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VARIABILITY SETS AND HAMILTON SEQUENCES***

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Abstract

This paper studies extremal quasiconformal mappings. Some properties of the variability set are obtained and the Hamilton sequences which are induced by point shift differentials are also discussed.

Keywords Quasiconformal mapping, Point shift differentials, Hamilton sequence, Extremal mapping
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§ 0 . Introduction

In this paper, the following notations will be used. $\mathcal{C}$={the finite complex plane}; $\triangle$={the unit disk in $\mathcal{C}$}; $\Gamma=\partial\Delta$.

The vector space of all holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ in Δ with L^1 norm

$$\|\varphi\| = \iint_\Delta |\varphi(z)|dxdy < \infty$$

will be denoted by $Q(\Delta)$ and the unit sphere in $Q(\Delta)$ will be denoted by $Q_0(\Delta)$.

Let $h : \Gamma \longrightarrow \Gamma$ be a sense-preserving homeomorphism. We call h quasisymmetric if there is a quasiconformal mapping $f : \Delta \longrightarrow \Delta$ such that $f|_\Gamma = h$, i.e, h has a quasiconformal extension f to Δ. For a quasisymmetric function h, set

$$[h] = \{f; f : \Delta \longrightarrow \Delta \text{ is a quasiconformal mapping with } f|_\Gamma = h\}.$$

In the following the maximal dilatation of a quasiconformal mapping f is denoted by $K(f)$ and the complex dilatation is denoted by μ_f. Now suppose that h is a quasisymmetric homeomorphism, set

$$K(h) = \inf\{K(f); f \in [h]\}.$$

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Let $f_0 \in [h]$. f_0 is said to be extremal if $K(f_0) = K(h)$. Set

$$[h]_e = \{f \in [h]; K(f) = K(h)\}.$$

It follows from the normal family argument that for every quasisymmetric function there always exists an extremal quasiconformal mapping f in $[h]$. A well-known criterion for $f \in [h]_e$ is the following theorem due to Hamilton-Krushkal-Reich-Strebel.

Theorem 0.1. (see [1]) *Let h be a quasisymmetric function. Then $f \in [h]_e$ if and only if*

$$\sup_{\varphi \in Q_0(\Delta)} \mathrm{Re} \iint_\Delta \mu_f(z)\varphi(z)dxdy = \|\mu_f\|_\infty.$$

A sequence $\{\varphi_n\}_1^\infty \subset Q_0(\Delta)$ attaining the above supremum is called a Hamilton sequence for f or for μ_f, that is, it satisfies

$$\lim_{n\to\infty} \mathrm{Re} \iint_\Delta \mu_f(z)\varphi_n(z)\,dxdy = \|\mu_f\|_\infty.$$

The Hamilton sequence $\{\varphi_n\}_1^\infty$ is called degenerating if $\lim\limits_{n\to\infty} \varphi_n(z) = 0$ locally uniformly.

§ 1. Some Properties of the Variability Set

Let $h : \Gamma \to \Gamma$ be a quasisymmetric homeomorphism and let $z_0 \in \Delta$. The variability set of z_0 with respect to h, which is introduced by Strebel [2], is defined as the set

$$V_h[z_0] = \{w \in \Delta \mid w = f(z_0), f \in [h]_e\}.$$

Strebel [3] proved that the variability set $V_h[z_0]$ is a compact and connected subset of Δ without holes. In this section, we shall study this set and find some new properties of the set.

Theorem 1.1. *Let $h : \Gamma \to \Gamma$ be a quasisymmetric homeomorphism and $z_0 \in \Delta$. Then there exists a constant M depending only on $K(h)$ such that the hyperbolic diameter of $V_h[z_0]$ is less than M.*

Proof. Let f_1 and f_2 be two extremal quasiconformal mappings in $[h]_e$ and suppose $w_j = f_j(z_0)$ $(j = 1, 2)$. Set $F = f_2 \circ f_1^{-1}$. Then F is a quasiconformal mapping with maximal dilatation at most $K^2(h)$. Now consider the Teichmüller shift mapping $T[w_1, w_2] : \Delta \to \Delta$, which is the extremal quasiconformal mapping with identity on Γ and maps w_1 to w_2. Define the maximal dilatation of $T[w_1, w_2]$ by K'. Then it is not hard to see that $K' \le K^2(h)$. It is known that the maximal dilatation of $T[w_1, w_2]$ depends only on the hyperbolic distance of w_1 and w_2. In fact, if we define

$$d_{K'} = \frac{1}{2}\log K'$$

and d_H by the hyperbolic distance of w_1 and w_2, then we have the following equality

$$\log \frac{e^{d_{K'}}+1}{e^{d_{K'}}-1} = \mu\Big(\frac{e^{d_H}-1}{e^{d_H}+1}\Big),$$

where $\mu(r)$ is the conformal module of the ring domain whose boundary components are the unit circle and the interval $\{x;\, 0 \le x \le r\,\}$. From this equality and the fact that $K' \le K^2(h)$, the theorem follows.

In the following we prove that as a set function the variability set is continuous in the Hausdorff topology.

Theorem 1.2. *Let $h : \Gamma \to \Gamma$ be a quasisymmetric homeomorphism and $z_0 \in \Delta$. Then $V_h[z_0]$ is continuous in Δ in the Hausdorff topology.*

Proof. Let z_0 be fixed and suppose that $z_n \to z_0$ $(n \to \infty)$. First we prove that $\limsup\limits_{z\to z_0} V_h[z] \subset V_h[z_0]$. Suppose $w_n \in V_h[z_n]$ $(n = 1, 2, \cdots)$ such that $w_n \to w_0$ $(n \to \infty)$. We need to prove that $w_0 \in V_h[z_0]$. For each n, we choose a mapping $f_n \in [h]_e$ such that $f_n(z_n) = w_n$. Since, for each n, the maximal dilatation of f_n is $K(h)$ and the family of the extremal mappings forms a normal family, the limit of any convergent subsequence of $\{f_n\}_1^\infty$ is a quasiconformal mapping f in $[h]_e$. Assume now the sequence $\{f_n\}_1^\infty$ itself converges to f, therefore $w_n = f_n(z_n) \to w_0 = f(z_0) \in V_h[z_0]$ $(n \to \infty)$.

To prove that $\liminf\limits_{z\to z_0} V_h[z] \supset V_h[z_0]$, let us fix a point $w_0 \in V_h[z_0]$ and arbitrarily take a quasiconformal mapping $f \in [h]_e$ such that $f(z_0) = w_0$. As $f(z)$ is continuous in Δ, we see that $\lim\limits_{z\to z_0} f(z) = w_0$. As $f(z) \in V_h[z]$, it follows that $\liminf\limits_{z\to z_0} V_h[z] \supset V_h[z_0]$. The proof of the theorem is completed.

Remark 1.1. From the property of the continuity of variability set, it is easy to see that when a point z_0 is such that the variability set $V_h[z_0]$ is not a single point, there is a neighborhood of z_0 such that every point z in this neighborhood has the property that the variability set $V_h[z]$ contains infinitely many points. Combining Strebel's result, we see that if the interior of $V_h[z_0]$ is not empty, then the interior of $V_h[z]$ is not empty either.

§ 2. Hamilton Sequences and Variability Set

Some authors have investigated the existence of Hamilton sequence. Recently, Strebel gives another way to form a Hamilton sequence for extremal Beltrami coefficient by making use of the point shift differentials. Let h be a quasisymmetric homeomorphism of the unit circle. For any $w \notin V_h[z_0]$, there is a uniquely determined holomorphic quadratic differentials φ_w on $\Delta\backslash\{z_0\}$ with L^1 norm:

$$\iint_\Delta |\varphi_w(z)|\, dxdy = 1$$

such that the Beltrami coefficient $\kappa\overline{\varphi}_w/|\varphi_w|$ (for some $0 < \kappa < 1$) is an extremal Beltrami coefficient for the boundary correspondence of $\Gamma \cup \{z_0\}$ onto $\Gamma \cup \{w_0\}$:

$$h_w^*(z) := \begin{cases} h(z), & z \in \partial\Delta; \\ w, & z = z_0. \end{cases}$$

The holomorphic quadratic differential φ_w in $\Delta\backslash\{z_0\}$ is called a point shift differential determined by w and h.

One of the main results of Strebel in [3] is the following

Theorem 2.1. (see [3]) *Suppose that $h_0 : \Gamma \longrightarrow \Gamma$ is a given quasisymmetric homeomorphism. Let w_0 be a boundary point of the variability set $V_{h_0}[z_0]$ and let $f_0 : \Delta \longrightarrow \Delta$ be an extremal quasiconformal mapping with $f_0|_\Gamma = h_0$ such that $f_0(z_0) = w_0$, the Beltrami coefficient of which is μ_0. Suppose that $\{w_n\}_1^\infty$ is a sequence of points in the set $\Delta\backslash V_{h_0}[z_0]$ with $w_n \to w_0$ $(n \to \infty)$ and that $\varphi_n = \varphi_{w_n}$ is a point shift differential determined by the point w_n and h_0 for each $n = 1, 2, \cdots$. Then $\{\varphi_n\}_1^\infty$ is a Hamilton sequence of μ_0, namely,*

$$\lim_{n\to\infty} \mathrm{Re} \iint_\Delta \mu_0\varphi_n \, dxdy = \|\mu_0\|_\infty.$$

The significance of this kind Hamilton sequence has two special aspects: one is that the Hamilton sequence depends only on one parameter and another is that these quadratic differentials in the Hamilton sequence are just induced from some quasiconformal mappings in the Teichmüller equivalence class $[h]$.

Now let $Q_h^P(\Delta)$ denote the set of quadratic differentials which have the following properties:

(1) every element $\varphi \in Q_h^P(\Delta)$ has norm 1 and is holomorphic except one simple pole in Δ ;

(2) for every $\varphi \in Q_h^P(\Delta)$, there exist a quasiconformal mapping $f \in [h]$ and a constant k $(0 < k < 1)$ such that the complex dilatation of f is $k\frac{\bar{\varphi}}{|\varphi|}$.

Now we pose the following questions.

Question 2.1. For every quasiconformal mapping $f \in [h]_e$, is there always a Hamilton sequences $\{\varphi_n\}_1^\infty \subset Q_h^P(\Delta)$? Furthermore is there always a common Hamilton sequence $\{\varphi_n\}_1^\infty \subset Q_h^P(\Delta)$?

In the following, we can solve this problem under some additional conditions.

First we prove the following lemma.

Lemma 2.1. *Let $h : \Gamma \to \Gamma$ be quasisymmetric and let $f_0 \in [h]_e$. Suppose that $\{\varphi_n\}_1^\infty$ is a degenerating Hamilton sequence for f_0 and that for each n, φ_n has finitely many simple poles $z_1, z_2, \cdots z_N$ in Δ. Then $\{\varphi_n\}_1^\infty$ is a common Hamilton sequence for all $f \in [h]_e$.*

Proof. For each n, according to the conditions on φ_n, we can write it as

$$\varphi_n = \sum_{j=1}^N \frac{\alpha_j^n}{z - z_j} + \psi_n, \tag{2.1}$$

where ψ_n is holomorphic in Δ and $\alpha_j^n,\ j=1,2,\cdots,N$ are constants.

Since $\{\varphi_n\}_1^\infty$ is a degenerating Hamilton sequence for μ_0, $\varphi_n\to 0$ $(n\to\infty)$ uniformly on any compact subset of $\Delta\backslash\{z_1,z_2,\cdots,z_N\}$.

Now, for each $1\le j_0\le N$, we take a circle $\Gamma_r^{j_0}=\{z\mid |z-z_{j_0}|=r\}$ with r small enough such that it is contained in Δ and the interior of it does not contain any pole of φ_n except z_{j_0}. As $\varphi_n\to 0$ $(n\to\infty)$ uniformly on $\Gamma_r^{j_0}$, we have

$$\lim_{n\to\infty}\int_{\Gamma_r^{j_0}}\varphi_n=0. \tag{2.2}$$

On the other hand,

$$\lim_{n\to\infty}\int_{\Gamma_r^{j_0}}\varphi_n=\lim_{n\to\infty}\int_{\Gamma_r^{j_0}}\Big(\sum_{j=1}^N\frac{\alpha_j^n}{z-z_j}+\psi_n\Big)=\lim_{n\to\infty}\int_{\Gamma_r^{j_0}}\frac{\alpha_{j_0}^n}{z-z_{j_0}}=2\pi i\lim_{n\to\infty}\alpha_{j_0}^n. \tag{2.3}$$

Therefore

$$\alpha_{j_0}^n\to 0,\qquad \text{as}\ \ n\to\infty. \tag{2.4}$$

From (2.1), we can see that

$$1-\Big\|\sum_{j=1}^N\frac{\alpha_j^n}{z-z_j}\Big\|\le\|\psi_n\|\le 1+\Big\|\sum_{j=1}^N\frac{\alpha_j^n}{z-z_j}\Big\|.$$

It follows from the above inequalities that $\|\psi_n\|\to 1,$ as $n\to\infty$.

Noting that

$$\|\mu_0\|_\infty=\lim_{n\to\infty}\Big(\mathrm{Re}\iint_\Delta\mu_0\sum_{j=1}^N\frac{\alpha_j^n}{z-z_j}+\mathrm{Re}\iint_\Delta\mu_0\psi_n\Big)=\lim_{n\to\infty}\mathrm{Re}\iint_\Delta\mu_0\psi_n,$$

we have

$$\lim_{n\to\infty}\mathrm{Re}\iint_\Delta\mu_0\psi_n/\,\|\psi_n\|=\|\mu_0\|_\infty. \tag{2.5}$$

We see that $\{\psi_n/\|\psi_n\|\}_1^\infty$ is a Hamilton sequence for μ_0. As, for each n, ψ_n is holomorphic, it follows from the result in [4] that $\{\psi_n/\|\psi_n\|\}_1^\infty$ is a Hamilton sequence for all $f\in[h]_e$. Now by elementary computation, we see easily that $\{\varphi_n\}$ is a Hamilton sequence for all $f\in[h]_e$. The proof of Lemma 2.1 is completed.

From the lemma we have the following result.

Theorem 2.2. *Let $h:\Gamma\longrightarrow\Gamma$ be a quasisymmetric homeomorphism and let w_0 be a boundary point of $V_h[z_0]$. Suppose that $f_0\in[h]_e$ and $f_0(z_0)=w_0$, the Beltrami coefficient of which is μ_0. Suppose further that $\{w_n\}_1^\infty$ is a sequence of points in the set $\Delta\backslash V_h[z_0]$ and $w_n\to w_0$ $(n\to\infty)$. Assume that $\varphi_n=\varphi_{w_n}$ $(n=1,2,\cdots)$ is a degenerating Hamilton sequence for μ_0, which is the point shift differentials determined by the point w_n and h. Then $\{\varphi_n\}_1^\infty$ is a Hamilton sequence for all $f\in[h]_e$.*

Proof. From the frame mapping criterion, we know that for each n, φ_n has a first order pole at z_0 and is holomorphic in $\Delta\backslash\{z_0\}$. So Theorem 2.2 follows from the lemma.

So if there is a degenerating Hamilton sequence induced by the point shift differentials, then Theorem 2.2 solves the question. If there is no such Hamilton sequence, we can prove the following result.

Theorem 2.3. *Under the above conditions of Theorem* 2.2, *suppose that* $\{\varphi_n\}_1^\infty$ *is not degenerating and* $[h]_e$ *contains more than one element. Then the interior of the variability set* $V_h[z_0]$ *is not empty, namely* $\overset{o}{V}_h[z_0] \neq \emptyset$.

Proof. Note that, for all n, $\|\varphi_n\| = 1$. By the normal family argument there exists a subsequence of $\{\varphi_n\}_1^\infty$ such that it converges uniformly in compact subsets of Δ. Without loss of generality, we can assume that the subsequence is $\{\varphi_n\}_1^\infty$ and its limit is φ_0. Because $\{\varphi_n\}_1^\infty$ is not degenerating, then φ_0 is not identical to zero and at most has a first order pole at z_0. We have the following two cases.

(I) First assume that φ_0 is holomorphic in Δ. From the Teichmüller unique theorem, we can see that $V_h[z]$ has only one point for all $z \in \Delta$, which contradicts our assumption.

(II) Now assume that φ_0 has a first order pole at z_0. In this case, we use the techniques to prove the parabolic lemma in [3]. One can find a Jordan domain $D \subset \Delta$ containing z_0 and a quasiconformal mapping $\tilde{f}$ such that $\tilde{f} = f_0, z \in \Delta\backslash D$ and the maximal dilatation of $\tilde{f}$ in D is strictly less than $K(h)$. It is obvious that $\tilde{f} \in [h]_e$. By using Lemma 2 in [5], we see that $\overset{o}{V}_h[z_0] \neq \emptyset$. The proof of Theorem 2.3 is completed.

Combining Theorem 2.2 and Theorem 2.3, we see that, if the answer to the question is negative, then, for every $z \in \Delta$, the variability set $V_h[z]$ is a simply connected closed domain with nonempty interior. Moreover, it is not hard to see from the proof of parabolic lemma in [3] that, for every boundary point w of the variability set $V_h[z]$, one can find a Jordan arc contained in the interior of $V_h[z]$ such that w is an endpoint of the arc. We believe that in this case the boundary of $V_h[z]$ is a Jordan curve.

It is easy to see from Theorem 1.1 that Euclidean diameter of the variability sets $V_h[z]$ tends to 0 as $z \to \Gamma$. If one can prove that the hyperbolic diameter tends to zero, then one can also solve the question.

Theorem 2.4. *Let* $h : \Gamma \to \Gamma$ *be quasisymmetric. Assume that there exists a sequence* $\{z_n\}_1^\infty \subset \Delta$ *such that the hyperbolic diameter of* $V_h[z_n]$ *tends to* 0 *as* $n \to \infty$. *Then we can find a Hamilton sequence* $\{\varphi_n\}_1^\infty \subset Q_h^P(\Delta)$ *for all* $f \in [h]_e$.

Proof. We can prove some further results. If there exists a Hamilton sequence which satisfies the conditions of Theorem 2.2, then the proposition follows. So we can assume that this case never happens. Now Theorem 2.3 tells us that for every $z \in \Delta$ and for every w on the boundary of $V_h[z]$, there is a quasiconformal mapping $f \in [h]_e$ such that $f(z) = w$ and $\mu_f = k\frac{\bar{\varphi}}{|\varphi|}$, where $\varphi_{(z,w)}$ is a quadratic differential with norm 1, holomorphic in Δ except a simple pole at z and $k = \frac{K(h)-1}{K(h)+1}$. Now we arbitrarily take a sequence $\varphi_n = \varphi_{(z_n,w_n)}$ which is induced by the quasiconformal mapping $f \in [h]_e$ such that $f(z_n) = w_n \in \partial V_h[z_n]$ for each $n = 1, 2, \cdots$, and arbitrarily fix a quasiconformal mapping $f_0 \in [h]_e$ with complex dilatation μ_0. By using the method of the proof of Theorem 2 in [3] or by directly using Theorem 2 in [6], we have the following estimate

$$0 \leq \|\mu_0\|_\infty - \mathrm{Re} \iint_\Delta \mu_0 \varphi_{(z_n,w_n)} \leq C \varrho(w_n, f_0(z_n)),$$

where ϱ is the hyperbolic distance and C is a universal constant. From this inequality, we see that $\varphi_n = \varphi_{(z_n,w_n)}$ is a Hamilton sequence for f_0. The theorem follows.

Finally we study the case of point shift quadratic differential which does not fix the quasisymetric function. Recall that the universal Teichmüller space T can be defined as the set of normalized quasisymmetric homeomorphism of Γ:

$$T := \{h : \Gamma \longrightarrow \Gamma \mid h \text{ is a quasisymmetric homeomorphism and } h(\pm 1) = \pm 1;\ h(i) = i\}.$$

The Teichmüller distance d_T of T is defined as follows:

$$d_T(h_1, h_2) := \frac{1}{2} \inf \left\{ \log \frac{1 + \delta(\mu_{f_1}, \mu_{f_2})}{1 - \delta(\mu_{f_1}, \mu_{f_2})};\ \begin{array}{l} f_j \text{ is quasiconformal mapping} \\ \text{and } f_j|_\Gamma = h_j,\ j = 1, 2 \end{array} \right\},$$

where

$$\delta(\mu_{f_1}, \mu_{f_2}) := \left\| \frac{\mu_{f_1} - \mu_{f_2}}{1 - \mu_{f_1} \bar{\mu}_{f_2}} \right\|.$$

Then we can prove the following result.

Theorem 2.5. *Suppose that $h_0 : \Gamma \longrightarrow \Gamma$ is a normalized quasisymmetric homeomorphism and that $z_0 \in \Delta$. Then there exist a sequence h_n of points in T and a sequence w_n of points in $\Delta \backslash V_{h_0}[z_0]$, for $n = 1, 2, \cdots$, such that*

(1) $d_T(h_n, h_0) \to 0$, *as* $n \to \infty$;

(2) *the point shift differentials φ_n determined by w_n and h_n form a Hamilton sequence for all $f \in [h_0]_e$.*

Proof. Again if there is a Hamilton sequence satisfying the condition of Theorem 2.2, Theorem 2.5 follows. Now if not, then there must exist an extremal quasiconformal mapping $f_0 \in [h_0]_e$ such that its complex dilatation μ_0 with $|\mu_0| \neq \text{constant, a.e.}$ Define $w_0 = f_0(z_0)$. By the results of [7], we know that there exists a sequence $\{h_n\}$ of points in T and a sequence $\{w_n\}$ of points in Δ with $w_n \in \Delta \backslash V_{h_n}[z_0]$, for $n = 1, 2, \cdots$, such that

(1) $d_T(h_n, h_0) \to 0$, as $n \to \infty$;

(2) $w_n \to w_0 \in \Delta$, as $n \to \infty$;

(3) the point shift differentials φ_n determined by w_n and h_n form a Hamilton sequence for the extremal mapping $f_0 : \Delta \backslash \{z_0\} \to \Delta \backslash \{w_0\}$.

It is easy to see that this sequence must be degenerating, so the theorem follows from the lemma.

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