Chin. Ann. Math.
26B:3(2005),551–558.

QUASI-LOCAL CONJUGACY THEOREMS IN BANACH SPACES***

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Abstract

Let $f : U(x_0) \subset E \longrightarrow F$ be a C^1 map and $f'(x_0)$ be the Frechet derivative of f at x_0. In local analysis of nonlinear functional analysis, implicit function theorem, inverse function theorem, local surjectivity theorem, local injectivity theorem, and the local conjugacy theorem are well known. Those theorems are established by using the properties: $f'(x_0)$ is double splitting and $R(f'(x)) \cap N(T_0^+) = \{0\}$ near x_0. However, in infinite dimensional Banach spaces, $f'(x_0)$ is not always double splitting (i.e., the generalized inverse of $f'(x_0)$ does not always exist), but its bounded outer inverse of $f'(x_0)$ always exists.

Only using the C^1 map f and the outer inverse $T_0^\#$ of $f'(x_0)$, the authors obtain two quasi-local conjugacy theorems, which imply the local conjugacy theorem if x_0 is a locally fine point of f. Hence the quasi-local conjugacy theorems generalize the local conjugacy theorem in Banach spaces.

Keywords Frechet derivative, Quasi-local conjugacy theorems, Outer inverse, Local conjugacy theorem
2000 MR Subject Classification 47H

§ 1 . Introduction

Let $f : U(x_0) \subset E \longrightarrow F$ be a C^1 map, where E and F are Banach spaces and $U(x_0)$ is an open set containing point $x_0 \in E$. Let $B(E, F)$ be the set of all bounded linear operators from E into F. First, we introduce an important concept in local linearization theory and nonlinear functional analysis.

Definition 1.1. (cf. [1]) *Suppose that $f : U(x_0) \subset E \longrightarrow F$ is C^1. By saying that f is locally conjugate to $f'(x_0)$ near x_0, we mean that there exist two neighborhoods U_0 at x_0 and V_0 at 0, with two maps u and v, such that*

(i) *$u : U_0 \to u(U_0)$ and $v : V_0 \to v(V_0)$ with $u(x_0) = 0,\ v(0) = f(x_0)$ are both diffeomorphisms,*

(ii) *$f(x) = (v \circ f'(x_0) \circ u)(x),\ \forall x \in U_0$.*

In order to solve the local linearization problem in nonlinear functional analysis, the famous mathematician Berger, M. posed a conjugacy problem (see [2, 3]): What properties

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***Project supported by the National Natural Science Foundation of China (No.10271053).

of $f(x)$ and $f'(x_0)$ ensure that f is conjugate to $f'(x_0)$ near x_0, which is intimately connected with rank theorem and construction of solution to the equation: $f(x) = y$ with $f(x_0) = y_0$ (see [2]). Professor Ma Jipu gave a complete answer to the local conjugacy problem: f is locally conjugate to $f'(x_0)$ near x_0 if and only if x_0 is a locally fine point of f (see [1]), i.e., $f'(x_0)$ has a generalized inverse T_0^+ and $R(f'(x)) \cap N(T_0^+) = \{0\}$ near x_0.

In this paper, $f'(x)$ is denoted by T_x, $f'(x_0)$ by T_0, the generalized inverse of a bounded operator T by T^+ and outer inverse of T by $T^\#$, respectively. The local conjugacy theorem and locally fine point are very important in global analysis (see [1–4]). Several kinds of the rank theorems in advanced calculus are also closely connected with local conjugacy theorem (see [1, 3, 5, 7, 8]).

It is well known that the bounded generalized inverse of a bounded linear operator T does not always exist. However, the bounded outer inverses of T always exist (see [6]). Using them, we obtain two quasi-local conjugacy theorems, i.e., Theorem 2.1,

$$(v \circ (T_0 T_0^\# T_0) \circ u)(x) - f(x) = \int_0^1 T_{z(t)}[(u'(z(t)))^{-1})(T_0^\# T_0 - I_E)(x - x_0)]\mathrm{d}t,$$

and Theorem 2.2,

$$T_0^\#(f(x)) = (T_0^\#(v \circ (T_0 T_0^\# T_0) \circ u))(x), \qquad \forall x \in U_0.$$

Theorem 2.1 and Theorem 2.2 generalize the local conjugacy theorem in Banach spaces.

Next, before proving the quasi-local conjugacy theorems, we need a lemma, which is established by Nashed, M. Z. and Chen, X.

Lemma 1.1. (cf. [6]) *Let $T_0 \in B(E, F)$ and $T_0^\# \in B(F, E)$ be an outer inverse of T_0. Suppose $T \in B(E, F)$ such that*

$$\|T_0^\#(T - T_0)\| < 1.$$

Then

$$T^\# = (I_E + T_0^\#(T - T_0))^{-1} T_0^\#$$

is a bounded outer inverse of T with

$$N(T^\#) = N(T_0^\#) \quad \text{and} \quad R(T^\#) = R(T_0^\#),$$

where I_E is the identity operator on E. Moreover

$$\|T^\# - T_0^\#\| \le \frac{\|T_0^\#(T - T_0)\| \|T_0^\#\|}{1 - \|T_0^\#(T - T_0)\|}.$$

From above, it is easy to get the following two facts:

$$N(T_0^\# T) = N(T^\# T), \tag{1.1}$$

and

$$N(T_0^\# T_x) = N(T_x^\# T_x) = (u'(x))^{-1} N(T_0^\# T_0) \tag{1.2}$$

near x_0.

Since

$$N(T_0^{\#}T) = N(T) + \{h \in E : \widetilde{T}[h] \in N(T_0^{\#})\}$$

and

$$N(T^{\#}T) = N(T) + \{h \in E : \widetilde{T}[h] \in N(T^{\#})\} = N(T) + \{h \in E : \widetilde{T}[h] \in N(T_0^{\#})\},$$

where $[h] \in E/N(T)$, $\widetilde{T}[h] = Th$ and $N(\cdot)$ denotes the null space of the operator in the parenthesis, (1.1) follows. To prove (1.2), let

$$u(x) = T_0^{\#}(f(x) - f(x_0)) + (I_E - T_0^{\#}T_0)(x - x_0).$$

By differentiation,

$$u'(x) = T_0^{\#}T_x + (I_E - T_0^{\#}T_0),$$

which induces that

$$u'(x)N(T_0^{\#}T_x) = (I_E - T_0^{\#}T_0)N(T_0^{\#}T_x), \qquad u'(x_0) = I_E.$$

Let $T_x^{\#} = (I_E + T_0^{\#}(T_x - T_0))^{-1}T_0^{\#}$ and $P_x = I_E - T_x^{\#}T_x$ near x_0. Then $T_x^{\#} \to T_0^{\#}$ by Lemma 1.1 and $P_x \to I_E - T_0^{\#}T_0$ as $x \to x_0$. The expression (1.1) and $(I - T_0^{\#}T_0)N(T_x^{\#}T_x) = N(T_0^{\#}T_0)$ show that

$$N(T_0^{\#}T_x) = N(T_x^{\#}T_x) = (u'(x))^{-1}N(T_0^{\#}T_0)$$

near x_0. So (1.2) follows.

§ 2. Main Results

The following two theorems are the main results in this paper, which are called quasi-local conjugacy theorems.

Theorem 2.1. (Quasi-local Conjugacy Theorem) *Let $f : U(x_0) \subset E \longrightarrow F$ be a C^1 map, then there exist two neighborhoods $U_0 \subset U(x_0)$ at x_0 and V_0 at 0, with two diffeomorphisms $u : U_0 \to u(U_0)$ and $v : V_0 \to v(V_0)$ such that*

(i) $u(x_0) = 0$, $u'(x_0) = I_E$ *and* $v(0) = f(x_0)$, $v'(0) = I_F$,

(ii) $$(v(T_0T_0^{\#}T_0)u)(x) - f(x) = \int_0^1 T_{z(t)}[(u'(z(t)))^{-1}(T_0^{\#}T_0 - I_E)(x - x_0)]\mathrm{d}t, \qquad (2.1)$$

$\forall\, x \in U_0$, where

$$z(t) = u^{-1}(T_0^{\#}(f(x) - f(x_0)) + (1 - t)(I_E - T_0^{\#}T_0)(x - x_0))$$

is a vector value function in E.

Proof. Let $u(x) = T_0^{\#}(f(x) - f(x_0)) + (I_E - T_0^{\#}T_0)(x - x_0)$. Obviously, $u(x_0) = 0$ and $u'(x_0) = I_E$. Moreover, we shall prove the following results:

(i) There exists an open disk $D_r^E(0)$ in E such that

$$u : u^{-1}(D_r^E(0)) \to D_r^E(0) \tag{2.2}$$

is a diffeomorphism.

(ii) There exists an open disk $D_\rho^E(x_0)$ in $u^{-1}(D_r^E(0))$ such that

$$T_0^{\#}(f(x) - f(x_0)) \in D_r^E(0), \qquad \forall\, x \in D_\rho^E(x_0), \tag{2.3}$$

$$u : D_\rho^E(x_0) \to u(D_\rho^E(x_0)) \tag{2.4}$$

is a diffeomorphism.

(iii) There exists an open disk $D_l^F(0)$ in F such that

$$T_0^{\#} y \in u(D_\rho^E(x_0)) \subset D_r^E(0), \qquad \forall\, y \in D_l^F(0). \tag{2.5}$$

In fact, by the inverse map theorem, (2.2) is direct. Since $D_\rho^E(x_0) \subset u^{-1}(D_r^E(0))$, (2.4) is clear. By (2.2), without loss of generality, we may say that (1.2) holds for all $x \in u^{-1}(D_r^E(0))$.

By the continuity of $T_0^{\#}(f(x) - f(x_0))$ at x_0, (2.3) is immediate. Since $T_0^{\#}0 = 0 \in u(D_\rho^E(x_0))$ and $T_0^{\#} \in B(F, E)$, (2.5) is obvious.

Let $x_1 = T_0^{\#}(f(x) - f(x_0))$, $\forall x \in D_\rho^E(x_0)$ and $x_2 = x_1 + (I_E - T_0^{\#}T_0)(x - x_0)$, respectively. So $x_2 = u(x)$. By (2.3), (2.5) and $D_\rho^E(x_0) \subset u^{-1}(D_r^E(0))$, we see that $x_1,\ x_2 \in D_r^E(0)$ for any $x \in D_\rho^E(x_0)$. By the convexity of $D_r^E(0)$,

$$w(t) = tx_1 + (1 - t)x_2 = T_0^{\#}(f(x) - f(x_0)) + (1 - t)(I_E - T_0^{\#}T_0)(x - x_0) \in D_r^E(0)$$

for any $x \in D_\rho^E(x_0)$ and $t \in [0, 1]$.

We now proceed to construct v required by Definition 1.1. Define

$$v(y) = (f \circ u^{-1} \circ T_0^{\#})y + (I_F - T_0 T_0^{\#})y, \qquad \forall\, y \in D_l^F(0).$$

Obviously, $v(0) = f(x_0)$, and

$$v'(0) = T_0 \circ (u^{-1})'(0) \circ T_0^{\#} + (I_F - T_0 T_0^{\#}) = I_F.$$

By the inverse map theorem, there is an open disk $D_m^F(0)$ with $0 < m < l$ such that

$$v : D_m^F(0) \to v(D_m^F(0))$$

is a diffeomorphism. Because of the boundedness of T_0, there is an open disk $D_q^E(x_0) \subset D_\rho^E(x_0)$ such that

$$y = T_0 x \in D_m^F(0), \qquad \forall x \in u(D_q^E(x_0)).$$

Finally, noting that (2.4) and (2.5) keep valid in $D_q^E(x_0)$, $\forall x \in D_q^E(x_0) := U_0$, we can write $(v \circ (T_0 T_0^{\#} T_0) \circ u)(x) - f(x)$ in integration form. Namely,

$$\begin{aligned}
&(v \circ (T_0 T_0^{\#} T_0) \circ u)(x) - f(x) \\
&= (v(T_0 T_0^{\#} T_0))(T_0^{\#}(f(x) - f(x_0)) + (I_E - T_0^{\#}T_0))(x - x_0)) - (f \circ u)(u^{-1}(x)) \\
&= v(T_0 T_0^{\#}(f(x) - f(x_0))) - (f \circ u^{-1})(T_0^{\#}(f(x) - f(x_0)) + (I_E - T_0^{\#}T_0)(x - x_0)) \\
&= (f \circ u^{-1})(T_0^{\#}(f(x) - f(x_0))) - (f \circ u^{-1})(T_0^{\#}(f(x) - f(x_0)) + (I_E - T_0^{\#}T_0)(x - x_0))
\end{aligned}$$

$$= \int_0^1 \frac{d}{dt}[(f \circ u^{-1})(T_0^{\#}(f(x) - f(x_0)) + (1-t)(I_E - T_0^{\#}T_0)(x - x_0))]dt$$
$$= \int_0^1 \frac{d}{dt}[(f \circ u^{-1})(tx_1 + (1-t)x_2)]dt$$
$$= \int_0^1 f'[u^{-1}(tx_1 + (1-t)x_2)](u')^{-1}[u^{-1}(tx_1 + (1-t)x_2)](T_0^{\#}T_0 - I_E)(x - x_0)dt$$
$$= \int_0^1 (T_{u^{-1}(w(t))})(u')^{-1}(u^{-1}(w(t)))(T_0^{\#}T_0 - I_E)(x - x_0)dt,$$

where $z(t) = u^{-1}(w(t)) \in u^{-1}(D_r^E(0))$. This completes the proof.

Theorem 2.2. (Quasi-local Conjugacy Theorem) *Let $f : U(x_0) \subset E \longrightarrow F$ be a C^1 map, then $T_0^{\#}(f(x))$ is conjugate to $T_0T_0^{\#}T_0$ near x_0, i.e.,*

$$T_0^{\#}(f(x)) = (T_0^{\#}(v \circ (T_0T_0^{\#}T_0) \circ u))(x), \qquad \forall\, x \in U_0,$$

where u and v are the same as in Theorem 2.1.

Proof. By Theorem 2.1, we have

$$(T_0^{\#}(v \circ (T_0T_0^{\#}T_0) \circ u))(x) - T_0^{\#}(f(x))$$
$$= \int_0^1 T_0^{\#}T_{z(t)}[(u'(z(t)))^{-1})(T_0^{\#}T_0 - I_E)(x - x_0)]dt.$$

Noting that

$$R\,(T_0^{\#}T_0 - I_E) = N\,(T_0^{\#}T_0)$$

and that (1.2) keeps valid for any $x \in u^{-1}$ $(D_r^E(0))$, we see that

$$T_0^{\#}T_z[(u'(z))^{-1})(T_0^{\#}T_0 - I_E)(x - x_0)]$$
$$\in T_0^{\#}T_z[(u'(z))^{-1}N(T_0^{\#}T_0)] = T_0^{\#}T_z(N(T_0^{\#}T_z)) = \{0\},$$

where $z = z(t) \in u^{-1}(D_r^E(0))$.

This gives

$$(T_0^{\#}(v \circ (T_0T_0^{\#}T_0) \circ u))(x) - T_0^{\#}(f(x)) = 0,$$

and completes the proof.

Remark 2.1. We adopt the notation that $f \sim f'(x_0)$ means that f is locally conjugate to $f'(x_0)$ near x_0. Then $f \sim f'(x_0)$ if and only if f satisfies

(1) $f'(x_0)$ has a generalized inverse T_0^+,

(2) $R(f'(x)) \cap N(T_0^+) = \{0\}$ near x_0.

However, Theorem 2.1 and Theorem 2.2 only need $f : U(x_0) \subset E \longrightarrow F$ to be a C^1 map. If the condition (1) above is satisfied, then we have the following Corollary 2.1. If both the conditions (1) and (2) are satisfied, then we have Corollary 2.2, i.e., the local conjugacy theorem is a special case of quasi-local conjugacy theorem.

Corollary 2.1. *If $T_0^{\#}$ is also an inner inverse of T_0, i.e., $T_0^{\#} = T_0^{+}$, then*

$$T_0^{+}(f(x)) = (T_0^{+}(v \circ f'(x_0) \circ u))(x), \qquad \forall\, x \in U_0.$$

Corollary 2.2. *If x_0 is a locally fine point of f, then*

$$f(x) = (v \circ f'(x_0) \circ u)(x), \qquad \forall\, x \in U_0.$$

Proof. By the equivalent conditions of locally fine points, there exist generalized inverses T_z^{+} such that $T_z^{+} \to T_0^{+}$ as $z \to x_0$. By $N(T_z) = N(T_z^{+}T_z)$ and (1.2), we see that

$$N(T_z) = (u'(z))^{-1} N(T_0^{\#}T_0), \qquad \forall z \in u^{-1}(D_r^{E}(0)).$$

For any $z = z(t) \in u^{-1}(D_r^{E}(0))$, we have

$$\begin{aligned} &T_z[(u'(z))^{-1})(T_0^{\#}T_0 - I_E)(x - x_0)] \\ &\in T_z[(u'(z))^{-1})R(T_0^{\#}T_0 - I_E)] = T_z[(u'(z))^{-1})N(T_0^{\#}T_0)] = T_z N(T_z^{+}T_z) = \{0\}. \end{aligned} \tag{2.6}$$

By (2.6) and (2.1) in Theorem 2.1, we have

$$(v \circ f'(x_0) \circ u)(x) - f(x) = 0,$$

which completes the proof.

The following simple Example 2.1 shows that Theorem 2.2 is a generalization of the local conjugacy theorem.

Example 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x_1,\ x_2) = (x_1^2 - x_2^2,\ x_1 + x_2)$. Obviously, $f(x)$ is a C^1 map. Let $x_0 = (0,\ 0)$. Then

$$T_0 = f'(x_0) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Given the generalized inverses of T_0 of rank 1 as follows,

$$T_0^{+} = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix},$$

where c is an arbitrary constant. By u in Theorem 2.1, we have

$$u(x_1,\ x_2) = (x_1,\ cx_1^2 - cx_2^2 + x_2),$$

where $(x_1,\ x_2) \in U_0$. Clearly, $u(x_0) = (0,\ 0),\ u'(x_0) = I_{\mathbb{R}^2}$ and

$$u^{-1}(x_1,\ x_2) = \Big(x_1,\ \frac{1 - \sqrt{1 - 4cx_2 + 4c^2x_1^2}}{2c}\Big),$$

where $(x_1,\ x_2) \in u(U_0)$. Since

$$f'(x_1,\ x_2) = \begin{pmatrix} 2x_1 & -2x_2 \\ 1 & 1 \end{pmatrix},$$

then $\mathrm{Rank}(f'(x_1,\ x_2))$ is not a constant in any neighbourhood at x_0. Hence x_0 is not a locally fine point of f, i.e., the local conjugacy theorem for f near x_0 does not hold. However, we have

$$T_0^+ f(x_1,\ x_2) = (0,\ cx_1^2 - cx_2^2 + x_1 + x_2). \tag{2.7}$$

By v in Theorem 2.1, then

$$v(y_1,\ y_2) = \Big(-\Big[\frac{1-\sqrt{1-4c(cy_1+y_2)}}{2c}\Big]^2 + y_1,\ \frac{1-\sqrt{1-4c(cy_1+y_2)}}{2c} - cy_1\Big),$$

which shows that

$$v(0,\ 0) = (0,\ 0), \qquad v'(0,\ 0) = I_{\mathbb{R}^2}$$

and

$$\begin{aligned}
&(v \circ f'(x_0) \circ u)(x_1,\ x_2)\\
&= v(f'(x_0)(x_1,\ cx_1^2 - cx_2^2 + x_2))\\
&= v((0,\ cx_1^2 - cx_2^2 + x_1 + x_2))\\
&= \Big(-\Big[\frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big]^2,\ \frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big).
\end{aligned}$$

Obviously

$$(v \circ f'(x_0) \circ u)(x_1,\ x_2) \neq f(x_1,\ x_2).$$

But

$$\begin{aligned}
&T_0^+(v \circ f'(x_0) \circ u)(x_1,\ x_2)\\
&= T_0^+\Big(-\Big[\frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big]^2,\ \frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big)\\
&= \Big(0,\ -c\Big[\frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big]^2 + \frac{1-\sqrt{1-4c(cx_1^2-cx_2^2+x_1+x_2)}}{2c}\Big)\\
&= (0,\ cx_1^2 - cx_2^2 + x_1 + x_2).
\end{aligned} \tag{2.8}$$

By (2.7) and (2.8), we obtain

$$T_0^+(f(x_1,\ x_2)) = (T_0^+(v \circ f'(x_0) \circ u))(x_1,\ x_2).$$

Remark 2.2. It is known that if $y \in F$ is a generalized regular value (see [1]) of f, then the preimage $S = f^{-1}(y)$ is a Banach submanifold in E. Example 2.1 shows that $y_0 = f(x_0) = (0,\ 0)$ is not a generalized regular value of f. However, the preimage $f^{-1}(y_0) = \{(x_1,\ x_2) : x_1 + x_2 = 0\}$ is a Banach submanifold in $\mathbb{R}^2$. So the generalized regular value y of f is only sufficient for $S = f^{-1}(y)$ to be a Banach submanifold in F. This is an interesting topic for constructing Banach submanifold in global analysis (see [5]).

Acknowledgement. The authors would like to express their cordial thanks to the referees for the suggestions.

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