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THE DISTANCE BETWEEN DIFFERENT COMPONENTS OF THE UNIVERSAL TEICHMULLER SPACE**

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Abstract

The model of the universal Teichmuller space by the derivatives of logarithm is the union of infinite disconnected components. In this paper, it is proved that the distance between different components is 0, and the distance from the center of a component to every other component is 2.

Keywords Universal Teichmuller space, Logarithmic derivative, Quasiconformal extension
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§ 1 . Introduction

Let B_i denote the Banach spaces of functions ϕ which are analytic in the unit disk Δ with the norms

$$\|\phi\|_i = \sup_{z\in\Delta}\{(1-|z|^2)^i|\phi(z)|\} < \infty \qquad (i=1,2).$$

For f holomorphic in Δ, let

$$[f] = \frac{f''}{f'} \ ,$$

which is called the derivative of logarithm of f, and

$$S_f = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2 \ ,$$

which is called the Schwarzian derivative of f.

Let $T = \{S_f \mid f$ is conformal in the unit disk Δ with quasiconformal extension to the Riemann sphere $C\}$. It is well known that T is the Bers universal Teichmuller space.

Let $T_1 = \{[f] \mid S_f \in T, f(0)=0, f'(0)=1\}$. T_1 is an alternative model of the universal Teichmuller space introduced in [1, 2, 3]. Astala and Gehring [6] gave a complete description of the closure of T_1, and Zhuravlev obtained an interesting result that T_1 is disconnected in the topology induced by the norm $\|\cdot\|_1$. He proved that $T_1 = L \cup \big(\bigcup_{\theta\in[0,2\pi)} L_\theta\big)$, where L and

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L_θ are connected components of T_1 with f bounded in Δ and $\lim\limits_{z\to e^{i\theta}} f(z)=\infty$ respectively, and $L\cap(L_\theta)=\phi$ and $L_{\theta_1}\cap L_{\theta_2}=\phi$ when $\theta_1\neq\theta_2$. Let

$$H_\theta=\frac{z}{1-e^{-i\theta}z}.$$

Then $[H_\theta]\in L_\theta$. $[H_\theta]$ plays an important role in the description of T_1 (see [1]), and is used as the center of the component L_θ. Let $\mathrm{id}=z,[\mathrm{id}]\in L$ is used as the center of L. Chen and Wei [2] and Wei [3] obtained some interesting results on the components L and L_θ, such as the distance between the centers of different components is 4, the distance from $[H_\theta]$ to L is 2 and the distance between different components is smaller than 1.

In this paper, we prove 4 theorems in Section 2 and Section 3. Theorems 2.1 and 2.2 imply that the distance between different components of T_1 is 0. And Theorems 3.1 and 3.2 imply that the distance from the center of one component to every other component is 2.

§ 2 . The Distance Between Components

First we prove

Theorem 2.1. $\mathrm{dist}(L_\theta,L)=0$.

In order to prove Theorem 2.1, we need the following lemmas.

Lemma 2.1. (see [4]) *f is conformal in the unit disk Δ with $f(0)=0$, $f'(0)=1$. If f satisfies* $\mathrm{Re}f'>0$. *Then* $[f]\in\mathrm{cl}(T_1)$.

Lemma 2.2. (see [5]) *f is a univalent holomorphic function with $f(0)=0$, $f'(0)=1$. Let $\|[f]\|_1=2a$.*

If $a<1$, then $f\in H^\infty$.

If $a>1$, then $f\in H^p$ for any $0<p<\frac{1}{a-1}$.

If $a=1$, then $f\in BMOA$.

Proof of Theorem 2.1. Without loss of generality, we may assume $\theta=0$. Let

$$f_\mu=\frac{1}{\mu}\Big(\frac{1+z}{1-z}\Big)^{\frac{\mu}{2}}-\frac{1}{\mu}\qquad(0<\mu\leq 2).\tag{2.1}$$

Then $f_\mu(0)=0,\ f'_\mu(0)=1$ and

$$[f_\mu]=\frac{\mu+2z}{1-z^2}.\tag{2.2}$$

It is easy to see that f_μ is analytic in Δ and $\lim\limits_{z\to1,z\in\Delta} f_\mu(z)=\infty$.

From

$$S_{f_\mu}=\Big(\frac{f''_\mu}{f'_\mu}\Big)'-\frac{1}{2}\Big(\frac{f''_\mu}{f'_\mu}\Big)^2=\Big(\frac{\mu+2z}{1-z^2}\Big)'-\frac{1}{2}\Big(\frac{\mu+2z}{1-z^2}\Big)^2=\frac{2-\frac{1}{2}\mu^2}{(1-z^2)^2},\tag{2.3}$$

we have

$$\|S_{f_\mu}\|_2=\sup_{z\in\Delta}\{(1-|z|^2)^2|S_{f_\mu}|\}=2-\frac{1}{2}\mu^2<2.$$

Hence, f_μ is conformal in the unit disk Δ with quasiconformal extension to the Riemann sphere C. So we obtain $[f_\mu] \in L_0$.

Let

$$f'_\nu = \frac{1}{(1-z^2)^\nu} \qquad (0 < \nu < 1). \tag{2.4}$$

Then

$$f''_\nu = \frac{2\nu z}{(1-z^2)^{1+\nu}}, \qquad [f_\nu] = \frac{f''_\nu}{f'_\nu} = \frac{2\nu z}{1-z^2}. \tag{2.5}$$

It is easy to know that

$$\operatorname{Re} f'_\nu = \operatorname{Re}\frac{1}{(1-z^2)^\nu} = \operatorname{Re}\frac{(1-\bar{z}^2)^\nu}{|1-z^2|^{2\nu}}.$$

Let $\bar{z}^2 = r(\cos\alpha + i\sin\alpha)$ $(r > 0)$. Then

$$\begin{aligned}
&\operatorname{Re}(1-\bar{z}^2)^\nu = \operatorname{Re}(1 - r(\cos\alpha + i\sin\alpha))^\nu \\
&= \operatorname{Re}\Big(\binom{\nu}{0} + \binom{\nu}{1}(-r)(\cos\alpha + i\sin\alpha) + \cdots + \binom{\nu}{n}(-r)^n(\cos n\alpha + i\sin n\alpha) + \cdots\Big) \\
&= \binom{\nu}{0} + \binom{\nu}{1}(-r)\cos\alpha + \cdots + \binom{\nu}{n}(-r)^n\cos n\alpha + \cdots .
\end{aligned}$$

From $0 < \nu < 1$ we have $\binom{\nu}{n}(-r)^n < 0$. Then

$$\begin{aligned}
&\binom{\nu}{0} + \binom{\nu}{1}(-r)\cos\alpha + \cdots + \binom{\nu}{n}(-r)^n\cos n\alpha + \cdots \\
&> \binom{\nu}{0} + \binom{\nu}{1}(-r) + \cdots + \binom{\nu}{n}(-r)^n + \cdots \\
&= (1-r)^\nu > 0.
\end{aligned}$$

So $\operatorname{Re}(1-\bar{z}^2)^\nu > 0$, and hence $\operatorname{Re} f'_\nu > 0$.

Applying Lemma 2.1, we see that $[f_\nu] \in \operatorname{cl}(T_1)$,

$$\|[f_\nu]\|_1 = \sup_{z\in\Delta}\Big\{\Big|\frac{2\nu z}{1-z^2}\Big|(1-|z|^2)\Big\} = 2\nu < 2.$$

Lemma 2.2 implies that $f \in H^\infty$, then $[f_\nu] \in \operatorname{cl}(L)$. Hence

$$\operatorname{dist}(L, L_\theta) = \operatorname{dist}(L, L_0) \le \lim_{\nu\to 1, \mu\to 0}\operatorname{dist}([f_\nu], [f_\mu]) = \lim_{\nu\to 1, \mu\to 0}(\mu + 2 - 2\nu) = 0,$$

which completes the proof of Theorem 2.1.

Next we prove

Theorem 2.2. $\operatorname{dist}(L_{\theta_1}, L_{\theta_2}) = 0$ *for* $\theta_1 \ne \theta_2$.

In order to prove Theorem 2.2, we need the following lemmas.

Lemma 2.3. (see [2])

$$\sup_{z\in\Delta}\Big\{(1-|z|^2)\Big|\frac{1-e^{-i\theta}}{(1-e^{-i\theta}z)(1-z)}\Big|\Big\}=2.$$

Denote

$$[f_\lambda]=\frac{f_\lambda''}{f_\lambda'}=\frac{\lambda(1-e^{-i\theta})+(1+e^{-i\theta})-2e^{-i\theta}z}{(1-e^{-i\theta}z)(1-z)}\qquad(-1\le\lambda<0,\ 0<\lambda\le 1).\tag{2.6}$$

It is easy to know that

$$[f_\lambda]=\frac{2}{1-z}=[H_0]\qquad\text{when}\ \ \lambda=1,$$
$$[f_\lambda]=\frac{2e^{-i\theta}}{1-e^{-i\theta}z}=[H_\theta]\qquad\text{when}\ \ \lambda=-1.$$

Lemma 2.4. $f_\lambda(z)$ *is analytic in* Δ.

Proof. If $\lambda>\frac{1}{2}$, then

$$\|[f_\lambda]-[H_0]\|_1=\sup_{z\in\Delta}\Big\{(1-|z|^2)\Big|\frac{(1-\lambda)(1-e^{-i\theta})}{(1-e^{-i\theta}z)(1-z)}\Big|\Big\}=2(1-\lambda)<1.\tag{2.7}$$

Hence $[f_\lambda]\in L_0$ and f_λ is analytic in Δ.

If $\lambda<-\frac{1}{2}$, then $\|[f_\lambda]-[H_\theta]\|_1<1$. Hence $[f_\lambda]\in L_\theta$ and f_λ is analytic in Δ.

Let $g_\lambda'(z)=(1-z)^2f_\lambda'(z)$. Then

$$[g_\lambda]=\frac{(\lambda-1)(1-e^{-i\theta})}{(1-e^{-i\theta}z)(1-z)}.$$

Let $g_\lambda(z)=\sum_{n=1}^{\infty}b_nz^n,\ b_1=1$. By comparing coefficient of z^{n-2}, we obtain

$$n(n-1)b_n=(\lambda-1)[b_1(1-e^{-(n-1)i\theta})+2b_2(1-e^{-(n-2)i\theta})+\cdots(n-1)b_{n-1}(1-e^{-i\theta})].$$

If $|\lambda|\le\frac{1}{2}$, then

$$n(n-1)|b_n|\le\frac{3}{2}2(|b_1|+2|b_2|+\cdots+(n-1)|b_{n-1}|).$$

It follows from $b_1=1$ that $|b_n|\le n$. So g_λ is analytic in Δ and hence f_λ is analytic in Δ. This completes the proof of Lemma 2.4.

Proof of Theorem 2.2. Without loss of generality, we may assume $\theta_1=0,\ \theta_2=\theta\ (0<\theta<2\pi)$.

$$S_{f_\lambda}=\Big(\frac{f_\lambda''}{f_\lambda'}\Big)'-\frac{1}{2}\Big(\frac{f_\lambda''}{f_\lambda'}\Big)^2=\frac{1}{2}\frac{(1-\lambda^2)(1-e^{-i\theta}z)^2}{(1-e^{-i\theta}z)^2(1-z)^2}.\tag{2.8}$$

Then

$$\|S_{f_\lambda}\|_2=\sup_{z\in\Delta}\{(1-|z|^2)^2|S_{f_\lambda}|\}=\frac{1}{2}(1-\lambda^2)\cdot 2^2<2\qquad(-1\le\lambda<0,0<\lambda\le 1).$$

Hence, f_λ is conformal in the unit disk Δ with quasiconformal extension to the Riemann sphere C. And we obtain that if $0 < \lambda \le 1$ then $[f_\lambda] \in L_0$ and if $-1 \le \lambda < 0$ then $[f_\lambda] \in L_\theta$. From

$$\lim_{\lambda\to 0} \|[f_\lambda] - [f_{-\lambda}]\|_1 = \lim_{\lambda\to 0} 4\lambda = 0,$$

we have $\mathrm{dist}(L_0, L_\theta) = 0$.

This completes the proof of Theorem 2.2.

§ 3. The Distance Between a Component and the Center of Every Other Component

Theorem 3.1. $\mathrm{dist}([\mathrm{id}], L_\theta) = 2.$

Proof. Without loss of generality, we may assume $\theta = 0$. From the proof of Theorem 2.1 we have $[f_\mu] \in L_0$ and

$$\|[f_\mu] - [\mathrm{id}]\|_1 = \sup_{z\in\Delta}\Big\{(1-|z|^2)\Big|\frac{\mu+2z}{1-z^2}\Big|\Big\} = 2+\mu. \tag{3.1}$$

Then

$$\lim_{\mu\to 0}\|[f_\mu] - [\mathrm{id}]\|_1 = 2.$$

Hence $\mathrm{dist}([\mathrm{id}], L_0) \le 2$.

Wei [3] obtained $\mathrm{dist}([\mathrm{id}], L_0) \ge 2$. Then we get $\mathrm{dist}([\mathrm{id}], L_0) = 2$.

This completes the proof of Theorem 3.1.

Theorem 3.2. $\mathrm{dist}([H_{\theta_1}], L_{\theta_2}) = 2$ *for* $\theta_1 \ne \theta_2$.

Proof. Without loss of generality, we may assume $\theta_1 = 0,\ \theta_2 = \theta\ (0 < \theta < 2\pi)$. First, we prove that $\mathrm{dist}([H_0], L_\theta) \le 2\ (\theta \ne 0)$.

From the proof of Theorem 2.2, we have $[f_\lambda] \in L_\theta,\ -1 \le \lambda < 0$.

$$\begin{aligned}\mathrm{dist}([H_0], L_\theta) &\le \lim_{\lambda\to 0}\|[f_\lambda] - [H_0]\|_1\\ &= \lim_{\lambda\to 0}\Big(\sup_{z\in\Delta}\Big\{(1-|z|^2)\Big|\frac{(\lambda-1)(1-e^{-i\theta})}{(1-e^{-i\theta}z)(1-z)}\Big|\Big\}\Big)\\ &= \lim_{\lambda\to 0} 2|\lambda-1| = 2.\end{aligned}$$

Hence $\mathrm{dist}([H_0], L_\theta) \le 2$.

Next, we proceed to prove that $\mathrm{dist}([H_0], L_\theta) \ge 2\ (\theta \ne 0)$.

Let $\mathrm{dist}([H_0], L_\theta) = k'$. If $k' < 2$, then we can find $[f] \in L_\theta$ such that

$$\mathrm{dist}([f], [H_0]) = \|[f] - [H_0]\|_1 \le k = \frac{2+k'}{2} < 2.$$

Let $g'(z) = (1-z)^2 f'(z)$. Then $\|[g]\|_1 = \|[f] - [H_0]\|_1 \le k$. Hence

$$\Big|\frac{g''(z)}{g'(z)}\Big| \le \frac{k}{1-|z|^2}. \tag{3.2}$$

By integration, we have

$$|g'(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{\frac{k}{2}}, \qquad |f'(z)(1-z)^2| \le \left(\frac{1+|z|}{1-|z|}\right)^{\frac{k}{2}}.$$

Then

$$|f'(te^{i\theta})| \le \left(\frac{1+t}{1-t}\right)^{\frac{k}{2}} \cdot \frac{1}{|1-te^{i\theta}|^2}, \tag{3.3}$$

and

$$\begin{aligned} |f(e^{i\theta})| &= \Big|\int_0^{e^{i\theta}} f'(z)dz\Big| = \Big|\int_0^1 f'(te^{i\theta})dt\Big| \le \int_0^1 |f'(te^{i\theta})|dt \\ &\le \frac{1}{|1-te^{i\theta}|^2}\int_0^1 \left(\frac{1+t}{1-t}\right)^{\frac{k}{2}} dt < +\infty. \end{aligned}$$

This contradicts $[f] \in L_\theta$.

Hence $\mathrm{dist}(L_\theta, [H_0]) = k' \ge 2$. Then we obtain $\mathrm{dist}([H_0], L_\theta) = 2$.

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