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CHARACTERIZATIONS OF JORDAN †-SKEW MULTIPLICATIVE MAPS ON OPERATOR ALGEBRAS OF INDEFINITE INNER PRODUCT SPACES***

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Abstract

Let H and K be indefinite inner product spaces. This paper shows that a bijective map $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ satisfies $\Phi(AB^\dagger+B^\dagger A)=\Phi(A)\Phi(B)^\dagger+\Phi(B)^\dagger\Phi(A)$ for every pair $A,B\in\mathcal{B}(H)$ if and only if either $\Phi(A)=cUAU^\dagger$ for all A or $\Phi(A)=cUA^\dagger U^\dagger$ for all A; Φ satisfies $\Phi(AB^\dagger A)=\Phi(A)\Phi(B)^\dagger\Phi(A)$ for every pair $A,B\in\mathcal{B}(H)$ if and only if either $\Phi(A)=UAV$ for all A or $\Phi(A)=UA^\dagger V$ for all A, where $A^\dagger$ denotes the indefinite conjugate of A, U and V are bounded invertible linear or conjugate linear operators with $U^\dagger U=c^{-1}I$ and $V^\dagger V=cI$ for some nonzero real number c.

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§ 1. Introduction

It is a surprising result of Matindale [6] that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Therefore, one can say that the multiplicative structure of rings of that kind completely determines the ring structure. This result was utilized by Šemrl in [9] to describe the form of the semigroup isomorphisms of standard operator algebras on Banach spaces. Some other results on the additivity of multiplicative maps (in fact, $*$-semigroup homomorphisms) between operator algebras can be found in [4, 5, 7, 8]. Besides additive and multiplicative maps (that is, ring homomorphisms) between rings, sometimes one has to consider Jordan homomorphisms and Jordan $*$-homomorphisms or Jordan †-homomorphisms. The structure of associative rings has been studied by many people in ring theory. Moreover, Jordan operator algebras have important applications in the mathematical foundations of quantum mechanics. Let $\mathcal{R}$ and $\mathcal{R}'$ be rings and let $\Phi:\mathcal{R}\to\mathcal{R}'$ be a map. Recall that Φ is called a Jordan homomorphism if

$$\Phi(A+B)=\Phi(A)+\Phi(B)$$

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and

$$\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$$

for all $A, B \in \mathcal{R}$. In the case that both $\mathcal{R}$ and $\mathcal{R}'$ have an involution operation $*$, Φ is called a Jordan $*$-homomorphism if it is a Jordan homomorphism and $\Phi(A^*) = \Phi(A)^*$ for all $A \in \mathcal{R}$. To ensure the additivity of a "multiplicative" map, one can go a little further by even more weakening the multiplicative property of the maps in question (see [5, 7, 8]). Let ϕ be a bijective map on a standard operator algebra. Molnár showed in [7] that if ϕ satisfies $\phi(ABA) = \phi(A)\phi(B)\phi(A)$, then ϕ is additive. Later, Molnár in [8] and then Lu in [5] considered the cases that ϕ preserves the operation $\frac{1}{2}(AB+BA)$ and $AB+BA$, respectively, and proved that such ϕ is also additive. Thus the Jordan multiplicative structure also determines the Jordan ring structure of the standard operator algebras. In the present paper, inspired by [1] and [2] we are interested in the questions of taking the indefinite structures and the involution structures in consideration.

Let H and K be real or complex complete indefinite inner product spaces with $\dagger$ standing for the conjugation related to the indefinite structures (see [3, 10]). The purpose of this paper is to show that if $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a bijective map and satisfies

$$\Phi(AB^\dagger + B^\dagger A) = \Phi(A)\Phi(B)^\dagger + \Phi(B)^\dagger\Phi(A) \tag{1.1}$$

for every pair $A, B \in \mathcal{B}(H)$ or

$$\Phi(AB^\dagger A) = \Phi(A)\Phi(B)^\dagger\Phi(A) \tag{1.2}$$

for every pair $A, B \in \mathcal{B}(H)$, then Φ is automatically linear or conjugate linear. Consequently, such Φ are classified completely. In fact we show that Φ satisfies the multiplicative property (1.1) if and only if there exist a nonzero real number c and a continuous invertible linear or conjugate linear operator U with $U^\dagger U = c^{-1}I$ such that either $\Phi(A) = cUAU^\dagger$ for all A or $\Phi(A) = cUA^\dagger U^\dagger$ for all A, that is, Φ is a $\dagger$-isomorphism, or a $\dagger$-anti-isomorphism, or a conjugate $\dagger$-isomorphism, or a conjugate $\dagger$-anti-isomorphism. While, Φ satisfies the multiplicative property (1.2) if and only if there exist a real number $c \neq 0$ and bounded invertible linear or conjugate linear operators U and V with $U^\dagger U = c^{-1}I$ and $V^\dagger V = cI$ such that either $\Phi(A) = UAV$ for all A or $\Phi(T) = UA^\dagger V$ for all A. Applications to Hilbert space case are also given. Let $\mathcal{A}$ be a factor C^*-algebra. A similar argument also reveals that, if a bijective map $\Phi : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{A} \otimes M_2(\mathbb{C})$ satisfies $\Phi(AB^* + B^*A) = \Phi(A)\Phi(B)^* + \Phi(B)^*\Phi(A)$ for every pair $A, B \in \mathcal{A} \otimes M_2(\mathbb{C})$, then Φ must be a C^*-isomorphism or conjugate C^*-isomorphism.

Let us recall some conceptions and fix some notations. Denote $\mathbb{F}$ the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. For every positive integer n, $M_n(\mathbb{F})$ denotes the matrix algebra of all $n \times n$ matrices over $\mathbb{F}$. Let $(H, [\cdot\,, \cdot])$ and $(K, [\cdot\,, \cdot])$ be complete indefinite inner product spaces and denote $\mathcal{B}(H, K)$ ($\mathcal{B}(H)$ if $H = K$) the set of all bounded linear operators from H into K. For any $T \in \mathcal{B}(H, K)$, the indefinite conjugate of T with respect to the indefinite inner products $[\cdot\,,\,\cdot]$ is an operator $T^\dagger \in \mathcal{B}(K,H)$ defined by the equation $[Tx,y]=[x,T^\dagger y]$ for all $x\in H$ and $y\in K$. On the other hand, assume that H_i $(i=1,2)$ are Hilbert spaces with inner product $\langle\cdot\,,\,\cdot\rangle$ and $J_i\in\mathcal{B}(H_i)$ are self-adjoint invertible operators. Then, for each $i=1,2$, $(H_i,[\cdot\,,\,\cdot]_{J_i})$ is a complete indefinite inner product space, where $[\cdot\,,\,\cdot]_{J_i}=\langle J_i(\cdot),\,\cdot\rangle$, which is induced by J_i. It is clear that, with respect to $[\cdot\,,\,\cdot]_{J_i}$, the indefinite conjugate $T^\dagger$ of an operator $T\in\mathcal{B}(H_1,H_2)$ is of the form $T^\dagger=J_1^{-1}T^*J_2$, in which T^* stands for the usual conjugate of T related to the inner products $\langle\cdot\,,\,\cdot\rangle$. Sometimes we also call $T^\dagger=J_1^{-1}T^*J_2$ the (J_1,J_2)-conjugate of T. If $H_1=H_2$ are the same Hilbert spaces and $J_1=J_2=J$, the (J_1,J_2)-conjugate of an operator T is often called the J-conjugate of T (see [3, 10]).

§ 2. Characterization of Jordan †-Skew Multiplicative Maps

In this section, the automatic additivity of Jordan †-skew multiplicative maps is proved, and then a characterization of Jordan †-isomorphisms of operator algebras on indefinite inner product spaces is given only by Jordan †-skew multiplicativity. The following is our main result.

Theorem 2.1. *Let H and K be (real or complex) complete indefinite inner product spaces and let $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ be a bijective map. If Φ satisfies*

$$\Phi(AB^\dagger+B^\dagger A)=\Phi(A)\Phi(B)^\dagger+\Phi(B)^\dagger\Phi(A) \tag{2.1}$$

for every pair $A,B\in\mathcal{B}(H)$, then Φ is additive.

Our approach is similar to that in [5] for the case that $\Phi(AB+BA)=\Phi(A)\Phi(B)+\Phi(B)\Phi(A)$. The main technique we will use is the following argument which will be termed a standard argument. Suppose $A,B,S\in\mathcal{A}$ are such that $\Phi(S)=\Phi(A)+\Phi(B)$. Multiplying this equality by $\Phi(T)^\dagger$ $(T\in\mathcal{A})$ from the right and the left, respectively, we get $\Phi(T)^\dagger\Phi(S)=\Phi(T)^\dagger\Phi(A)+\Phi(T)^\dagger\Phi(B)$ and $\Phi(S)\Phi(T)^\dagger=\Phi(A)\Phi(T)^\dagger+\Phi(B)\Phi(T)^\dagger$. Summing them, we get

$$\Phi(S)\Phi(T)^\dagger+\Phi(T)^\dagger\Phi(S)=\Phi(A)\Phi(T)^\dagger+\Phi(T)^\dagger\Phi(A)+\Phi(T)^\dagger\Phi(B)+\Phi(B)\Phi(T)^\dagger.$$

It follows from (2.1) that

$$\Phi(ST^\dagger+T^\dagger S)=\Phi(AT^\dagger+T^\dagger A)+\Phi(BT^\dagger+T^\dagger B).$$

Moreover, if

$$\Phi(AT^\dagger+T^\dagger A)+\Phi(BT^\dagger+T^\dagger B)=\Phi(AT^\dagger+T^\dagger A+BT^\dagger+T^\dagger B),$$

then by injectivity of Φ, we have

$$ST^\dagger+T^\dagger S=AT^\dagger+T^\dagger A+BT^\dagger+T^\dagger B.$$

Now assume that Φ satisfies the assumption in Theorem 2.1, we have to show that Φ is additive. We divide the proof into several lemmas. For the sake of simplicity, without loss of the generality, we assume that $H = K$.

Lemma 2.1. $\Phi(0) = 0$.

Proof. Since Φ is surjective, we can find an $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. Therefore $\Phi(0) = \Phi(0A + A0) = 0$.

Let $H = H_+ \oplus H_-$ be a regular decomposition of H. We may assume that both H_+ and H_- are nontrivial. In the sequel, we denote $P_1 \in \mathcal{B}(H)$ be the fixed non-trivial self-adjoint idempotent operator which has the matrix form

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

according to the regular decomposition, where I is the identity operator on H_+ (In the case that H_+ or H_- is $\{0\}$, say $H_+ = 0$, one may take any nonzero projection P_1 in $\mathcal{B}(H_-)$ with inner product $\langle \cdot, \cdot \rangle = -[\cdot, \cdot]$, and the proof is almost the same). Let $P_2 = I - P_1$ and set $\mathcal{A}_{ij} = P_i \mathcal{B}(H) P_j$, $i, j = 1, 2$. Then we have $\mathcal{B}(H) = \mathcal{A}_{11} \dot{+} \mathcal{A}_{12} \dot{+} \mathcal{A}_{21} \dot{+} \mathcal{A}_{22}$. In what follows, we write $A_{ij}, B_{ij}, \cdots$ for the elements in $\mathcal{A}_{ij}$.

Lemma 2.2. *Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$. The following statements are true.*

(i) *For $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le i, j \le 2)$, we have $T_{ij}S + ST_{ij} = T_{ij}S_{j1} + T_{ij}S_{j2} + S_{1i}T_{ij} + S_{2i}T_{ij}$.*

(ii) *If $T_{ij}S_{jk} = 0$ holds for every $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le i, j, k \le 2)$, then $S_{jk} = 0$. Dually, if $S_{ki}T_{ij} = 0$ for all $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le i, j, k \le 2)$, then $S_{ki} = 0$.*

(iii) *If $T_{ij}S + ST_{ij} \in \mathcal{A}_{ij}$, for every $T_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$, then $S_{ji} = 0$.*

(iv) *If $S_{ii}T_{ii} + T_{ii}S_{ii} = 0$ for every $T_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$, then $S_{ii} = 0$.*

(v) *If $T_{jj}S + ST_{jj} \in \mathcal{A}_{ij}$ for every $T_{jj} \in \mathcal{A}_{jj}$ $(1 \le i \ne j \le 2)$, then $S_{ji} = 0$ and $S_{jj} = 0$. Dually, if $T_{jj}S + ST_{jj} \in \mathcal{A}_{ji}$ for every $T_{jj} \in \mathcal{A}_{jj}$ $(1 \le i \ne j \le 2)$, then $S_{ij} = 0$ and $S_{jj} = 0$.*

Proof. Obvious.

Lemma 2.3. $\Phi(A_{ii} + A_{ij}) = \Phi(A_{ii}) + \Phi(A_{ij})$ $(1 \le i \ne j \le 2)$.

Proof. Since Φ is surjective, we may find an element $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(A_{ij}). \tag{2.2}$$

For $T_{ji}^\dagger \in \mathcal{A}_{ij}$, applying a standard argument to (2.2), we get $\Phi(T_{ji}^\dagger S + ST_{ji}^\dagger) = \Phi(T_{ji}^\dagger A_{ii} + A_{ii}T_{ji}^\dagger) + \Phi(A_{ij}T_{ji}^\dagger + T_{ji}^\dagger A_{ij}) = \Phi(A_{ii}T_{ji}^\dagger)$. Therefore, $T_{ji}^\dagger S + ST_{ji}^\dagger = A_{ii}T_{ji}^\dagger$ holds for every $T_{ji}^\dagger \in \mathcal{A}_{ij}$. It follows from Lemma 2.2(iii) that $S_{ji} = 0$. Hence by Lemma 2.2(i), we see that

$$T_{ji}^\dagger S_{jj} + S_{ii}T_{ji}^\dagger = A_{ii}T_{ji}^\dagger \tag{2.3}$$

for all $T_{ji}^\dagger \in \mathcal{A}_{ij}$. For $T_{jj}^\dagger \in \mathcal{A}_{jj}$, applying a standard argument to (2.2) again, we obtain that $T_{jj}^\dagger S + ST_{jj}^\dagger = A_{ij}T_{jj}^\dagger$. Then Lemma 2.2(v) entails that $S_{ij} = 0$ and $S_{jj} = 0$. Hence by Lemma 2.2(i), we get $S_{ij}T_{jj}^{\dagger} = A_{ij}T_{jj}^{\dagger}$ for every $T_{jj}^{\dagger} \in \mathcal{A}_{jj}$ and thus $A_{ij} = S_{ij}$ by Lemma 2.2(ii). Moreover from (2.3) we have that $S_{ii}T_{ji}^{\dagger} = A_{ii}T_{ji}^{\dagger}$ holds for every $T_{ji}^{\dagger} \in \mathcal{A}_{ij}$. Hence by Lemma 2.2(ii) we see that $A_{ii} = S_{ii}$. Consequently, $S = A_{ii} + A_{ij}$, as desired.

Lemma 2.4. $\Phi(A_{ii} + A_{ji}) = \Phi(A_{ii}) + \Phi(A_{ji})$ $(1 \leq i \neq j \leq 2)$.

The proof of this lemma is similar to that of Lemma 2.3, and we omit it here.

Lemma 2.5. $\Phi(A_{12} + C_{12}A_{22}) = \Phi(A_{12}) + \Phi(C_{12}A_{22})$.

Proof. Since

$$A_{12} + B_{21}^{\dagger}A_{22} = (A_{12} + A_{22})(P_1 + B_{21}^{\dagger}) + (P_1 + B_{21}^{\dagger})(A_{12} + A_{22}),$$

by using (2.1), Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\Phi(A_{12} + B_{21}^{\dagger}A_{22}) &= \Phi(A_{12} + A_{22})\Phi(P_1 + B_{21})^{\dagger} + \Phi(P_1 + B_{21})^{\dagger}\Phi(A_{12} + A_{22}) \\
&= (\Phi(A_{12}) + \Phi(A_{22}))(\Phi(P_1) + \Phi(B_{21}))^{\dagger} \\
&\quad + ((\Phi(P_1) + \Phi(B_{21}))^{\dagger}(\Phi(A_{12}) + \Phi(A_{22})) \\
&= \Phi(A_{12})\Phi(P_1)^{\dagger} + \Phi(A_{12})\Phi(B_{21})^{\dagger} + \Phi(A_{22})\Phi(P_1)^{\dagger} \\
&\quad + \Phi(A_{22})\Phi(B_{21})^{\dagger} + \Phi(P_1)^{\dagger}\Phi(A_{12}) \\
&\quad + \Phi(B_{21})^{\dagger}\Phi(A_{12}) + \Phi(P_1)^{\dagger}\Phi(A_{22}) + \Phi(B_{21})^{\dagger}\Phi(A_{22}) \\
&= \Phi(A_{12}P_1 + P_1A_{12}) + \Phi(A_{22}B_{21}^{\dagger} + B_{21}^{\dagger}A_{22}) \\
&\quad + \Phi(A_{12}B_{21}^{\dagger} + B_{21}^{\dagger}A_{12}) + \Phi(A_{22}P_1 + P_1A_{22}) \\
&= \Phi(A_{12}) + \Phi(B_{21}^{\dagger}A_{22}).
\end{aligned}$$

Lemma 2.6. $\Phi(A_{21} + A_{22}D_{21}) = \Phi(A_{21}) + \Phi(A_{22}D_{21})$.

Proof. Note that

$$A_{21} + A_{22}B_{12}^{\dagger} = (A_{21} + A_{22})(P_1 + B_{12}^{\dagger}) + (P_1 + B_{12}^{\dagger})(A_{21} + A_{22}).$$

Now we can complete the proof using a computation similar to that in the proof of Lemma 2.5.

Lemma 2.7. Φ *is additive on* $\mathcal{A}_{12}$.

Proof. Let $A_{12}, B_{12} \in \mathcal{A}_{12}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(B_{12}). \tag{2.4}$$

For $T_{22}^{\dagger} \in \mathcal{A}_{22}$, applying a standard argument to the equation (2.4) and using Lemma 2.5, we get

$$\Phi(T_{22}^{\dagger}S + ST_{22}^{\dagger}) = \Phi(A_{12}T_{22}^{\dagger}) + \Phi(B_{12}T_{22}^{\dagger}) = \Phi((A_{12} + B_{12})T_{22}^{\dagger}).$$

Hence

$$T_{22}^{\dagger}S + ST_{22}^{\dagger} = (A_{12} + B_{12})T_{22}^{\dagger} \tag{2.5}$$

for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. It follows from Lemma 2.2(v) that $S_{22} = S_{21} = 0$. Moreover, by (2.5) and Lemma 2.2(i) we have $S_{12}T_{22}^{\dagger} = (A_{12} + B_{12})T_{22}^{\dagger}$ for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. And then $S_{12} = A_{12} + B_{12}$.

Now there remains to prove that $S_{11} = 0$. For $T_{21}^{\dagger} \in \mathcal{A}_{12}$, applying a standard argument to (2.4) again, we get that $T_{21}^{\dagger}S + ST_{21}^{\dagger} = 0$. Since $S_{22} = S_{21} = 0$, we see that $S_{11}T_{21}^{\dagger} = 0$ for every $T_{21}^{\dagger} \in \mathcal{A}_{12}$. Hence from Lemma 2.2(ii) we get $S_{11} = 0$.

Lemma 2.8. Φ *is additive on* $\mathcal{A}_{21}$.

Proof. Let $A_{21}, B_{21} \in \mathcal{A}_{21}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{21}) + \Phi(B_{21}). \tag{2.6}$$

For $T_{22}^{\dagger} \in \mathcal{A}_{22}$, applying a standard argument to the equation (2.6) and using Lemma 2.6, we get

$$T_{22}^{\dagger}S + ST_{22}^{\dagger} = T_{22}^{\dagger}(A_{21} + B_{21}) \tag{2.7}$$

for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. It follows from Lemma 2.2(v) that $S_{22} = S_{12} = 0$. Hence (2.7) becomes $T_{22}^{\dagger}S_{21} = T_{22}^{\dagger}(A_{21} + B_{21})$ for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. This implies that $S_{21} = A_{21} + B_{21}$.

To prove $S_{11} = 0$, let $T_{12}^{\dagger} \in \mathcal{A}_{21}$. Applying a standard argument to (2.6) again, we get that $T_{12}^{\dagger}S + ST_{12}^{\dagger} = 0$. Since we already have $S_{22} = S_{12} = 0$, we get that $T_{12}^{\dagger}S_{11} = 0$ for every $T_{12}^{\dagger} \in \mathcal{A}_{21}$. Hence it follows from Lemma 2.2(ii) that $S_{11} = 0$.

Lemma 2.9. Φ *is additive on* $\mathcal{A}_{ii}$ $(i = 1, 2)$.

Proof. Let $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ and choose $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{ii}) + \Phi(B_{ii}). \tag{2.8}$$

Let $j \neq i$. For $T_{jj}^{\dagger} \in \mathcal{A}_{jj}$, applying a standard argument to (2.8), we get $T_{jj}^{\dagger}S + ST_{jj}^{\dagger} = 0$. It follows from Lemma 2.2(v) that $S_{ij} = S_{ji} = S_{jj} = 0$.

There remains to prove that $S_{ii} = A_{ii} + B_{ii}$. For $T_{ji}^{\dagger} \in \mathcal{A}_{ij}$, applying a standard argument to (2.8) again, we get

$$\Phi(ST_{ji}^{\dagger} + T_{ji}^{\dagger}S) = \Phi(A_{ii}T_{ji}^{\dagger}) + \Phi(B_{ii}T_{ji}^{\dagger}).$$

Hence by Lemmas 2.7 and 2.8, one sees that

$$ST_{ji}^{\dagger} + T_{ji}^{\dagger}S = (A_{ii} + B_{ii})T_{ji}^{\dagger}$$

for every $T_{ji}^{\dagger} \in \mathcal{A}_{ij}$. Since $S_{ij} = S_{ji} = S_{ii} = 0$, it follows that $S_{ii}T_{ji}^{\dagger} = (A_{ii} + B_{ii})T_{ji}^{\dagger}$ for every $T_{ji}^{\dagger} \in \mathcal{A}_{ij}$. Hence, by Lemma 2.2(ii) we have that $S_{ii} = A_{ii} + B_{ii}$.

Lemma 2.10. Φ *is additive on* $P_1\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12}$.

Proof. Let $A_{11}, B_{11} \in \mathcal{A}_{ii}$ and $A_{12}, B_{12} \in \mathcal{A}_{12}$. Then by Lemmas 2.3, 2.7 and 2.9 we see that

$$\begin{aligned}\Phi((A_{11}+A_{12})+(B_{11}+B_{12})) &= \Phi((A_{11}+B_{11})+(A_{12}+B_{12})) \\ &= \Phi(A_{11}+A_{12})+\Phi(B_{11}+B_{12}) \\ &= \Phi(A_{11})+\Phi(A_{12})+\Phi(B_{11})+\Phi(B_{12}) \\ &= \Phi(A_{11}+A_{12})+\Phi(B_{11}+B_{12}).\end{aligned}$$

Lemma 2.11. $\Phi(A_{11}+A_{22}) = \Phi(A_{11})+\Phi(A_{22})$.

Proof. Let $S = S_{11}+S_{12}+S_{21}+S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{11})+\Phi(A_{22}). \tag{2.9}$$

For P_1, applying a standard argument to (2.9) we have

$$\Phi(P_1S+SP_1) = \Phi(A_{11}P_1+P_1A_{11}).$$

This implies that $P_1S+SP_1 = 2A_{11}$. Multiplying P_2 from the left and the right in the above equality, we get that $S_{12} = S_{21} = 0$ and $S_{11} = A_{11}$. Furthermore, a standard argument to (2.9) yields that

$$\Phi(ST_{22}^{\dagger}+T_{22}^{\dagger}S) = \Phi(A_{22}T_{22}^{\dagger}+T_{22}^{\dagger}A_{22})$$

for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. Hence by the injectivity of Φ, we have $ST_{22}^{\dagger}+T_{22}^{\dagger}S = A_{22}T_{22}^{\dagger}+T_{22}^{\dagger}A_{22}$. Since $S_{12} = S_{21} = 0$, in combination with the above equality, we get

$$(S_{22}-A_{22})T_{22}^{\dagger}+T_{22}^{\dagger}(S_{22}-A_{22}) = 0$$

for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. Hence $S_{22}-A_{22} = 0$. Consequently $S = A_{11}+A_{22}$.

Lemma 2.12. $\Phi(A_{12}+A_{21}) = \Phi(A_{12})+\Phi(A_{21})$.

Proof. Let $S = S_{11}+S_{12}+S_{21}+S_{22} \in \mathcal{B}(H)$ be such that

$$\Phi(S) = \Phi(A_{12})+\Phi(A_{21}). \tag{2.10}$$

For $T_{21}^{\dagger} \in \mathcal{A}_{12}$, applying a standard argument to (2.10), we have

$$\begin{aligned}\Phi(ST_{21}^{\dagger}+T_{21}^{\dagger}S) &= \Phi(A_{12}T_{21}^{\dagger}+T_{21}^{\dagger}A_{12})+\Phi(A_{21}T_{21}^{\dagger}+T_{21}^{\dagger}A_{21}) \\ &= \Phi(A_{21}T_{21}^{\dagger}+T_{21}^{\dagger}A_{21}).\end{aligned}$$

Hence by injectivity of Φ, we obtain

$$ST_{21}^{\dagger}+T_{21}^{\dagger}S = A_{21}T_{21}^{\dagger}+T_{21}^{\dagger}A_{21}$$

for every $T_{21}^{\dagger} \in \mathcal{A}_{12}$. Multiplying this equality by P_1 from the right, we get that $T_{21}^{\dagger} S_{21} = T_{21}^{\dagger} A_{21}$ for every $T_{21}^{\dagger} \in \mathcal{A}_{12}$. Then it follows from Lemma 2.2(ii) that $S_{21} = A_{21}$. Hence by Lemma 2.2(i),

$$T_{21}^{\dagger} S_{22} + S_{11} T_{21}^{\dagger} = 0 \tag{2.11}$$

holds for every $T_{21}^{\dagger} \in \mathcal{A}_{12}$. An argument similar to what has led to the equality $S_{21} = A_{21}$ shows that $S_{12} = A_{12}$.

Applying a standard argument to (2.10) again, we get

$$\begin{aligned}\Phi(SP_1 + P_1 S) &= \Phi(P_1 A_{12} + A_{12} P_1) + \Phi(P_1 A_{21} + A_{21} P_1) \\ &= \Phi(A_{21}) + \Phi(A_{12}) = \Phi(S).\end{aligned}$$

Therefore $S = SP_1 + P_1 S$. This implies that $S_{11} = 0$. Hence we deduce from (2.11) and Lemma 2.2(ii) that $S_{22} = 0$. Consequently, $S = A_{12} + A_{21}$.

Lemma 2.13. $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.

Proof. Let $S = S_{11} + S_{12} + S_{21} + S_{22} \in \mathcal{B}(H)$ be such that $\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$. Then, by Lemma 2.3 and Lemma 2.4, we have

$$\Phi(S) = \Phi(A_{11} + A_{12}) + \Phi(A_{21}) \tag{2.12}$$

and

$$\Phi(S) = \Phi(A_{11} + A_{21}) + \Phi(A_{12}). \tag{2.13}$$

For $T_{12}^{\dagger} \in \mathcal{A}_{21}$, applying a standard argument to (2.12) we get

$$T_{12}^{\dagger} S + S T_{12}^{\dagger} = A_{12} T_{12}^{\dagger} + T_{12}^{\dagger} A_{11} + T_{12}^{\dagger} A_{12}. \tag{2.14}$$

Multiplying this equality by P_1 from the left, we see that $S_{12} T_{12}^{\dagger} = A_{12} T_{12}^{\dagger}$ for every $T_{12}^{\dagger} \in \mathcal{A}_{21}$. So $S_{12} = A_{12}$. Similarly, for $T_{21}^{\dagger} \in \mathcal{A}_{12}$, applying a standard argument to (2.13), we get $S_{21} = A_{21}$. Multiplying (2.14) by P_2 and P_1 from the left and from the right, we get

$$S_{22} T_{12}^{\dagger} + T_{12}^{\dagger} S_{11} = T_{12}^{\dagger} A_{11} \tag{2.15}$$

for every $T_{12}^{\dagger} \in \mathcal{A}_{21}$. Let $T_{22}^{\dagger} \in \mathcal{A}_{22}$. Applying a standard argument to (2.12), we can easily see that

$$\Phi(S T_{22}^{\dagger} + T_{22}^{\dagger} S) = \Phi(A_{12} T_{22}^{\dagger}) + \Phi(T_{22}^{\dagger} A_{21}) = \Phi(A_{12} T_{22}^{\dagger} + T_{22}^{\dagger} A_{21})$$

by making use of Lemma 2.12. Therefore

$$S T_{22}^{\dagger} + T_{22}^{\dagger} S = A_{12} T_{22}^{\dagger} + T_{22}^{\dagger} A_{21}$$

for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. Multiplying this equality by P_2 from the left and from the right, we get that $S_{22} T_{22}^{\dagger} + T_{22}^{\dagger} S_{22} = 0$ for every $T_{22}^{\dagger} \in \mathcal{A}_{22}$. It follows from Lemma 2.2(iv) that $S_{22} = 0$. Hence, by (2.15), we have $S_{11} = A_{11}$. Consequently, $S = A_{11} + A_{12} + A_{21}$.

Proof of Theorem 2.1. It suffices to check that $\Phi(A_{11}+A_{12}+A_{21}+A_{22}) = \Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})$.

Let $S = S_{11}+S_{12}+S_{21}+S_{22} \in \mathcal{B}(H)$ such that

$$\Phi(S) = \Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}). \tag{2.16}$$

Then we have

$$\begin{aligned}\Phi(P_1S+SP_1) &= \Phi(P_1)\Phi(S)+\Phi(S)\Phi(P_1)\\ &= \Phi(P_1)(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))\\ &\quad + (\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}))\Phi(P_1)\\ &= \Phi(2A_{11})+\Phi(A_{12})+\Phi(A_{21})\\ &= \Phi(2A_{11}+A_{12}+A_{21}),\end{aligned}$$

making use of Lemma 2.13 in the last equality. It follows that $P_1S+SP_1 = 2A_{11}+A_{12}+A_{21}$, that is, $2S_{11}+S_{12}+S_{21} = 2A_{11}+A_{12}+A_{21}$. Multiplying the equality by P_2 from the right and the left, we get that $S_{11} = A_{11}$, $S_{21} = A_{21}$ and $S_{12} = A_{12}$.

Let $T_{21}^{\dagger} \in \mathcal{A}_{12}$. Applying a standard argument to (2.16) again we see that

$$\Phi(T_{21}^{\dagger}S+ST_{21}^{\dagger}) = \Phi(A_{11}T_{21}^{\dagger})+\Phi(A_{21}T_{21}^{\dagger}+T_{21}^{\dagger}A_{21})+\Phi(T_{21}^{\dagger}A_{22}).$$

Furthermore, applying a standard argument to the above equality, we obtain

$$\begin{aligned}\Phi(P_1ST_{21}^{\dagger}+P_1T_{21}^{\dagger}S+T_{21}^{\dagger}SP_1) &= \Phi(P_1A_{11}T_{21}^{\dagger})+\Phi(2T_{21}^{\dagger}A_{21}P_1)+\Phi(T_{21}^{\dagger}A_{22})\\ &= \Phi(A_{11}T_{21}^{\dagger}+2T_{21}^{\dagger}A_{21}+T_{21}^{\dagger}A_{22}),\end{aligned}$$

making use of Lemma 2.10. Hence we have

$$T_{21}^{\dagger}S_{22}+S_{11}T_{21}^{\dagger}+2T_{21}^{\dagger}S_{21} = A_{11}T_{21}^{\dagger}+2T_{21}^{\dagger}A_{21}+T_{21}^{\dagger}A_{22}.$$

Since we have shown that $S_{11} = A_{11}$ and $S_{21} = A_{21}$, it follows that $T_{21}^{\dagger}S_{22} = T_{21}^{\dagger}A_{22}$ for every $T_{21}^{\dagger} \in \mathcal{A}_{12}$, and hence $S_{22} = A_{22}$. Consequently, $S = A_{11}+A_{12}+A_{21}+A_{22}$.

Now it is clear that Φ is additive, completing the proof.

The following result further gives a thorough classification of the maps which satisfies the assumption in Theorem 2.1.

Theorem 2.2. *Let H and K be (real or complex) complete indefinite inner product spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then Φ satisfies (2.1), that is,*

$$\Phi(AB^{\dagger}+B^{\dagger}A) = \Phi(A)\Phi(B)^{\dagger}+\Phi(B)^{\dagger}\Phi(A)$$

for every pair $A, B \in \mathcal{B}(H)$, if and only if there exist a nonzero real number c and a linear or conjugate linear bounded invertible operator $U \in \mathcal{B}(H,K)$ satisfying $U^{\dagger}U = c^{-1}I$ such that $\Phi(A) = cUAU^{\dagger}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = cUA^{\dagger}U^{\dagger}$ for all $A \in \mathcal{B}(H)$.

Proof. We need only check the "only if" part. Assume that Φ satisfies the equation (2.1) for every pair $A, B \in \mathcal{B}(H)$. By Theorem 2.1, Φ is additive. Note that

$$\Phi(I) = \Phi\Big(I\Big(\frac{1}{2}I\Big)^{\dagger} + \Big(\frac{1}{2}I\Big)^{\dagger} I\Big) = \Phi(I)\Phi\Big(\frac{1}{2}I\Big)^{\dagger} + \Phi\Big(\frac{1}{2}I\Big)^{\dagger}\Phi(I)$$
$$= \Phi\Big(\Big(\frac{1}{2}I\Big)I^{\dagger} + I^{\dagger}\frac{1}{2}I\Big) = \Phi\Big(\frac{1}{2}I\Big)\Phi(I)^{\dagger} + \Phi(I)^{\dagger}\Phi\Big(\frac{1}{2}I\Big).$$

This yields that $\Phi(I)^{\dagger} = \Phi(I)$. Letting $A = I, B = I$ in Equation (2.1), we get that $\Phi(I)^2 = \Phi(I)$ by the additivity of Φ. Letting $B = I$ in Equation (2.1), we have

$$2\Phi(A) = \Phi(A)\Phi(I) + \Phi(I)\Phi(A).$$

Multiplying this equality by $\Phi(I)$ from the left and the right, we get $\Phi(I)\Phi(A) = \Phi(A)\Phi(I)$, which implies that $\Phi(I) = I$ since $\Phi(I)^{\dagger} = \Phi(I)$ and $\Phi(I)^2 = \Phi(I)$. Furthermore, $\Phi(B^{\dagger}) = \Phi(B)^{\dagger}$.

Since $\mathcal{B}(H)$ is a prime ring, as a Jordan ring automorphism, Φ must be a ring automorphism or a ring anti-automorphism. If H is of infinite dimension, then by a result in [7] Φ is linear or conjugate linear. Therefore there exists a linear or conjugate linear bounded invertible operator U on H such that $\Phi(A) = UAU^{-1}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UJA^*JU^{-1} = UA^{\dagger}U^{-1}$ for all $A \in \mathcal{B}(H)$ (notice that $A^* = JA^{\dagger}J$ with $J = 2P_1 - I$, where P_1 is the projection from H onto H_+ along H_-). Because Φ preserves the $\dagger$ operation, we see that $U^{\dagger}U = \alpha I$ for some nonzero real number α. Let $c = \alpha^{-1}$. Then we have either $\Phi(A) = cUAU^{\dagger}$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = cUA^{\dagger}U^{\dagger}$ for all $T \in \mathcal{B}(H)$.

Assume now that H is finite dimensional with $\dim H = n$ and $\dim H_+ = k$. According to the regular decomposition $H = H_+ \oplus H_-$, we take orthonormal bases $\{e_1, e_2, \cdots, e_k\}$ in H_+ and $\{e_{k+1}, e_{k+2}, \cdots, e_n\}$ in H_- respectively. Then H is isomorphism to $\mathbb{F}^n$ equipped with the indefinite inner product $[\,\cdot\,,\,\cdot\,] = \langle J(\,\cdot\,), \,\cdot\,\rangle$, where $J = 2P_1 - I$ with $P_1 = \sum\limits_{i=1}^{k} E_{ii} \in M_n(\mathbb{F})$ and $E_{ij} \in M_n(\mathbb{F})$ the unit matrix with entry 1 at (i,j) position and 0 elsewhere. So $\mathcal{B}(H)$ is isomorphism to $M_n(\mathbb{F})$ and we can regard Φ as a Jordan ring isomorphism of $M_n(\mathbb{F})$. By a result in [7] there exist an automorphism τ of $\mathbb{F}$ and an invertible matrix $U \in M_n(\mathbb{F})$ such that $\Phi(A) = UA_{\tau}U^{-1}$ or $\Phi(A) = UA_{\tau}^{t}U^{-1}$ for all $U \in M_n(\mathbb{F})$. Here A^t denotes the transpose of A and A_{τ} denotes the matrix obtained from A by applying τ on every entries of it. If Φ takes the form $\Phi(A) = UA_{\tau}U^{-1}$ for every A, then, since $\Phi(A^{\dagger}) = \Phi(A)^{\dagger}$, we have $U(A^{\dagger})_{\tau}U^{-1} = (U^{-1})^{\dagger}(A_{\tau})^{\dagger}U^{\dagger}$. It follows that $U^{\dagger}U(A^{\dagger})_{\tau} = (A_{\tau})^{\dagger}U^{\dagger}U$. Furthermore, $JU^*JU(JA^*J)_{\tau} = J(A_{\tau})^*JJU^*JU$. This entails that $U^*JUJ(A^*)_{\tau} = (A_{\tau})^*U^*JUJ$. Replacing A by E_{ij} in this equality, we have $U^*JUJE_{ij} = E_{ij}U^*JUJ$, which implies that $U^*JUJ = \alpha I$ for some nonzero real number α. Thus we get $(A^*)_{\tau} = (A_{\tau})^*$ for all $A \in M_n(\mathbb{F})$ which is clearly implies that $\overline{\tau(\lambda)} = \tau(\overline{\lambda})$. Therefore, either $\tau(\lambda) = \lambda$ for every $\lambda \in \mathbb{F}$ or $\tau(\lambda) = \overline{\lambda}$ for every $\lambda \in \mathbb{F}$. Let $c = \alpha^{-1}$. Then we have either $\Phi(A) = cUAU^{\dagger}$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = cU\overline{A}U^{\dagger}$ for all $A \in M_n(\mathbb{F})$. Here $\overline{A}$ denotes the matrix obtained by taking conjugate operation on every entries of A. For the second case, a similar argument shows that $\Phi(A) = cUA^{t}U^{\dagger}$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = cU(\overline{A})^t U^\dagger$ for all $A \in M_n(\mathbb{F})$. However, it is easily checked that these forms are equivalent to the forms stated in the theorem. The proof is completed.

Theorem 2.2 can be restated in another way.

Theorem 2.2′. *Let H_i be (real or complex) Hilbert spaces and let $J_i \in \mathcal{B}(H_i)$ be invertible self-adjoint operators, $i = 1, 2$. Assume that $\Phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2)$ is a bijective map. Then Φ satisfies*

$$\Phi(AJ_1^{-1}B^*J_1 + J_1^{-1}B^*J_1A) = \Phi(A)J_2^{-1}\Phi(B)^*J_2 + J_2^{-1}\Phi(B)^*J_2\Phi(A)$$

for every pair $A, B \in \mathcal{B}(H_1)$ if and only if there exist a nonzero real number c and a linear or conjugate linear bounded invertible operator $U \in \mathcal{B}(H_1, H_2)$ satisfying $J_1^{-1}U^*J_2U = c^{-1}I$ such that $\Phi(A) = cUAJ_1^{-1}U^*J_2$ for all $A \in \mathcal{B}(H_1)$ or $\Phi(A) = cUJ_1^{-1}A^*U^*J_2$ for all $A \in \mathcal{B}(H_1)$.

In particular, we have

Corollary 2.1. *Let H and K be (real or complex) Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then, Φ satisfies*

$$\Phi(AB^* + B^*A) = \Phi(A)\Phi(B)^* + \Phi(B)^*\Phi(A)$$

for every pair $A, B \in \mathcal{B}(H)$ if and only if there exists a unitary or a conjugate unitary operator $U \in \mathcal{B}(H, K)$ such that either $\Phi(A) = UAU^*$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UA^*U^*$ for all $A \in \mathcal{B}(H)$.

Remark 2.1. Theorem 2.2 still holds if we replace the equation (2.1) by the following equation

$$\Phi\Big(\frac{1}{2}(AB^\dagger + B^\dagger A)\Big) = \frac{1}{2}\Phi(A)\Phi(B)^\dagger + \frac{1}{2}\Phi(B)^\dagger\Phi(A) \tag{2.17}$$

for every pair $A, B \in \mathcal{B}(H)$. Analogues of Theorem 2.2′ and Corollary 2.1 are also true. The proofs are similar and omitted here.

Corollary 2.1 can also be generalized slightly to the C^*-algebra case.

Theorem 2.3. *Let $\mathcal{A}$ be a factor C^*-algebra and let $\Phi : \mathcal{A} \otimes M_2(\mathbb{C}) \to \mathcal{A} \otimes M_2(\mathbb{C})$ be a bijective map. Then*

$$\Phi(AB^* + B^*A) = \Phi(A)\Phi(B)^* + \Phi(B)^*\Phi(A) \tag{2.18}$$

for every pair $A, B \in \mathcal{A} \otimes M_2(\mathbb{C})$ if and only if Φ is a C^*-isomorphism or a conjugate C^*-isomorphism.

Proof. Denote by E_{ij} the unit matrix with entry 1 at (i, j) position and 0 elsewhere, $i, j = 1, 2$. Let $\mathcal{B}_{ij} = \mathcal{A} \otimes E_{ij}$. Then $\mathcal{B} = \mathcal{A} \otimes M_2(\mathbb{C}) = \mathcal{B}_{11} \dot{+} \mathcal{B}_{12} \dot{+} \mathcal{B}_{21} \dot{+} \mathcal{B}_{22}$. Replacing † operation by $*$ operation and applying the similar argument as that in the proof of Theorem 2.1, we see that Φ is additive. Moreover, $\Phi(I) = I$, $\Phi(A^*) = \Phi(A)^*$ and $\Phi(A^2) = \Phi(A)^2$.

For $\lambda \in \mathbb{R}^+$, we note that

$$\Phi(\lambda I) = \Phi\Big(\sqrt{\frac{\lambda}{2}}I\sqrt{\frac{\lambda}{2}}I + \sqrt{\frac{\lambda}{2}}I\sqrt{\frac{\lambda}{2}}I\Big) = \Phi\Big(\sqrt{\frac{\lambda}{2}}I\Big)\Phi\Big(\sqrt{\frac{\lambda}{2}}I\Big)^* + \Phi\Big(\sqrt{\frac{\lambda}{2}}I\Big)\Phi\Big(\sqrt{\frac{\lambda}{2}}I\Big)^* \geq 0$$

and $\Phi(\rho I) = \rho I$ when ρ is a rational number. For every $\lambda \in \mathbb{R}$, there exist two rational number sequences $\{r_n\}$, $\{s_n\}$ such that $r_n \leq \lambda \leq s_n$ and $\lim r_n = \lim s_n = \lambda$ when $n \to \infty$. Thus by the additivity of Φ, we have

$$r_n I = \Phi(r_n I) \leq \Phi(\lambda I) \leq \Phi(s_n I) = s_n I,$$

therefore $\Phi(\lambda I) = \lambda I$. Further we have

$$\Phi(\lambda A) = \Phi\Big(\Big(\frac{\lambda}{2}I\Big)A + A\Big(\frac{\lambda 1}{2}I\Big)\Big) = \Phi\Big(\frac{\lambda}{2}I\Big)\Phi(A) + \Phi(A)\Phi\Big(\frac{\lambda}{2}I\Big) = \lambda\Phi(A),$$

that is, Φ is real linear. The facts

$$\Phi(iI)^* = \Phi(-iI) = -\Phi(iI) \quad \text{and} \quad \Phi(iI)^*\Phi(iI) = \Phi(iI)\Phi(iI)^* = 1$$

together imply that $\Phi(iI)$ is unitary. Noting that

$$\Phi(A(iI)^* + (iI)^*A) = -\Phi(A)\Phi(iI) - \Phi(iI)\Phi(A),$$

we have

$$2\Phi(iA) = \Phi(A)\Phi(iI) + \Phi(iI)\Phi(A). \tag{2.19}$$

Multiplying (2.19) by $\Phi(iI)^*$ from both sides, we get that $\Phi(iI)\Phi(iA) = \Phi(iA)\Phi(iI)$. Therefore $\Phi(iI) = \pm iI$. Moreover by (2.19) we have $\Phi(iA) = \pm i\Phi(A)$. Consequently Φ is linear or conjugate linear. So Φ is a C^*-isomorphism or a conjugate C^*-isomorphism. The proof of this theorem is completed.

Particularly, since every von Neumann algebra is locally matrix, we have the following

Corollary 2.2. *Let $\mathcal{A}$ be a factor von Neumann algebra and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a bijective map. Then*

$$\Phi(AB^* + B^*A) = \Phi(A)\Phi(B)^* + \Phi(B)^*\Phi(A)$$

holds for every pair $A, B \in \mathcal{A}$ if and only if Φ is a C^*-isomorphism or a conjugate C^*-isomorphism.

§ 3. Jordan †-Skew Triple-Multiplicative Maps

In this section, we turn to the discussion of the maps which preserve the †-skew triple-product $AB^\dagger A$ of operators on indefinite inner product spaces. We show that such maps are automatically additive and can be classified completely, too.

Theorem 3.1. *Let H and K be two (real or complex) complete indefinite inner product spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then Φ satisfies*

$$\Phi(AB^\dagger A) = \Phi(A)\Phi(B)^\dagger\Phi(A) \tag{3.1}$$

for every pair $A, B \in \mathcal{B}(H)$ if and only if there exist a real number $c \neq 0$ and bounded invertible linear or conjugate linear operators $U : H \to K$ and $V : K \to H$ with $U^\dagger U = c^{-1}I$ and $V^\dagger V = cI$ such that either $\Phi(A) = UAV$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UA^\dagger V$ for all $A \in \mathcal{B}(H)$.

Proof. Only the "only if" part needs to be checked. Assume that Φ satisfies (3.1). Let us show that Φ has the desired form.

Since Φ is surjective, we can find an $A \in \mathcal{B}(H)$ such that $\Phi(A) = 0$. Therefore $\Phi(0) = \Phi(A0^\dagger A) = \Phi(A)\Phi(0)^\dagger\Phi(A) = 0$.

We assert that $\Phi(I)$ is invertible and $\Phi(I)\Phi(I)^\dagger = \Phi(I)^\dagger\Phi(I) = I$. Let $A = I$ and B be arbitrary in the equality (3.1). We have that

$$\Phi(B^\dagger) = \Phi(I)\Phi(B)^\dagger\Phi(I). \tag{3.2}$$

This yields that $\Phi(I)$ is invertible by the bijectivity of Φ. Multiplying the equality $\Phi(I) = \Phi(I^\dagger) = \Phi(I)\Phi(I)^\dagger\Phi(I)$ by $\Phi(I)^{-1}$, we get that $\Phi(I)\Phi(I)^\dagger = \Phi(I)^\dagger\Phi(I) = I$.

Let $\Psi(A) = \Phi(A)\Phi(I)^\dagger$. Then $\Psi(I) = I$, and

$$\begin{aligned}\Psi(AB^\dagger A) &= \Phi(AB^\dagger A)\Phi(I)^\dagger = \Phi(A)\Phi(B)^\dagger\Phi(A)\Phi(I)^\dagger\\ &= \Phi(A)\Phi(B)^\dagger\Psi(A) = \Psi(A)\Psi(B)^\dagger\Psi(A).\end{aligned}$$

It is obvious that $\Psi(A^\dagger) = \Psi(A)^\dagger$, $\Psi(A^2) = \Psi(A)^2$. Furthermore, we have

$$\Psi(ABA) = \Psi(A(B^\dagger)^\dagger A) = \Psi(A)\Psi(B^\dagger)^\dagger\Psi(A) = \Psi(A)\Psi(B)\Psi(A)$$

for every pair $A, B \in \mathcal{B}(H)$. Thus, by [7], Ψ is a Jordan ring isomorphism. The same reason as that in the proof of Theorem 2.2 gives that Ψ has the form $\Psi(A) = cUAU^\dagger$ ($A \in \mathcal{B}(H)$) or $\Psi(A) = cUA^\dagger U^\dagger$ ($A \in \mathcal{B}(H)$), where $U : H \to K$ is a bounded invertible linear or conjugate linear operator with $U^\dagger U = c^{-1}I$ for some real number $c \neq 0$. Let $V = cU^\dagger\Phi(I)$. Then $V^\dagger V = cI$, and either $\Phi(A) = UAV$ for all $A \in \mathcal{B}(H)$ or $\Phi(T) = UA^\dagger V$ for all $A \in \mathcal{B}(H)$, and this completes the proof.

Like Theorem 2.2 can be restated as Theorem 2.2′, Theorem 3.1 also has a restatement in terms of Hilbert space operators which we omit here. Particularly, we have the following

Corollary 3.1. *Let H and K be two (real or complex) Hilbert spaces and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a bijective map. Then Φ satisfies*

$$\Phi(AB^* A) = \Phi(A)\Phi(B)^*\Phi(A)$$

for every pair $A, B \in \mathcal{B}(H)$ if and only if there exist unitaries or conjugate unitaries $U : H \to K$ and $V : K \to H$ such that either $\Phi(A) = UAV$ for all $A \in \mathcal{B}(H)$ or $\Phi(A) = UA^*V$ for all $A \in \mathcal{B}(H)$.

It follows from Theorem 3.1 that we can get a characterization of Jordan †-isomorphisms by Jordan †-skew triple-multiplicativity. In fact, a bijective map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a Jordan †-isomorphism if and only if it is unital and $\Phi(AB^\dagger A) = \Phi(A)\Phi(B)^\dagger\Phi(A)$ for every A, B.

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