Kac of infinite-dimensional simple linearly compact Lie superalgebras (see [8]).

In last ten years, many important results of modular Lie superalgebras also have been obtained (see [19–23]). Now the complete classification of the finite-dimensional modular simple Lie superalgebras remains an open problem. It is well known that the primary decomposition theorem of modular Lie algebras has played an important role in the classification of the finite-dimensional simple modular Lie algebras (see [5, 6]). In the present paper, we give some identities of associative Lie superalgebras and some properties of modular Lie superalgebras. Furthermore, we obtain the primary decomposition theorem of modular Lie superalgebras, which may play an important role in the open classification of the finite-dimensional simple modular Lie superalgebras.

Throughout this thesis, let all spaces and all algebras be finite-dimensional over a field $\mathbf{F}$ of positive characteristic $p \geq 3$. Our notation and terminology are standard as may be found in [3, 11] .

§ 2 . Main Results

Let $A = A_{\bar{0}} \bigoplus A_{\bar{1}}$ be an associative Lie superalgebra over $\mathbf{F}$, and define a multiplication on A by $[a, b] := ab - (-1)^{\alpha\beta} ba$, where $a \in A_\alpha, b \in A_\beta, \alpha, \beta \in Z_2 = \{\bar{0}, \bar{1}\}$. The product $[a, b]$ is referred to as the commutator of a and b and the Lie superalgebra $(A, [\,,\,])$ is denoted by A^-. We always let $\alpha, \beta, \gamma \in Z_2 = \{\bar{0}, \bar{1}\}$.

Lemma 2.1. *Let $A = A_{\bar{0}} \bigoplus A_{\bar{1}}$ be an associative Lie superalgebra over $\mathbf{F}$. Then the following identities hold in A^-.*

(1) $[xy, z] = (-1)^{\beta\gamma}[x, z]y + x[y, z]$, $\forall\, x \in A_\alpha, \forall\, y \in A_\beta, \forall\, z \in A_\gamma$;

(2) $[x^2, y] = (\mathrm{ad} x^2)(y) = (\mathrm{ad} x)^2(y)$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_\beta$;

(3) $[x^m, y] = \sum\limits_{i=0}^{m-1} (-1)^{\alpha\beta(m+i-1)} x^i [x, y] x^{m-1-i}$, $\forall\, x \in A_\alpha, \forall\, y \in A_\beta$.

Proof. (1) We have $[xy, z] = xyz - (-1)^{\gamma(\alpha+\beta)} zxy$ and

$$\begin{aligned}(-1)^{\beta\gamma}[x, z]y + x[y, z] &= (-1)^{\beta\gamma} xzy - (-1)^{(\beta\gamma+\alpha\gamma)} zxy + xyz - (-1)^{\beta\gamma} xzy \\ &= xyz - (-1)^{\gamma(\alpha+\beta)} zxy.\end{aligned}$$

So $[xy, z] = (-1)^{\beta\gamma}[x, z]y + x[y, z]$.

(2) For $\forall\, x \in A_{\bar{1}}$, $\forall\, y \in A_{\beta}$, by Lemma 2.1(1) we have

$$[x^2, y] = x[x, y] + (-1)^{\beta}[x, y]x = [x, [x, y]] = (\mathrm{ad}x)^2(y).$$

(3) It will be proved by induction on m.

For the case $m = 1$, we have

$$\sum_{i=0}^{1-1}(-1)^{\alpha\beta(1+i-1)}x^i[x, y]x^{1-1-i} = [x, y],$$

i.e., the identity is true for $m = 1$.

Suppose that the identity is true for the case m, i.e.,

$$[x^m, y] = \sum_{i=0}^{m-1}(-1)^{\alpha\beta(m+i-1)}x^i[x, y]x^{m-1-i}.$$

For the case $m + 1$, by Lemma 2.1(1) we obtain

$$\begin{aligned}
[x^{m+1}, y] &= x^m[x, y] + (-1)^{\alpha\beta}[x^m, y]x \\
&= x^m[x, y] + (-1)^{\alpha\beta}\Big(\sum_{i=0}^{m-1}(-1)^{\alpha\beta(m+i-1)}x^i[x, y]x^{m-1-i}\Big)x \\
&= (-1)^{\alpha\beta(m+m)}x^m[x, y] + \sum_{i=0}^{m-1}(-1)^{\alpha\beta(m+i)}x^i[x, y]x^{m-i} \\
&= \sum_{i=0}^{m}(-1)^{\alpha\beta(m+i)}x^i[x, y]x^{m-i}.
\end{aligned}$$

Hence the identity holds for every $m \in N$.

Lemma 2.2. *Let $A = A_{\bar{0}} \bigoplus A_{\bar{1}}$ be an associative Lie superalgebra over $\mathbf{F}$. Then the following identities hold in A^-.*

(1) $yx^k = \sum\limits_{i=0}^{k}(-1)^{k-i}C_k^i x^i(\mathrm{ad}x)^{k-i}(y)$, $\forall\, x \in A_{\bar{0}}$, $y \in A_{\beta}$;

(2) $x^k y = \sum\limits_{i=0}^{k} C_k^i(\mathrm{ad}x)^i(y)x^{k-i}$, $\forall\, x \in A_{\bar{0}}$, $y \in A_{\beta}$;

(3) $yx^{2k} = \sum\limits_{i=0}^{k}(-1)^i C_k^i x^{2k-2i}(\mathrm{ad}x)^{2i}(y)$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_{\beta}$;

(4) $x^{2k}y = \sum\limits_{i=0}^{k} C_k^i(\mathrm{ad}x)^{2k-2i}(y)x^{2i}$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_{\beta}$.

Proof. (1) and (2) can be proved by induction on k.

(3) If $x \in A_{\bar{1}}$, then $x^2 \in A_{\bar{0}}$. By Lemma 2.1(2) and Lemma 2.2(1), we have

$$yx^{2k} = y(x^2)^k = \sum_{i=0}^{k}(-1)^{k-i}C_k^i(x^2)^i(\mathrm{ad}x^2)^{k-i}(y)$$

$$= \sum_{i=0}^{k}(-1)^{k-i}C_k^i x^{2i}(\mathrm{ad}x)^{2k-2i}(y) = \sum_{i=0}^{k}(-1)^{i}C_k^i x^{2k-2i}(\mathrm{ad}x)^{2i}(y).$$

Hence $yx^{2k} = \sum\limits_{i=0}^{k}(-1)^i C_k^i x^{2k-2i}(\mathrm{ad}x)^{2i}(y)$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_\beta$.

(4) If $x \in A_{\bar{1}}$, then $x^2 \in A_{\bar{0}}$. By Lemma 2.1(2) and Lemma 2.2(2), we have

$$x^{2k}y = (x^2)^k y = \sum_{i=0}^{k} C_k^i (\mathrm{ad}x^2)^i(y)(x^2)^{k-i}$$

$$= \sum_{i=0}^{k} C_k^i (\mathrm{ad}x)^{2i}(y)x^{2k-2i} = \sum_{i=0}^{k} C_k^i (\mathrm{ad}x)^{2k-2i}(y)x^{2i}.$$

Hence $x^{2k}y = \sum\limits_{i=0}^{k} C_k^i(\mathrm{ad}x)^{2k-2i}(y)x^{2i}$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_\beta$.

Lemma 2.3. (see [21]) *Let $A = A_{\bar{0}} \bigoplus A_{\bar{1}}$ be an associative Lie superalgebra over $\mathbf{F}$. Then the following identities hold in A^-.*

(1) $(\mathrm{ad}x)^k(y) = \sum\limits_{i=0}^{k}(-1)^{k-i}C_k^i x^i y x^{k-i}$, $\forall\, x \in A_{\bar{0}}, \forall\, y \in A_\beta$;

(2) $(\mathrm{ad}x)^{2k}(y) = \sum\limits_{i=0}^{k}(-1)^{k-i}C_k^i x^{2i} y x^{2k-2i}$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_\beta$;

(3) $(\mathrm{ad}x)^{2k-1}(y) = \sum\limits_{i=0}^{2k-1}(-1)^{t_i}C_{k-1}^{[i/2]} x^i y x^{2k-1-i}$, $\forall\, x \in A_{\bar{1}}, \forall\, y \in A_\beta$,

where $t_i = (1+i)(1+\beta) + (k-1) + i(i-1)/2$, $[i/2]$ denotes the integer part of $i/2$.

Theorem 2.1. *Let $\mathrm{Pl}(V)$ be the general linear Lie superalgebra of a finite-dimensional Z_2-graded vector space V over $\mathbf{F}$. Suppose that there exist positive integers m and n such that $(\mathrm{ad}A)^m(B) = 0$ and $A^n(x) = 0$, where $A \in \mathrm{Pl}(V)_\alpha$, $B \in \mathrm{Pl}(V)_\beta$, $x \in V$. Then $A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = 0$.*

Proof. Case I. $\alpha = \bar{0}$

By Lemma 2.3(1), we have

$$0 = (\mathrm{ad}A)^m(B) = \sum_{i=0}^{m}(-1)^{m-i}C_m^i A^i B A^{m-i} = A^m B + \sum_{i=0}^{m-1}(-1)^{m-i}C_m^i A^i B A^{m-i}.$$

Then $A^m B = -\sum\limits_{i=0}^{m-1}(-1)^{m-i}C_m^i A^i B A^{m-i}$.

Using Lemma 2.3(1), we obtain

$$A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = A\Big(\sum_{i=0}^{m-1}(-1)^{m-1-i}C_{m-1}^i A^i B A^{m-1-i}\Big)A^{n-1}(x)$$

$$= A\Big(A^{m-1}B + \sum_{i=0}^{m-2}(-1)^{m-1-i}C_{m-1}^i A^i B A^{m-1-i}\Big)A^{n-1}(x)$$

$$= (A^m B)A^{n-1}(x) + \sum_{i=0}^{m-2}(-1)^{m-1-i}C_{m-1}^i A^{i+1} B A^{m+n-2-i}(x)$$

$$= \Big(- \sum_{i=0}^{m-1} (-1)^{m-i} C_m^i A^i B A^{m-i} \Big) A^{n-1}(x) + \sum_{i=0}^{m-2} (-1)^{m-1-i} C_{m-1}^i A^{i+1} B A^{m+n-2-i}(x)$$

$$= - \sum_{i=0}^{m-1} (-1)^{m-i} C_m^i A^i B A^{m+n-1-i}(x) + \sum_{i=0}^{m-2} (-1)^{m-1-i} C_{m-1}^i A^{i+1} B A^{m+n-2-i}(x).$$

Since $A^n(x) = 0$, we obtain

$$\sum_{i=0}^{m-1} (-1)^{m-i} C_m^i A^i B A^{m+n-1-i}(x) = 0, \quad \sum_{i=0}^{m-2} (-1)^{m-1-i} C_{m-1}^i A^{i+1} B A^{m+n-2-i}(x) = 0.$$

So $A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = 0$.

Case II. $\alpha = \bar{1}$ and $m = 2k$

By Lemma 2.3(2), we have

$$0 = (\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k}(B) = \sum_{i=0}^{k} (-1)^{k-i} C_k^i A^{2i} B A^{2k-2i}$$

$$= A^{2k}B + \sum_{i=0}^{k-1} (-1)^{k-i} C_k^i A^{2i} B A^{2k-2i}.$$

Then $A^{2k}B = - \sum_{i=0}^{k-1} (-1)^{k-i} C_k^i A^{2i} B A^{2k-2i}$.

Using Lemma 2.3(3), we obtain

$$(\mathrm{ad}A)^{m-1}(B) = (\mathrm{ad}A)^{2k-1}(B) = \sum_{i=0}^{2k-1} (-1)^{t_i} C_{k-1}^{[i/2]} A^i B A^{2k-1-i},$$

where $t_i = (1+i)(1+\beta) + (k-1) + i(i-1)/2$, $[i/2]$ denotes the integer part of $i/2$.

Using Lemma 2.3(2), we obtain

$$A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = A\Big(\sum_{i=0}^{2k-1} (-1)^{t_i} C_{k-1}^{[i/2]} A^i B A^{2k-1-i} \Big) A^{n-1}(x)$$

$$= (A^{2k}B)A^{n-1}(x) + \sum_{i=0}^{2k-2} (-1)^{t_i} C_{k-1}^{[i/2]} A^{i+1} B A^{2k+n-2-i}(x)$$

$$= -\Big(\sum_{i=0}^{k-1} (-1)^{k-i} C_k^i A^{2i} B A^{2k-2i} \Big) A^{n-1}(x) + \sum_{i=0}^{2k-2} (-1)^{t_i} C_{k-1}^{[i/2]} A^{i+1} B A^{2k+n-2-i}(x)$$

$$= - \sum_{i=0}^{k-1} (-1)^{k-i} C_k^i A^{2i} B A^{2k+n-1-2i}(x) + \sum_{i=0}^{2k-2} (-1)^{t_i} C_{k-1}^{[i/2]} A^{i+1} B A^{2k+n-2-i}(x).$$

Since $A^n(x) = 0$, we obtain

$$\sum_{i=0}^{k-1} (-1)^{k-i} C_k^i A^{2i} B A^{2k+n-1-2i}(x) = 0, \quad \sum_{i=0}^{2k-2} (-1)^{t_i} C_{k-1}^{[i/2]} A^{i+1} B A^{2k+n-2-i}(x) = 0.$$

So $A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = 0$.

Case III. $\alpha = \bar{1}$ and $m = 2k-1$

By Lemma 2.3(3), we have

$$0 = (\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k-1}(B) = \sum_{i=0}^{2k-1}(-1)^{t_i}C_{k-1}^{[i/2]}A^iBA^{2k-1-i}$$
$$= A^{2k-1}B + \sum_{i=0}^{2k-2}(-1)^{t_i}C_{k-1}^{[i/2]}A^iBA^{2k-1-i},$$

where $t_i = (1+i)(1+\beta) + (k-1) + i(i-1)/2$, $[i/2]$ denotes the integer part of $i/2$.

Then $A^{2k-1}B = -\sum\limits_{i=0}^{2k-2}(-1)^{t_i}C_{k-1}^{[i/2]}A^iBA^{2k-1-i}$.

Using Lemma 2.3(2), we obtain

$$(\mathrm{ad}A)^{m-1}(B) = (\mathrm{ad}A)^{2k-2}(B) = \sum_{i=0}^{k-1}(-1)^{k-1-i}C_{k-1}^{i}A^{2i}BA^{2k-2-2i}.$$

Then

$$A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = A\Big(\sum_{i=0}^{k-1}(-1)^{k-1-i}C_{k-1}^{i}A^{2i}BA^{2k-2-2i}\Big)A^{n-1}(x)$$
$$= A\Big(A^{2k-2}B + \sum_{i=0}^{k-2}(-1)^{k-1-i}C_{k-1}^{i}A^{2i}BA^{2k-2-2i}\Big)A^{n-1}(x)$$
$$= (A^{2k-1}B)A^{n-1}(x) + \sum_{i=0}^{k-2}(-1)^{k-1-i}C_{k-1}^{i}A^{2i+1}BA^{2k+n-3-2i}(x)$$
$$= \Big(-\sum_{i=0}^{2k-2}(-1)^{t_i}C_{k-1}^{[i/2]}A^iBA^{2k-1-i}\Big)A^{n-1}(x) + \sum_{i=0}^{k-2}(-1)^{k-1-i}C_{k-1}^{i}A^{2i+1}BA^{2k+n-3-2i}(x).$$

Since $A^n(x) = 0$, we have

$$\Big(-\sum_{i=0}^{2k-2}(-1)^{t_i}C_{k-1}^{[i/2]}A^iBA^{2k-1-i}\Big)A^{n-1}(x) = 0,$$
$$\sum_{i=0}^{k-2}(-1)^{k-1-i}C_{k-1}^{i}A^{2i+1}BA^{2k+n-3-2i}(x) = 0.$$

Hence $A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = 0$.

Theorem 2.2. *Let* $\mathrm{Pl}(V)$ *be the general linear Lie superalgebra of a finite-dimensional* Z_2*-graded vector space* V *over* $\mathbf{F}$*. Suppose that there exists a positive integer* m *such that* $(\mathrm{ad}A)^m(B) = 0$*, where* $V_{0A} = \{x \in V | A^i(x) = 0, \exists i \in N\}, V_{1A} = \bigcap\limits_{i=1}^{\infty} A^iV$*,* $A \in \mathrm{Pl}(V)_\alpha, B \in \mathrm{Pl}(V)_\beta$*. Then the Fitting components* V_{0A}, V_{1A} *of* V *relative to* A *are invariant under* B*.*

Proof. We have $V \supseteq A(V) \supseteq A^2(V) \supseteq \cdots$. Then there is r such that $A^r(V) = A^{r+1}(V) = \cdots = V_{1A}$ since V is finite-dimensional. Let $W_i = \{v \in V \mid A^i(v) = 0\}$. Then $W_1 \subseteq W_2 \subseteq \cdots$. So there is s such that $W_{s+1} = W_{s+2} = \cdots = V_{0A}$ since V is finite-dimensional. Let $t = \max(r, s)$. Then $W_t = V_{0A}$ and $V_{1A} = A^t(V)$.

Let $x \in V$. Then $A^t(x) = A^{2t}(y)$ for some y since $A^t(V) = A^{2t}(V)$. Then $x = (x - A^t(y)) + A^t(y)$ and $A^t(y) \in V_{1A}$, while $A^t(x - A^t(y)) = 0$, so $(x - A^t(y)) \in V_{0A}$. Hence $V = V_{0A} + V_{1A}$. Let $z \in V_{0A} \cap V_{1A}$. Then there is $w \in V$ such that $z = A^t(w)$ and $0 = A^t(z) = A^{2t}(w)$. So we have $w \in V_{0A} = W_t$ and $A^t(w) = 0$. Hence $z = 0$ and $V_{0A} \cap V_{1A} = \{0\}$. Thus $V = V_{0A} \dot{+} V_{1A}$.

Case I. Let $x \in V_{0A}$. There exists $n \in N$ such that $A^n(x) = 0$, and let $k = m + n - 1$. We consider three cases.

Case i. If $A, B \in \mathrm{Pl}(V)_{\bar{0}}$, then we have the following identity since $(\mathrm{ad}A)^m(B) = 0$ and $A^n(x) = 0$ by Lemma 2.2(2):

$$\begin{aligned} A^k B(x) &= \sum_{i=0}^{k} C_k^i (\mathrm{ad}A)^i(B) A^{k-i}(x) = \sum_{i=0}^{m+n-1} C_{m+n-1}^i (\mathrm{ad}A)^i(B) A^{m+n-1-i}(x) \\ &= \sum_{i=0}^{m-1} C_{m+n-1}^i (\mathrm{ad}A)^i(B) A^{m+n-1-i}(x) + \sum_{i=m}^{m+n-1} C_{m+n-1}^i (\mathrm{ad}A)^i(B) A^{m+n-1-i}(x) \\ &= 0. \end{aligned}$$

So $B(x) \in V_{0A}$.

Case ii. Let $[i/2]$ denote the integer part of $i/2$. If $A \in \mathrm{Pl}(V)_{\bar{1}}, B \in \mathrm{Pl}(V)_{\beta}$ and $k = 2k_1$, then we have the following identity since $(\mathrm{ad}A)^m(B) = 0$ and $A^n(x) = 0$ by Lemma 2.2(4):

$$\begin{aligned} A^k Bx = A^{2k_1} B(x) &= \sum_{i=0}^{k_1} C_{k_1}^i (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i}(x) \\ &= \sum_{i=0}^{[(n+1)/2]-1} C_{k_1}^i (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i}(x) + \sum_{i=[(n+1)/2]}^{k_1} C_{k_1}^i (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i}(x) \\ &= 0. \end{aligned}$$

So $B(x) \in V_{0A}$.

Case iii. If $A \in \mathrm{Pl}(V)_{\bar{1}}, B \in \mathrm{Pl}(V)_{\beta}$ and $k = 2k_1 + 1 = m + n - 1$, then we have the following identity since $(\mathrm{ad}A)^m(B) = 0$ and $A^n(x) = 0$ by Lemma 2.2(4):

$$\begin{aligned} A^k Bx = A^{2k_1+1} B(x) &= A\Big(\sum_{i=0}^{k_1} C_{k_1}^i (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i}(x) \Big) \\ &= \Big(\sum_{i=0}^{k_1} C_{k_1}^i A (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i} \Big)(x). \end{aligned}$$

Since $2k_1 + 1 = m + n - 1$, both m and n are odd or even.

If m and n are even, then we have

$$\begin{aligned} A^k B(x) &= A^{2k_1+1} B(x) \\ &= \Big(\sum_{i=0}^{n/2-1} C_{k_1}^i A (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i} \Big)(x) + \Big(\sum_{i=n/2}^{k_1} C_{k_1}^i A (\mathrm{ad}A)^{2k_1-2i}(B) A^{2i} \Big)(x) \\ &= 0. \end{aligned}$$

If m and n are odd, then we have

$$\begin{aligned}
A^kB(x) &= A^{2k_1+1}B(x) \\
&= \Big(\sum_{i=0}^{(n+1)/2-2} C_{k_1}^i A(\mathrm{ad}A)^{2k_1-2i}(B)A^{2i}\Big)(x) \\
&\quad + \Big(\sum_{i=(n+1)/2}^{k_1} C_{k_1}^i A(\mathrm{ad}A)^{2k_1-2i}(B)A^{2i}\Big)(x) \\
&\quad + C_{k_1}^{(n+1)/2-1} A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) \\
&= C_{k_1}^{(n+1)/2-1} A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x).
\end{aligned}$$

By virtue of Theorem 2.1, we have $A(\mathrm{ad}A)^{m-1}(B)A^{n-1}(x) = 0$.

Then $A^kB(x) = 0$. So $B(x) \in V_{0A}$.

Case II. Next let $x \in V_{1A}$. If t is the integer used in the above proof, then we can write $x = A^{t+m-1}(y)$. So $B(x) = BA^{t+m-1}(y)$. We consider three cases.

Case i. If $A, B \in \mathrm{Pl}(V)_{\bar{0}}$, then we have the following identity since $(\mathrm{ad}A)^m(B) = 0$ by Lemma 2.2(1):

$$\begin{aligned}
B(x) &= BA^{t+m-1}(y) \\
&= \sum_{i=0}^{t+m-1} (-1)^{t+m-1-i} C_{t+m-1}^i A^i (\mathrm{ad}A)^{t+m-1-i}(B)(y) \\
&= \sum_{i=t}^{t+m-1} (-1)^{t+m-1-i} C_{t+m-1}^i A^i (\mathrm{ad}A)^{t+m-1-i}(B)(y) \in A^t(V) = V_{1A}.
\end{aligned}$$

Case ii. If $A \in \mathrm{Pl}(V)_{\bar{1}}, B \in \mathrm{Pl}(V)_\beta$ and $t+m-1 = 2k_1$, then we have the following identity by Lemma 2.2(3):

$$B(x) = BA^{t+m-1}(y) = BA^{2k_1}(y) = \sum_{i=0}^{k_1} (-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y).$$

If m is an even number, then there exists $k_2 \in N$ such that $m = 2k_2$. Since $(\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k_2}(B) = 0$, we have

$$\begin{aligned}
B(x) &= \sum_{i=0}^{k_1} (-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \\
&= \sum_{i=0}^{k_2-1} (-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) + \sum_{i=k_2}^{k_1} (-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \\
&= \sum_{i=0}^{k_2-1} (-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \in A^t(V) = V_{1A}.
\end{aligned}$$

If m is an odd number, then there exists $k_2 \in N$ such that $m = 2k_2+1$. Since $(\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k_2+1}(B) = 0$, we have

$$\begin{aligned} B(x) &= \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \\ &= \sum_{i=0}^{k_2}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) + \sum_{i=k_2+1}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \\ &= \sum_{i=0}^{k_2}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)(y) \in A^t(V) = V_{1A}. \end{aligned}$$

Hence $B(V_{1A}) \subseteq V_{1A}$.

Case iii. If $A \in \mathrm{Pl}(V)_{\bar{1}}, B \in \mathrm{Pl}(V)_\beta$ and $t+m-1 = 2k_1+1$, then we have the following identity by Lemma 2.2(3):

$$B(x) = BA^{t+m-1}(y) = (BA^{2k_1})A(y) = \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y).$$

If m is an even number, then there exists $k_2 \in N$ such that $m = 2k_2$. Since $(\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k_2}(B) = 0$, we have

$$\begin{aligned} B(x) &= \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) \\ &= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) + \sum_{i=k_2}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) \\ &= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) \in A^t(V) = V_{1A}. \end{aligned}$$

If m is an odd number, then there exists $k_2 \in N$ such that $m = 2k_2 + 1$.
Since $(\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{2k_2+1}(B) = 0$, we have

$$\begin{aligned} B(x) &= \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) \\ &= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) + (-1)^{k_2} C_{k_1}^{k_2} A^{2k_1-2k_2}(\mathrm{ad}A)^{2k_2}(B)A(y) \\ &\quad + \sum_{i=k_2+1}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) \\ &= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) + (-1)^{k_2} C_{k_1}^{k_2} A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y) \\ &= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{t+(2k_2-1)-2i}(\mathrm{ad}A)^{2i}(B)A(y) + (-1)^{k_2} C_{k_1}^{k_2} A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y). \end{aligned}$$

By means of Lemma 2.3(3) and $(\mathrm{ad}A)^m(B)=(\mathrm{ad}A)^{2k_2+1}(B)=0$, we have

$$\begin{aligned}0&=(\mathrm{ad}A)^m(B)=(\mathrm{ad}A)^{2k_2+1}(B)=\sum_{i=0}^{2k_2+1}(-1)^{t_i}C_{k_2}^{[i/2]}A^iBA^{2k_2+1-i}\\&=(-1)^{1+k_2+\beta}BA^{2k_2+1}+\sum_{i=1}^{2k_2+1}(-1)^{t_i}C_{k_2}^{[i/2]}A^iBA^{2k_2+1-i},\end{aligned}$$

where $t_i=(1+i)(1+\beta)+k_2+i(i-1)/2$, $[i/2]$ denotes the integer part of $i/2$, $B\in \mathrm{Pl}(V)_\beta$. Then

$$\begin{aligned}BA^{2k_2+1}&=\sum_{i=1}^{2k_2+1}(-1)^{t_i+k_2+\beta}C_{k_2}^{[i/2]}A^iBA^{2k_2+1-i},\\A^{t-1}BA^{2k_2+1}(y)&=A^{t-1}(BA^{2k_2+1})(y)\\&=A^{t-1}\Big(\sum_{i=1}^{2k_2+1}(-1)^{t_i+k_2+\beta}C_{k_2}^{[i/2]}A^iBA^{2k_2+1-i}\Big)(y)\\&=\sum_{i=1}^{2k_2+1}(-1)^{t_i+k_2+\beta}C_{k_2}^{[i/2]}A^{i+t-1}BA^{2k_2+1-i}(y)\\&=\sum_{i=1}^{2k_2+1}(-1)^{t_i+k_2+\beta}C_{k_2}^{[i/2]}A^{i+t-1}(BA^{2k_2+1-i}(y))\in A^t(V)=V_{1A}.\end{aligned}$$

By Lemma 2.3(2), we obtain

$$\begin{aligned}&(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y)=(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}((\mathrm{ad}A)^{2k_2}(B))A(y)\\&=(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}\Big(\sum_{i=0}^{k_2}(-1)^{k_2-i}C_{k_2}^iA^{2i}BA^{2k_2-2i}\Big)A(y)\\&=(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}BA^{2k_2}A(y)+C_{k_1}^{k_2}A^{t-1}\Big(\sum_{i=1}^{k_2}(-1)^{2k_2-i}C_{k_2}^iA^{2i}BA^{2k_2-2i}A(y)\Big)\\&=(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}BA^{2k_2+1}(y)+\sum_{i=1}^{k_2}(-1)^{2k_2-i}C_{k_1}^{k_2}C_{k_2}^iA^{2i+t-1}BA^{2k_2-2i}A(y)\\&=(-1)^{k_2}C_{k_1}^{k_2}A^{t-1}\Big(\sum_{i=1}^{2k_2+1}(-1)^{t_i+k_2+\beta}C_{k_2}^{[i/2]}A^iBA^{2k_2+1-i}\Big)(y)\\&\quad+\sum_{i=1}^{k_2}(-1)^{2k_2-i}C_{k_1}^{k_2}C_{k_2}^iA^{2i+t-1}BA^{2k_2-2i}A(y)\\&=\sum_{i=1}^{2k_2+1}(-1)^{t_i+2k_2+\beta}C_{k_1}^{k_2}C_{k_2}^{[i/2]}A^{t+i-1}BA^{2k_2+1-i}(y)\\&\quad+\sum_{i=1}^{k_2}(-1)^{2k_2-i}C_{k_1}^{k_2}C_{k_2}^iA^{2i+t-1}BA^{2k_2-2i}A(y)\in A^t(V)=V_{1A}.\end{aligned}$$

Hence

$$\begin{aligned}
B(x) &= \sum_{i=0}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y)\\
&= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) + (-1)^{k_2}C_{k_1}^{k_2}A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y)\\
&\quad + \sum_{i=k_2+1}^{k_1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y)\\
&= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{2k_1-2i}(\mathrm{ad}A)^{2i}(B)A(y) + (-1)^{k_2}C_{k_1}^{k_2}A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y)\\
&= \sum_{i=0}^{k_2-1}(-1)^i C_{k_1}^i A^{t+(2k_2-1)-2i}(\mathrm{ad}A)^{2i}(B)A(y)\\
&\quad + (-1)^{k_2}C_{k_1}^{k_2}A^{t-1}(\mathrm{ad}A)^{m-1}(B)A(y) \in A^t(V) = V_{1A}.
\end{aligned}$$

Then $B(V_{1A}) \subseteq V_{1A}$. Thus the Fitting components V_{0A}, V_{1A} of V relative to A are invariant under B.

Theorem 2.3. *Let* $\mathrm{Pl}(V)$ *be the general linear Lie superalgebra of a finite-dimensional* Z_2*-graded vector space* V *over* $\mathbf{F}$*. Suppose that there exists a positive integer* m *such that* $(\mathrm{ad}A)^m(B) = 0$*, where* $A \in \mathrm{Pl}(V)_\alpha, B \in \mathrm{Pl}(V)_\beta$*. Assume that* $P \in F[X]$ *is a polynomial. Then* $V(P(A)) = \{x \in V \mid \exists k \in N, (P(A))^k(x) = 0\}$ *is invariant under* B*.*

Proof. Note that $\{x \in V \mid \exists k \in N, (P(A))^k(x) = 0\} = \{x \in V \mid \exists n \in N, (P(A))^{2np}(x) = 0\}$. It suffices to verify that $V(P(A)) = \{x \in V \mid \exists n \in N, (P(A))^{2np}(x) = 0\}$ is invariant under B.

We shall use induction on m. For the case $m = 1$, i.e., $[A, B] = 0$. We have $AB = (-1)^{\alpha\beta}BA$. We readily conclude that $A^{2i}B = BA^{2i}$ for every $i \in N$.

Let $P_0(A) = \sum_{i=0}^{k_0} a_i A^{2i}, P_1(A) = \sum_{i=0}^{k_1} b_i A^{2i+1}, a_i, b_i \in \mathbf{F}, k_0, k_1 \in N_0$. Then there exist $P_0(A)$ and $P_1(A)$ such that $P(A) = P_0(A) + P_1(A)$ for any $P(A)$. Since $\mathbf{F}$ is characteristic p, we have $(P(A))^{2np} = [P_0(A) + P_1(A)]^{2np} = (P_0(A))^{2np} + (P_1(A))^{2np}$.

By virtue of $A^{2i}B = BA^{2i}$ for every $i \in N$, we have

$$\begin{aligned}
&P(A)^{2np}B = (P_0(A))^{2np}B + (P_1(A))^{2np}B = \Big(\sum_{i=0}^{k_0} a_i A^{2i}\Big)^{2np}B + \Big(\sum_{i=0}^{k_1} b_i A^{2i+1}\Big)^{2np}B\\
&= \sum_{i=0}^{k_0} a_i^{2np}A^{4npi}B + \sum_{i=0}^{k_1} b_i^{2np}A^{4npi+2np}B = B\Big(\sum_{i=0}^{k_0} a_i^{2np}A^{4npi}\Big) + B\Big(\sum_{i=0}^{k_1} b_i^{2np}A^{4npi+2np}\Big)\\
&= B(P_0(A))^{2np} + B(P_1(A))^{2np} = B(P(A))^{2np}.
\end{aligned}$$

If x is an element of $V(P(A))$, then for a suitable n we obtain

$$(P(A))^{2np}B(x) = B(P(A))^{2np}(x) = 0,$$

which qualifies $B(x)$ as an element of $V(P(A))$.

For the case $m > 1$, we have

$$0 = (\mathrm{ad}A)^m(B) = (\mathrm{ad}A)^{m-1}(\mathrm{ad}A(B)) = (\mathrm{ad}A)^{m-1}([A, B]).$$

The inductive hypothesis yields $[A, B](V(P(A))) \subseteq V(P(A))$.

If $v \in V(P(A))$, then $(P(A))^k A(v) = A(P(A))^k(v) = 0$.

So $V(P(A))$ is A-invariant. Using Lemma 2.1(3), we see that the following statement holds for any $t \in N$:

$$[A^t, B](V(P(A))) \subseteq \sum_{i=0}^{t-1}(-1)^{\alpha\beta(t+i-1)} A^i[A, B]A^{t-1-i}(V(P(A))) \subseteq V(P(A)).$$

Let $x \in V(P(A))$ and $(P(A))^{2np}(x) = 0$. By $[A^t, B](V(P(A))) \subseteq V(P(A))$ for any $t \in N$ and Lemma 2.1(2), we have

$$(P(A))^{2np}(B(x)) = [(P(A))^{2np}, B](x) + B((P(A))^{2np}(x)) = [(P(A))^{2np}, B](x) \in V(P(A)).$$

Consequently, we can choose some $s \in N$ such that

$$0 = (P(A))^{2ps}(P(A))^{2np}(B(x)) = (P(A))^{2np+2ps}(B(x)).$$

Thus $B(x) \in V(P(A))$ and $V(P(A)) = \{x \in V \mid \exists k \in N, (P(A))^k(x) = 0\}$ is invariant under B.

Let H be a Lie superalgebra and V be a finite-dimensional Z_2-graded vector space over $\mathbf{F}$, and let $\rho : H \to \mathrm{Pl}(V)$ be a graded representation of H and let $\pi : H \to F[X], h \longmapsto \pi_h$ be a mapping. Then we define $V_\pi := \{x \in V \mid \forall h \in H, \exists n(h, x) \in N : (\pi_h(\rho(h)))^{n(h,x)}(x) = 0\}$.

Lemma 2.4. *Let f be an endomorphism of a finite-dimensional Z_2-graded vector space V over $\mathbf{F}$ and let X be a polynomial such that $X(f) = 0$. Then the following statements hold:*

(1) *If $X = q_1 \cdot q_2$ and q_1, q_2 are relatively prime, then V decomposes into a direct sum of f-invariant subspace $V = U \bigoplus W$ such that $q_1(f)(U) = q_2(f)(W) = \{0\}$.*

(2) *V decomposes into a direct sum of f-invariant subspaces $V = V_0 \bigoplus V_1$, for which $f|_{V_0}$ is nilpotent and $f|_{V_1}$ is invertible.*

Proof. It is similar to Lemma 3.8 (see [18, p.22]).

Remark 2.1. (1) Note that in the case where V is finite-dimensional we may choose X to be the characteristic polynomial of f. The decomposition of (2) of Lemma 2.4 is called the Fitting decomposition with respect to f. V_0 and V_1 are referred to as the Fitting-0 and Fitting-1 components of V, respectively.

(2) Let L be a finite-dimensional Lie superalgebra over $\mathbf{F}$. Let $A = \bigcap_{i=1}^{\infty} \mathrm{ad}a^i(L)$ and $B = \bigcup_{i=1}^{\infty} B_i$ where $B_i = \{x \in L \,|\, \mathrm{ad}a^i(x) = 0\}$ for all $a \in L$ and $i \in N$. Then $A = [a, A]$ and $(\mathrm{ad}a)^n(B) = \{0\}$ for some $n \in N$ because of the finite dimensionality of L. Thus $L = A \bigoplus B$ by Lemma 2.4.

Theorem 2.4. (Primary Decomposition Theorem) *Let H be a nilpotent Lie superalgebra and V be a finite-dimensional Z_2-graded vector space over $\mathbf{F}$. Let $\rho : H \to \mathrm{Pl}(V)$ be a finite-dimensional graded representation. Then there exists a finite set $S \subset \mathrm{Map}(H, F[X])$ such that*

(1) *π_h is irreducible for all $\pi \in S, h \in H$;*

(2) *V_π is an H-submodule for all $\pi \in S$;*

(3) *$V = \bigoplus_{\pi \in S} V_\pi$.*

Proof. We shall use induction on $\dim_{\mathbf{F}} V$. If V is one-dimensional, then $V = \mathbf{F}v$. Let $\rho : H \to \mathrm{Pl}(V), \rho(h)v = \alpha(h)v, \alpha(h) \in \mathbf{F}$. For all $w = \beta v \in \mathbf{F}V, \beta \in \mathbf{F}$, we have $\rho(h)w = \rho(h)(\beta v) = \beta\rho(h)v = \beta\alpha(h)v = \alpha(h)w$. So $\rho(h) = \alpha(h)\mathrm{id}_V$. Define $\pi_h := X - \alpha(h)$ for all $h \in H$. Then $V = V_\pi$ holds.

Suppose that $\dim_{\mathbf{F}} V \geq 2$. For every $h \in H$, let X_h denote the characteristic polynomial of $\rho(h)$. If X_h is a prime power for all $h \in H$, for example, $X_h = (\pi_h)^n$, π_h is prime, then we obtain $V = V_\pi$. Otherwise, there is an element $h_0 \in H$ such that $X_{h_0} = q_1 \cdot q_2$, where q_1 and q_2 are nonconstant and have the greatest common divisor 1. Lemma 2.4 yields a decomposition $V = U \bigoplus W$ such that $q_1(\rho(h))(U) = q_2(\rho(h))(W) = \{0\}$. Note that $U, W \neq \{0\}$. According to Theorem 2.3, U and W are H-submodules of V and the induction hypothesis gives rise to two finite sets $S_1, S_2 \in \mathrm{Map}(H, P[X])$ such that

$$U = \bigoplus_{\pi \in S_1} U_\pi, \qquad W = \bigoplus_{\pi \in S_2} W_\pi.$$

For $\pi \in S_1 \bigcup S_2$, we have $V_\pi = U_\pi \bigoplus W_\pi$. Then $V = \bigoplus_{\pi \in S_1 \bigcup S_2} V_\pi$ is the desired decomposition of V.

Remark 2.2. The primary decomposition is particularly important in case the base field $\mathbf{F}$ is algebraically closed. Then every function π fulfilling the condition (1) of Theorem 2.4 is of the form $\pi_h = X - \gamma(h), \gamma(h) \in \mathbf{F}$. It is customary to write the corresponding space $V_\gamma := \{x \in V \mid \exists n(h,x) \in N : (\rho(h) - \gamma(h)\mathrm{id}_V)^{n(h,x)} = 0\}$. The mapping $\gamma : H \to \mathbf{F}$ is called a weight and V_γ the weight space if $V_\gamma \neq 0$. With this notion, we have the following corollary.

Corollary 2.1. *Let $\rho : H \to \mathrm{Pl}(V)$ be a finite-dimensional graded representation of nilpotent Lie superalgebra H and let $\mathbf{F}$ be algebraically closed. Then V is the direct sum of its weight spaces $V = \bigoplus_{\gamma} V_\gamma$.*

Theorem 2.5. *Let T be a nilpotent Lie superalgebra over $\mathbf{F}$. Then any T-invariant subspace $W \subseteq L$ decomposes $W = C_W(T) + [T, W]$.*

Proof. The adjoint graded representation gives W the structure of a T-module. According to Theorem 2.4, we may write $W = \bigoplus_{\pi \in S} W_\pi$. Let π_0 be the function with $\pi_{0h} = x, \forall h \in T$. Then $W_{\pi_0} \subseteq C_W(T)$ and $[T, W_\pi] = W_\pi, \forall \pi \neq \pi_0$. Hence $W = C_W(T) + [T, W]$.

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