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BOUNDEDNESS OF MAXIMAL SINGULAR INTEGRALS***

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Abstract

The authors study the singular integrals under the Hörmander condition and the measure not satisfying the doubling condition. At first, if the corresponding singular integral is bounded from L^2 to itself, it is proved that the maximal singular integral is bounded from L^∞ to $RBMO$ except that it is infinite μ-a.e. on R^d. A sufficient condition and a necessary condition such that the maximal singular integral is bounded from L^2 to itself are also obtained. There is a small gap between the two conditions.

Keywords Maximal singular integral, $RBMO$

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§ 1. Introduction

Given a positive Radon measure μ on R^d satisfying the linear growth condition

$$\mu(B(x,r)) \le Cr^n, \qquad x \in R^d,\ r>0, \tag{1.1}$$

where n is a fixed number with $0<n\le d$. In this note we always assume that μ satisfies condition (1.1). Let $k(x,y)$ be a locally integrable function on $R^d\times R^d\setminus\{(x,y): x=y\}$ and

$$|k(x,y)|+|k(y,x)| \le \frac{C}{|x-y|^n}. \tag{1.2}$$

Given a locally integrable function f on R^d, set

$$T_{\varepsilon,N}f(x)=\int_{\varepsilon<|x-y|<N} k(x,y)f(y)d\mu(y),$$

$$T^*_{\varepsilon,N}f(x)=\sup_{\varepsilon\le\delta,R\le N}|T_{\delta,R}f(x)|,$$

$$T^*f(x)=\sup_{N>\varepsilon>0}|T_{\varepsilon,N}f(x)|.$$

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Here $T^*f(x)$ may be infinite. $T^*_{\varepsilon,N}f$ converges to T^*f increasingly as $\varepsilon \to 0, N \to \infty$. The doubling condition on μ, $\mu(B(x,2r)) \le C\mu(B(x,r))$ $(\forall\, x \in R^d,\ r > 0)$, is an essential assumption in most of the results of classical Calderón-Zygmund theory. However, recently it has been shown that a big part of the classical theory remains valid if the doubling assumption on μ is substituted by the size condition (1.1). For example, T1 Theorem and Tb Theorem have been proved in [6, 8, 12–15, 17]. More literatures and related topics can be found in [18].

In [7, 11], some Cotlar type inequality and weak type (1,1) estimate for the maximal CZO have been proved. But to the best of our knowledge there is no result about the maximal CZO on L^∞ even if μ is Lebesgue measure. We will give the boundedness of the maximal singular integral on L^∞ under a weaker assumption.

In the above mentioned papers the CZKs considered always satisfy the Lipschitz type condition. In this note we will study the singular integrals under the Hörmander condition. If μ is Lebesgue measure, some work has been done to relax Lipschitz condition in T(1) Theorem, such as [2, 3, 5, 19]. But an example given in [20] has shown that T(1) Theorem may be not valid under the Hörmander condition. Here we will develop the ideas in [13] and [4] to get a sufficient condition and a necessary condition for the L^2 boundedness of the singular integral under the Hörmander condition. It is remarkable that it seems that our necessary condition is sharper than the sufficient and necessary condition given in T1 Theorem (see [6, 10]).

Now we give the definition of $RBMO$. More details can be found in [13].

Given a fixed number $\rho > 1$, Q and R are two cubes such that $Q \subset R$. Suppose that $\rho^{m-1}l(Q) \le l(R) < \rho^m l(Q), f \in L^1_{\rm loc}(R^d, d\mu)$. Set

$$k_{Q,R} = 1 + \sum_{j=0}^{m} \frac{\mu(\rho^j Q)}{(l(\rho^j Q))^n},$$
$$m_Q f = \frac{1}{\mu(Q)} \int_Q f(y)d\mu(y).$$

We say $f \in RBMO$ if for any cube Q there exists a number f_Q such that

$$\frac{1}{\mu(\rho Q)} \int_Q |f(y) - f_Q| d\mu(y) \le C$$

and

$$|f_R - f_Q| \le C k_{Q,R}, \qquad Q \subset R,$$

where the smallest number C satisfying these inequalities is called the $RBMO$ norm of f. If ρ is replaced by another $\rho' > 1$, we will get an equivalent norm. From now on we always set $\rho = 6$.

We will use the interpolation proved in [13] to get the L^2 boundedness. Though it is very easy to prove the boundedness from $H^{1,\infty}_{atb}$ to L^1, for the purpose of this note we will prove the boundedness from L^∞ to $RBMO$ and use the dual argument to give the boundedness from $H^{1,\infty}_{atb}$ to L^1. The definition of $H^{1,\infty}_{atb}$ can be found in [13, 16].

The theorems below are main results in this note.

Theorem 1.1. *Besides* (1.2), *suppose that there hold the Hörmander condition*

$$\int_{|x-y|>2|y-y'|} |k(y,x)-k(y',x)|d\mu(x) \le C, \tag{1.3}$$

and the following uniformly boundedness for a bounded function a with $\mathrm{supp}(a) \subset Q$ *where Q is a cube*

$$\int |T_{\epsilon,N}a|^2 d\mu \le C\|a\|_\infty^2 \mu(Q). \tag{1.4}$$

Then we have

$$\|T^*_{\epsilon,N}f\|_{RBMO} \le C\|f\|_\infty. \tag{1.5}$$

So T^*f is either infinite μ-a.e. or finite μ-a.e. Furthermore, if T^*f is finite μ-a.e., then $\|T^*f\|_{RBMO} \le C\|f\|_\infty$.

Remark 1.1. In the proof of Theorem 1.1 it is easy to check that $T^*_{\varepsilon,N}f$ can be replaced by $T_{\varepsilon,N}f$, so we have

$$\|T_{\varepsilon,N}f\|_{RBMO} \le C\|f\|_\infty.$$

It is crucial in the proof of Theorem 1.2.

Theorem 1.2. *For convenience, suppose that k is a real-valued function. Besides* (1.2), *suppose that there hold the Hörmander condition*

$$\int_{|x-y|>2|y-y'|} (|k(x,y)-k(x,y')| + |k(y,x)-k(y',x)|)d\mu(x) \le C, \tag{1.6}$$

and for any $N > \varepsilon > 0$,

$$\iint_{S_\varepsilon Q} |U_{\varepsilon,N}(x,y)|d\mu(x)d\mu(y) \le C\mu(Q), \tag{1.7}$$

$$\iint_{S_\varepsilon Q} |U'_{\varepsilon,N}(x,y)|d\mu(x)d\mu(y) \le C\mu(Q), \tag{1.8}$$

where $S_\varepsilon Q = \{(x,y) \in Q\times Q : |x-y| > \varepsilon\}$,

$$U_{\varepsilon,N}(x,y) = \int_{\varepsilon<|x-z|<N,\, \varepsilon<|y-z|<N} k(z,x)k(z,y)d\mu(z),$$

$$U'_{\varepsilon,N}(x,y) = \int_{\varepsilon<|x-z|<N,\, \varepsilon<|y-z|<N} k(x,z)k(y,z)d\mu(z).$$

We can conclude that

$$\|T^*f\|_2 \le C\|f\|_2. \tag{1.9}$$

On the other hand, if (1.9) *holds, then*

$$\int \Big| \int_{S_\varepsilon Q} U_{\varepsilon,N}(x,y)d\mu(x)\Big| d\mu(y) \le C\mu(Q), \tag{1.10}$$

$$\int \Big| \int_{S_\varepsilon Q} U'_{\varepsilon,N}(x,y)d\mu(x)\Big| d\mu(y) \le C\mu(Q). \tag{1.11}$$

Remark 1.2. Under the Lipschitz condition, if for any $N > \varepsilon > 0$,

$$\iint_{S_\varepsilon Q} U_{\varepsilon,N}(x,y)d\mu(x)d\mu(y) \leq C\mu(Q), \tag{1.12}$$

$$\iint_{S_\varepsilon Q} U'_{\varepsilon,N}(x,y)d\mu(x)d\mu(y) \leq C\mu(Q), \tag{1.13}$$

then from [10, Theorem 4, p.306] and [6], we have $\|T^*f\|_2 \leq C\|f\|_2$. This fact shows that under the Lipschitz condition (1.10) and (1.11) are equivalent to (1.12) and (1.13).

The letter C in this note denotes a positive constant which only depends on n, d and may be variant in different cases.

§ 2 . Some Lemmas

In this section we always suppose that $k(x,y)$ satisfies conditions (1.2) and (1.3).

We say a ball $B = B(x,r)$ is $(6, 6^{d+1})$-ball if $\mu(6B) \leq 6^{d+1}\mu(B)$. For any ball B, let k be the smallest positive integer such that $B(x, 6^k r)$ is a $(6, 6^{d+1})$-ball. The existence of k is obviously.

Lemma 2.1. *The notations are as above. Then*

$$\int_{B(x,6^k r)\setminus B(x,r)} \frac{d\mu(y)}{|x-y|^n} \leq C. \tag{2.1}$$

The proof is easy. For convenience, we give the proof here.

Proof. From (1.1) and the definition of k, we have

$$\begin{aligned}
\int_{B(x,6^k r)\setminus B(x,r)} \frac{d\mu(y)}{|x-y|^n} &= \sum_{j=1}^{k} \int_{B(x,6^j r)\setminus B(x,6^{j-1}r)} \frac{d\mu(y)}{|x-y|^n} \\
&\leq \sum_{j=1}^{k} \frac{\mu(B(x,6^j r))}{(6^{j-1}r)^n} \\
&\leq C\sum_{j=1}^{k} \frac{6^{(j-k)(d+1)}\mu(B(x,6^k r))}{(6^j r)^n} \\
&\leq C\sum_{i=0}^{\infty} 6^{-i(d+1-n)} \frac{\mu(B(x,6^k r))}{(6^k r)^n} \\
&\leq C.
\end{aligned}$$

Lemma 2.2. *Let*

$$T'_{\varepsilon,N}f(x) = \int_{\varepsilon<|x-y|<N} k(y,x)f(y)d\mu(y)$$

and η be a Borel measure which is supported in a ball $B = B(y_0, r)$ and $\eta(B) = 0$. Then

$$\int_{(6B)^c} |T'_{\varepsilon,N}\eta|d\mu \leq C\|\eta\|. \tag{2.2}$$

Proof. Set

$$A = \{x : \max\{6r, \varepsilon - r\} < |x - y_o| < \varepsilon + r \quad \text{or} \quad \max\{6r, N - r\} < |x - y_o| < N + r\}.$$

If $x \in A^c$,

$$B(y_0, r) \subseteq \{y : \varepsilon < |y - x| < N\} \quad \text{or} \quad B(y_0, r) \subseteq \{y : |y - x| < \varepsilon \text{ or } N < |y - x|\},$$

so

$$\begin{aligned} |T'_{\varepsilon,N}\eta(x)| &= \Big| \int_{\varepsilon<|x-y|<N} k(y,x) d\eta(y) \Big| \\ &= \Big| \int_{\varepsilon<|x-y|<N} (k(y,x) - k(y_0,x)) d\eta(y) \Big| \\ &\le \int_{\varepsilon<|x-y|<N} |k(y,x) - k(y_0,x)| d|\eta(y)|. \end{aligned}$$

Now it can be derived from the Hörmander condition that

$$\int_{A^c \cap (6B)^c} |T'_{\varepsilon,N}\eta| d\mu \le C\|\eta\|. \tag{2.3}$$

If $x \in A$,

$$|T'_{\varepsilon,N}\eta(x)| \le C \int_{B(y_0,r)} |k(y,x)| d|\eta(y)| \le C \int_{B(y_0,r)} \frac{d|\eta(y)|}{|x-y|^n}.$$

It is obvious that for any $y \in B(y_0, r)$,

$$\int_A \frac{d\mu(x)}{|x-y|^n} \le C.$$

So it can be shown that

$$\int_A |T'_{\varepsilon,N}\eta(x)| d\mu \le C\|\eta\|. \tag{2.4}$$

From (2.3) and (2.4), we get the desired inequality.

Lemma 2.3. (A Variant Inequality of [1])

$$T^*_{\varepsilon,N}\varphi(x) \le C(M(T_{\varepsilon,N}\varphi)(x) + \|\varphi\|_\infty), \tag{2.5}$$

where M is the centered H-L operator and $\varphi \in L^\infty$.

Proof. Let k be the smallest positive integer such that $B(x, R) = B(x, 6^k r)$ is a $(6, 6^{d+1})$-ball. Since

$$|T_{r,s}\varphi(x)| \le |T_{r,N}\varphi(x)| + |T_{s,N}\varphi(x)| \qquad \text{for any} \quad \varepsilon < r < s < N,$$

it is sufficient to estimate $|T_{r,N}\varphi(x)|$.

$$\begin{aligned}|T_{r,N}\varphi(x) - T_{6R,N}\varphi(x)| &\le \int_{B(x,6R)\setminus B(x,r)} \frac{|\varphi(y)|}{|x-y|^n} d\mu(y) \\ &\le \|\varphi\|_\infty \int_{B(x,6R)\setminus B(x,r)} \frac{d\mu(y)}{|x-y|^n} \\ &\le C\|\varphi\|_\infty. \end{aligned} \tag{2.6}$$

The last inequality can be obtained by Lemma 2.1. Set

$$U_R(x) = \frac{1}{\mu(B(x,R))} \int_{B(x,R)} T_{\varepsilon,N}\varphi(y)d\mu(y),$$

which is clearly controlled by $M(T_{\varepsilon,N}\varphi)(x)$.

$$\begin{aligned} &|T_{6R,N}\varphi(x) - U_R(x)| \\ =& \Big|T_{\varepsilon,N}\varphi\chi_{\{y:6R<|x-y|\}}(x) - \int \frac{\chi_{B(x,R)}}{\mu(B(x,R))} T_{\varepsilon,N}\varphi d\mu\Big| \\ =& \Big|\int T'_{\varepsilon,N}\delta_x(z)\varphi\chi_{\{z:6R<|x-z|\}}(z)d\mu(z) - \int T'_{\varepsilon,N}\Big(\frac{\chi_{B(x,R)}}{\mu(B(x,R))}\Big)(z)\varphi(z)d\mu(z)\Big| \\ \le& \int \Big|T'_{\varepsilon,N}\Big(\delta_x - \frac{\chi_{B(x,R)}}{\mu(B(x,R))}\Big)(z)\varphi\chi_{\{z:6R<|x-z|\}}(z)\Big|d\mu(z) \\ &+ \frac{1}{\mu(B(x,R))}\int \chi_{B(x,R)}(y)|T_{\varepsilon,N}(\varphi\chi_{\{y:|x-y|<6R\}})|d\mu(y), \end{aligned}$$

where $T'_{\varepsilon,N}$ denotes the dual of $T_{\varepsilon,N}$. The first term does not exceed $C\|\varphi\|_\infty$ by Lemma 2.2, while the second can be controlled by

$$\begin{aligned} &\frac{1}{\mu(B(x,R))}\|\chi_{B(x,R)}\|_{L^2(\mu)}\Big(\int_R |T_{\varepsilon,N}(\varphi\chi_{B(x,6R)})|^2 d\mu\Big)^{\frac12} \\ \le& C\Big(\frac{\mu(B(x,6R))}{\mu(B(x,R))}\Big)^{\frac12}\|\varphi\|_\infty \le C\|\varphi\|_\infty, \end{aligned}$$

so the proof of the lemma is completed.

Lemma 2.4. *Q and R are two cubes and $Q \subset R$. There holds*

$$\int_Q\int_R |T^*_{\varepsilon,N}f(x) - T^*_{\varepsilon,N}f(y)|d\mu(y)d\mu(x) \le Ck_{Q,R}\|f\|_\infty(\mu(Q)\mu(6R) + \mu(6Q)\mu(R)). \tag{2.7}$$

Proof. Set $f_1 = f\chi_{6R}, f_2 = f\chi_{(6R)^c}$. From

$$|T^*_{\varepsilon,N}f(x) - T^*_{\varepsilon,N}f(y)| \le |T^*_{\varepsilon,N}f_2(x) - T^*_{\varepsilon,N}f_2(y)| + T^*_{\varepsilon,N}f_1(x) + T^*_{\varepsilon,N}f_1(y)$$

we have

$$
\begin{aligned}
&\int_Q \int_R |T^*_{\varepsilon,N} f(x) - T^*_{\varepsilon,N} f(y)| d\mu(y) d\mu(x) \\
\le &\int_Q \int_R |T^*_{\varepsilon,N} f_2(x) - T^*_{\varepsilon,N} f_2(y)| d\mu(y) d\mu(x) \\
&+ \mu(Q) \int_R T^*_{\varepsilon,N} f_1(y) d\mu(y) + \mu(R) \int_Q T^*_{\varepsilon,N} f_1(x) d\mu(x). \qquad (2.8)
\end{aligned}
$$

By Lemma 2.3 and (1.4), we have

$$
\begin{aligned}
\mu(Q) \int_R T^*_{\varepsilon,N} f_1(y) d\mu(y) &\le C\mu(Q) \int_R [M(T_{\varepsilon,N} f_1)(y) + \|f\|_\infty] d\mu(y) \\
&\le C\mu(Q) \Big((\mu(R))^{\frac{1}{2}} \Big(\int_R (T_{\varepsilon,N} f_1)^2 d\mu \Big)^{\frac{1}{2}} + \mu(R)\|f\|_\infty \Big) \\
&\le C\mu(Q)(\mu(R))^{\frac{1}{2}} (\mu(6R))^{\frac{1}{2}} \|f\|_\infty + C\mu(Q)\mu(R)\|f\|_\infty \\
&\le C\mu(Q)\mu(6R)\|f\|_\infty. \qquad (2.9)
\end{aligned}
$$

Take m such that $6^{m-1} l(Q) \le l(R) < 6^m l(Q)$. It is easy to check that

$$R \subset 6^{m+1} Q, \qquad 6R \subset 6^{m+2} Q.$$

For any $x \in Q$,

$$
\begin{aligned}
T^*_{\varepsilon,N} f_1(x) &\le \int_{6^{m+2}Q \setminus 6Q} |k(x,y)| d\mu(y) \|f\|_\infty + T^*_{\varepsilon,N}(f_1 \chi_{6Q})(x) \\
&\le \sum_{j=1}^{m+1} \frac{\mu(6^{j+1}Q)}{(l(6^j Q))^n} \|f\|_\infty + T^*_{\varepsilon,N}(f_1 \chi_{6Q})(x) \\
&\le C(k_{Q,R}\|f\|_\infty + T^*_{\varepsilon,N}(f_1 \chi_{6Q})(x)). \qquad (2.10)
\end{aligned}
$$

So by Lemma 2.3 and (1.4)

$$
\begin{aligned}
\mu(R) \int_Q T^*_{\varepsilon,N} f_1(x) d\mu(x) &\le C\mu(R) \Big(\mu(Q) k_{Q,R} \|f\|_\infty + \int_Q T^*_{\varepsilon,N}(f_1 \chi_{6Q}) d\mu \Big) \\
&\le C\mu(R)(\mu(Q) k_{Q,R} \|f\|_\infty + \mu(6Q)\|f\|_\infty) \\
&\le C\mu(R)\mu(6Q) k_{Q,R} \|f\|_\infty. \qquad (2.11)
\end{aligned}
$$

Now there is nothing left to deal with except the first term in (2.8).

It is obvious that we can take $\varepsilon(x)$, $N(x)$ such that

$$|T_{\varepsilon(x),N(x)} f_2(x)| = T^*_{\varepsilon,N} f_2(x). \qquad (2.12)$$

Obviously

$$
\begin{aligned}
&|T^*_{\varepsilon,N} f_2(x) - T^*_{\varepsilon,N} f_2(y)| \\
\le &|T_{\varepsilon(x),N(x)} f_2(x) - T_{\varepsilon(x),N(x)} f_2(y)| + |T_{\varepsilon(y),N(y)} f_2(x) - T_{\varepsilon(y),N(y)} f_2(y)|. \qquad (2.13)
\end{aligned}
$$

For convenience we may denote $r = \varepsilon(x), s = N(x)$. Because of symmetry, it is sufficient to compute one term. Set

$$A_x = \{z \in (6R)^c : r < |x - z| < s\}, \qquad A_y = \{z \in (6R)^c : r < |y - z| < s\}.$$

we have

$$\begin{aligned}|T_{r,s}f_2(x) - T_{r,s}f_2(y)| &= \Big| \int_{A_x} k(x,z)f_2(z)d\mu(z) - \int_{A_y} k(y,z)f_2(z)d\mu(z)\Big| \\ &\le \int_{A_x\cap A_y} |(k(x,z) - k(y,z))f_2(z)|d\mu(z) \\ &\quad + \int_{A_x\setminus A_y} \frac{|f_2(z)|}{|x-z|^n}d\mu(z) + \int_{A_y\setminus A_x} \frac{|f_2(z)|}{|y-z|^n}d\mu(z). \end{aligned} \tag{2.14}$$

Noticing that $x, y \in R, z \in (6R)^c$, from (1.3) we have

$$\int_{A_x\cap A_y} |(k(x,z) - k(y,z))f_2(z)|d\mu(z) \le C\|f\|_\infty. \tag{2.15}$$

Besides, it is easy to check that when $x, y \in R$,

$$\int_{A_x\setminus A_y} \frac{1}{|x-z|^n}d\mu(z) \le C, \qquad \int_{A_y\setminus A_x} \frac{1}{|y-z|^n}d\mu(z) \le C.$$

Now we can conclude that

$$|T_{r,s}f_2(x) - T_{r,s}f_2(y)| \le C\|f\|_\infty,$$

$$\int_Q\int_R |T^*_{\varepsilon,N}f_2(x) - T^*_{\varepsilon,N}f_2(y)|d\mu(x)d\mu(y) \le C\mu(Q)\mu(R)\|f\|_\infty. \tag{2.16}$$

Let everything together, this lemma has been proved.

§ 3. Proof of Theorem 1.1

Let $Q = R$ in Lemma 2.4. We can get

$$\begin{aligned}&\frac{1}{\mu(6Q)}\int_Q |T^*_{\varepsilon,N}f(y) - m_Q T^*_{\varepsilon,N}f|d\mu(y) \\ &\le \frac{1}{\mu(Q)\mu(6Q)}\int_Q\int_Q |T^*_{\varepsilon,N}f(x) - T^*_{\varepsilon,N}f(y)|d\mu(y)d\mu(x) \\ &\le C\frac{k_{Q,Q}\|f\|_\infty \mu(Q)\mu(6Q)}{\mu(Q)\mu(6Q)} \\ &= C\|f\|_\infty. \end{aligned} \tag{3.1}$$

On the other hand, it can be derived that

$$\begin{aligned}|m_R T^*_{\varepsilon,N}f - m_Q T^*_{\varepsilon,N}f| &\le \frac{1}{\mu(Q)\mu(R)}\int_Q\int_R |T^*_{\varepsilon,N}f(x) - T^*_{\varepsilon,N}f(y)|d\mu(y)d\mu(x) \\ &\le C\|f\|_\infty k_{Q,R}\Big(\frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)}\Big). \end{aligned} \tag{3.2}$$

Using the result of [13, Lemma 2.10, p.101], we have

$$\|T^*_{\epsilon,N}f\|_{RBMO} \le C\|f\|_\infty.$$

§ 4. Proof of Theorem 1.2

At first we show that it is sufficient to prove (1.4) for $T_{\epsilon,N}$ and $T'_{\epsilon,N}$. In the proof of Theorem 1.1 it can be derived that $\|T_{\epsilon,N}f\|_{RBMO} \le C\|f\|_\infty$. Also we have $\|T'_{\epsilon,N}f\|_{RBMO} \le C\|f\|_\infty$, so $\|T_{\epsilon,N}f\|_1 \le C\|f\|_{H^{1,\infty}_{atb}}$ by the dual argument. Using the interpolation proved in [13], we have $\|T_{\epsilon,N}f\|_2 \le C\|f\|_2$. In [9] we have proved that under the Hörmander condition if $\|T_\epsilon f\|_2 \le C\|f\|_2$ for any $\varepsilon > 0$, then $\|T^*f\|_p \le C\|f\|_p$ for any $1 < p < \infty$ and $\|T^*f\|_{1,\infty} \le C\|f\|_1$.

Because of symmetry it is enough for us to check that (1.4) holds for $T_{\epsilon,N}$. Let a be a bounded function with $\mathrm{supp}(a) \subset Q$ where Q is a cube. As

$$\begin{aligned}
&\int_Q \int_{|x-y|<\varepsilon} |U_{\varepsilon,N}(x,y)| d\mu(x)\mu(y) \\
&\le C \int_Q \int_{|x-y|<\varepsilon} \int_{|x-z|>2\varepsilon} \frac{1}{|x-z|^{2n}} d\mu(z) d\mu(x) d\mu(y) \\
&\quad + C \int_Q \int_{|x-y|<\varepsilon} \int_{\varepsilon<|x-z|<2\varepsilon} \frac{1}{\varepsilon^{2n}} d\mu(z) d\mu(x) d\mu(y) \\
&\le C\mu(Q),
\end{aligned} \tag{4.1}$$

we have

$$\begin{aligned}
\|T_{\epsilon,N}a\|_2^2 &= \iint_{\{x\in Q: N>|z-x|>\varepsilon\}} \int_{\{y\in Q: N>|z-y|>\varepsilon\}} k(z,x)k(z,y)a(x)a(y) d\mu(x) d\mu(y) d\mu(z) \\
&= \iint_{S_\varepsilon Q} U_{\varepsilon,N}(x,y) a(x) a(y) d\mu(x) d\mu(y) + O(\|a\|_\infty^2 \mu(Q)) \\
&\le C\|a\|_\infty^2 \mu(Q).
\end{aligned} \tag{4.2}$$

Also we can get $\|T'_{\epsilon,N}a\|_2^2 \le C\|a\|_\infty^2 \mu(Q)$.

Now, we show that (1.9) is necessary. For two bounded functions a and b supported on a cube Q, we have

$$\int_Q T_{\epsilon,N}a T_{\epsilon,N}b = \iint U(x,y)a(x)b(y) d\mu(x) d\mu(y).$$

Set

$$a = \chi_Q, \qquad b(y) = \mathrm{sign}\Big(\int_Q U(x,y) d\mu(x)\Big)\chi_Q(y).$$

It is easy to see that (1.9) is necessary for the L^2 boundedness of T^*, also for (1.10).

The proof is completed.

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