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THE REGULAR SOLUTIONS OF THE ISENTROPIC EULER EQUATIONS WITH DEGENERATE LINEAR DAMPING***

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Abstract

The regular solutions of the isentropic Euler equations with degenerate linear damping for a perfect gas are studied in this paper. And a critical degenerate linear damping coefficient is found, such that if the degenerate linear damping coefficient is larger than it and the gas lies in a compact domain initially, then the regular solution will blow up in finite time; if the degenerate linear damping coefficient is less than it, then under some hypotheses on the initial data, the regular solution exists globally.

Keywords Compressible isentropic Euler equations, Degenerate linear damping, Regular solution, Blow-up, Global existence
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§ 1. Introduction

In this paper we consider the following Cauchy problem of the isentropic Euler equations with degenerate linear damping:

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0,\\ \rho(\partial_t+(u\cdot\nabla))u+\nabla p=-\alpha(t)\rho u\end{cases}\tag{1.1}$$

with initial data

$$\rho(x,0)=\rho_0(x),\qquad u(x,0)=u_0(x),\tag{1.2}$$

where ρ, u, p represent the density, velocity and pressure of a polytropic gas respectively, the equation of state is $p=k^2\rho^\gamma$, $1<\gamma\le 3$, γ is the adiabatic coefficient, $k>0$. $x\in\mathbb{R}^n, n\ge 1$. $\alpha(t)$ is the damping coefficient, so $\alpha(t)\ge 0$, $\alpha(t)\to 0$ when $t\to\infty$, we say that the damping $\alpha(t)\rho u$ is a degenerate linear damping. We assume that $\alpha(t)$ is Lipschitz continuous in this paper.

Previous works in the field of Euler equations with damping are concentrated in equations with constant damping coefficient, that is, $\alpha(t)$ is a constant α. When vacuum does not

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occurs, especially the density has a positive infimum, the global classical solutions exist if the initial data is small enough in some sense (see [10, 11]). The damping $\alpha\rho u$ plays an important and positive role in the existence of global classical solutions, since the supremum of the initial data for the global existence of a classical solution is related to α. When some functionals related to the initial data are large enough, the classical solution does not exist globally, or we say the classical solution blows up in finite time (see [9, 10]).

When vacuum occurs, we usually study the regular solutions of (1.1), such as [5, 7]. Take

$$\pi = \frac{2k}{\gamma - 1}\sqrt{\gamma}\ \rho^{(\gamma-1)/2}.$$

Then the equations (1.1) can be written as:

$$\begin{cases} (\partial_t + u\cdot\nabla)\pi + \dfrac{\gamma-1}{2}\pi \operatorname{div} u = 0, \\ (\partial_t + (u\cdot\nabla))u + \dfrac{\gamma-1}{2}\pi\nabla\pi = -\alpha(t)u. \end{cases} \tag{1.3}$$

Let $\rho = 0$, that is $\pi = 0$. Then the velocity u satisfies $(\partial_t + (u\cdot\nabla))u = -\alpha(t)u$. This leads to the following definition.

Definition 1.1. *We say a solution* (ρ, u) *of* (1.1) *is a regular solution on* $\mathbb{R}^n \times [0, T)$ *if and only if*

(i) $(\rho, u) \in C^1(\mathbb{R}^n \times [0, T)),\ \rho \geq 0$,

(ii) $p\,(\rho, S)(x, t) \in C^1(\mathbb{R}^n \times [0, T))$,

(iii) *in outside of the support of* ρ, u *satisfies*

$$(\partial_t + (u\cdot\nabla))u = -\alpha(t)u. \tag{1.4}$$

We first investigate the case $\alpha(t) \equiv \alpha > 0$ in (1.1). We prove that if the gas locates initially in a compact domain, then the regular solution of (1.1) does not exist globally (see Theorem 2.1). But when $\alpha(t) \equiv 0$ and the initial gas locates in a compact domain, then under some assumptions on initial velocity (see (H1), (H2)) and the initial density, the regular solution of (1.1) exists globally (see [2, Theorem 1]). This motivates us to consider the Euler equations with degenerate linear damping, that is, $t \to \infty,\ \alpha(t) \to 0$. The other forms of degenerate damping are not considered in this paper. When $\alpha(t)$ tends to zero slowly, we can also prove the regular solution of (1.1) blows up in finite time (see Theorem 2.2). When $\alpha(t)$ tends to zero fast enough, under the same assumptions on the initial data as that of it in [2], we prove that the regular solution of (1.1) exists globally (see Theorem 3.1). Furthermore, $\alpha(t)$ discussed in Theorem 2.2 and Theorem 3.1 is almost complete.

Remark 1.1. From our work we discover that the damping plays a negative role in the existence of regular solution of Euler equations if the vacuum occurs. This is quite different from the case of non-vacuum.

§ 2 . Non-existence of the Global Regular Solution

2.1. Linear damping case

We first consider the Euler equations with linear damping. Our result is as follows.

Theorem 2.1. *Suppose* (ρ, u) *is a regular solution of* (1.1) *on* $\mathbb{R}^n \times [0,T)$ *corresponding to initial density* ρ_0*,* $\mathrm{supp}\rho_0$ *is compact and the total mass* $m = \int \rho_0(x)dx > 0,\ \alpha(t) \equiv \alpha > 0$*. Then the life span* T *must be finite.*

Proof. The proof is similar to that in [5, 7]. Let $\Omega(t) = \mathrm{supp}\rho(x,t)$, $\Omega_0 = \mathrm{supp}\rho_0, R_0 = \sup\limits_{x\in\Omega_0} |x|$. For every $x' \in \partial\Omega(t_0)$, there exists a curve $x(t)$ and a point $x_0 \in \partial\Omega_0$ such that

$$\frac{dx(t)}{dt} = u(x(t),t), \quad x(0) = x_0, \quad x(t_0) = x'.$$

So

$$\frac{d^2x(t)}{dt^2} = u_t(x(t),t) + (u(x(t),t)\cdot\nabla)u(x(t),t) = -\alpha u(x(t),t) = -\alpha\frac{dx(t)}{dt},$$

and

$$x(t) = x_0 + \frac{1-e^{-\alpha t}}{\alpha}u_0(x_0). \tag{2.1}$$

This indicates that $\Omega(t)$ always stays in a compact domain. Let $R^* = R_0 + \sup\limits_{x\in\partial\Omega_0} \frac{\{|u_0(x)|\}}{\alpha}$. Then from (2.1) we have

$$R(t) = \sup_{x\in\Omega(t)} |x| = \sup_{x\in\partial\Omega(t)} |x| \le R^*.$$

We define

$$H(t) = \frac{1}{2}\int_{\Omega(t)} \rho(x,t)|x|^2 dx.$$

Note that $\rho|_{\partial\Omega(t)} = p|_{\partial\Omega(t)} = 0$. From the equations (1.1) we have

$$\begin{aligned} H'(t) &= \frac{1}{2}\int_{\Omega(t)} \rho_t(x,t)|x|^2dx = -\frac{1}{2}\int_{\Omega(t)} \mathrm{div}(\rho \mathrm{u})|x|^2dx = \int_{\Omega(t)} \rho u\cdot xdx, \\ H''(t) &= \int_{\Omega(t)} (\rho u)_t\cdot xdx = \int_{\Omega(t)} -[\mathrm{div}(\rho\mathrm{u})u + (\rho u\cdot\nabla)u + \nabla p + \alpha\rho u]\cdot xdx \\ &= \int_{\Omega(t)} (\rho(x,t)|u|^2 + np(\rho,s))dx - \alpha\int_{\Omega(t)} \rho u\cdot xdx. \end{aligned}$$

So we get

$$H''(t) + \alpha H'(t) = \int_{\Omega(t)} (\rho(x,t)|u|^2 + np(\rho,s))dx \ge \int_{\Omega(t)} np(\rho,s)dx.$$

By Hölder inequality

$$m = \int_{\Omega(t)} \rho dx \le \Big(\int_{\Omega(t)} \rho^\gamma dx\Big)^{1/\gamma}\Big(\int_{\Omega(t)} dx\Big)^{1/\gamma'} = (V(t))^{1/\gamma'}\Big(\int_{\Omega(t)} \rho^\gamma dx\Big)^{1/\gamma},$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Then we get from the equation of state

$$n\int_{\Omega(t)} p(\rho,s)dx = nk^2\int_{\Omega(t)} \rho^\gamma dx \ge nk^2m^\gamma(V(t))^{-\gamma/\gamma'} \ge nk^2m^\gamma(\omega_n(R^*)^n)^{-\gamma/\gamma'} = \eta > 0,$$

where ω_n represents the volume of the unit ball in $\mathbb{R}^n$. So we get the following estimate:

$$H''(t) + \alpha H'(t) \geq \eta.$$

Solving this inequality we obtain

$$H(t) \geq H(0) + \frac{1}{\alpha}\Big(H'(0) - \frac{\eta}{\alpha}\Big)(1 - e^{-\alpha t}) + \frac{\eta}{\alpha}t.$$

On the other hand,

$$H(t) = \frac{1}{2}\int_{\Omega(t)} \rho(x,t)|x|^2 dx \leq \frac{m}{2}(R^*)^2.$$

So

$$\frac{\eta}{\alpha}t \leq \frac{m}{2}(R^*)^2 - H(0) - \frac{1}{\alpha}\Big(H'(0) - \frac{\eta}{\alpha}\Big)(1 - e^{-\alpha t}) \leq \frac{m}{2}(R^*)^2 + \frac{1}{\alpha}\Big(|H'(0)| + \frac{\eta}{\alpha}\Big).$$

Then T must be finite.

Theorem 2.1 points out that no matter what the positive constant α is, the regular solution of (1.1) must blow up in finite time. As indicated in the first section, the Euler equations with compactly supported initial density may possess global regular solution, so we consider the degenerate linear damping.

2.2. Degenerate linear damping case

The equality (2.1) implies that the smaller α is, the larger R^* is. This is also true for $\alpha(t)$. So, if $\alpha(t)$ tends to zero slowly enough, the regular solution will also blow up in finite time.

Theorem 2.2. *Suppose that (ρ, u) is a regular solution of (1.1) on $\mathbb{R}^n \times [0,T)$, $\alpha(t) \sim \frac{A}{(1+t)^\theta}$ $(t \to \infty)$, $0 < \theta < 1$, $A > 0$ or $\alpha(t) = \frac{A}{1+t}$, $A > \frac{n(\gamma-1)}{2+n(\gamma-1)}$, $\mathrm{supp}\rho_0$ is compact and the total mass $m > 0$. Then the life span T must be finite.*

Proof. The definitions of $\Omega(t), \Omega_0, R_0, x(t)$ are exactly as those in the proof of Theorem 2.1. We have $\frac{d^2x(t)}{dt^2} = -\alpha(t)\frac{dx(t)}{dt}$ and

$$x(t) = x_0 + u_0(x_0)\int_0^t e^{-\int_0^s \alpha(\tau)d\tau} ds = x_0 + u_0(x_0)G(t). \tag{2.2}$$

If $\alpha(t) \to 0$ $(t \to \infty)$ and $G(t) = \int_0^t e^{-\int_0^s \alpha(\tau)d\tau} ds$ is bounded, then $\mathrm{supp}\,\Omega(t)$ will stay in a compact domain $\overline{B(O,R^*)}$, and we will easily prove the blow-up of the regular solution in a way similar to the proof of Theorem 2.1. We also define $H(t) = \frac{1}{2}\int_{\Omega(t)} \rho(x,t)|x|^2 dx$. Then we can get

$$H''(t) + \alpha(t)H'(t) = \int_{\Omega(t)} (\rho(x,t)|u|^2 + np(\rho,s))dx \geq \int_{\Omega(t)} np(\rho,s)dx.$$

From the equation of state and the boundedness of $G(t)$, we have

$$n\int_{\Omega(t)} p(\rho,s)dx \geq nk^2\Big(\int_{\Omega(t)} \rho dx\Big)^\gamma \Big(\int_{\Omega(t)} dx\Big)^{-\frac{\gamma}{\gamma'}} \geq nk^2 m^\gamma (V(t))^{-\gamma/\gamma'} \geq \eta > 0,$$

so

$$H''(t) + \alpha(t)H'(t) \geq \eta.$$

Multiplying the last inequality by $e^{\int_0^t \alpha(s)ds}$ and then integrating it on $[0,t]$, we have

$$e^{\int_0^t \alpha(s)ds} H'(t) - H'(0) \geq \eta \int_0^t e^{\int_0^s \alpha(\tau)d\tau} ds.$$

By the boundedness of $G(t)$ we have $e^{-\int_0^t \alpha(s)ds} \to 0$ $(t \to \infty)$, so $e^{\int_0^t \alpha(s)ds} \to \infty$ $(t \to \infty)$, and we can easily prove $\lim_{t\to\infty} \frac{\int_0^t e^{\int_0^s \alpha(\tau)d\tau} ds}{e^{\int_0^t \alpha(s)ds}} = +\infty$. There exists $T_0 > 0$ such that $H'(t) \geq H'(0)e^{-\int_0^t \alpha(s)ds} + 2 \geq 1$ when $t > T_0$. Then by integrating it on $[T_0, t]$ we obtain

$$H(t) \geq H(T_0) + t - T_0.$$

On the other hand, we have

$$H(t) = \frac{1}{2}\int_{\Omega(t)} \rho(x,t)|x|^2 dx \leq \frac{m}{2}(R^*)^2.$$

So t must be finite.

If $\alpha(t) = \frac{A}{1+t}$, $A > 1$, then $g(t) = e^{-\int_0^t \alpha(s)ds} = \frac{1}{(1+t)^A}$, $G(t) = \int_0^t g(s)ds < \int_0^\infty g(t)dt < +\infty$. Or if $\alpha(t) \sim \frac{A}{(1+t)^\theta}$ $(t \to \infty)$, $0 < \theta < 1$, $A > 0$, then it is easy to deduce that $\int_0^\infty g(t)dt < \infty$ by the fact that $\lim_{t\to\infty} \frac{\int_0^t \alpha(s)ds}{\frac{A}{1-\theta}[(1+t)^{(1-\theta)} - 1]} = \lim_{t\to\infty} \frac{\alpha(t)}{\frac{A}{(1+t)^\theta}} = 1$. $G(t)$ is bounded on both these occasions.

Then we consider the case of $\alpha(t) = \frac{A}{1+t}$, $\frac{n(\gamma-1)}{2+n(\gamma-1)} < A \leq 1$. First we suppose $A < 1$. Then $g(t) = \frac{1}{(1+t)^A}$, $G(t) = \frac{1}{1-A}[(1+t)^{(1-A)} - 1]$, and

$$R(t) = \sup_{x\in\Omega(t)} |x| = \sup_{x\in\partial\Omega(t)} |x| \leq R_0 + G(t) \sup_{x\in\partial\Omega(t)} |u_0(x)| \leq R_0 + BG(t) = d^*(t).$$

Repeat the deduction above we obtain

$$\begin{aligned} H''(t) + \alpha(t)H'(t) &= \int_{\Omega(t)} (\rho(x,t)|u|^2 + np(\rho,s))dx \geq \int_{\Omega(t)} np(\rho,s)dx \geq nk^2 m^\gamma (V(t))^{-\gamma/\gamma'} \\ &\geq nk^2 m^\gamma (\omega_n (d^*(t))^n)^{-(\gamma-1)} = C(R_0 + BG(t))^{-n(\gamma-1)}, \end{aligned}$$

so we have

$$\frac{d}{dt}\left(e^{\int_0^t \alpha(s)ds} H'(t)\right) \geq Ce^{\int_0^t \alpha(s)ds}(R_0 + BG(t))^{-n(\gamma-1)}.$$

Integrating the last inequality on $[0,t]$, we have

$$\begin{aligned} H'(t) &\geq e^{-\int_0^t \alpha(s)ds} H'(0) + e^{-\int_0^t \alpha(s)ds} \int_0^t Ce^{\int_0^s \alpha(\tau)d\tau}(R_0 + BG(s))^{-n(\gamma-1)} ds \\ &= H'(0)g(t) + F(t). \end{aligned}$$

Again integrating the last inequality on $[0,t]$, we have

$$H(t) \geq H(0) + H'(0)G(t) + \int_0^t F(s)ds. \tag{2.3}$$

On the other hand, we have

$$H(t) = \frac{1}{2}\int_{\Omega(t)} \rho(x,t)|x|^2 dx \leq \frac{m}{2}(d^*(t))^2 \leq \frac{m}{2}(R_0 + BG(t))^2. \tag{2.4}$$

We calculate the following limit

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_0^t F(s)ds}{\frac{m}{2}(R_0+BG(t))^2} &= \lim_{t\to\infty}\frac{F(t)}{m(R_0+BG(t))g(t)}\\ &= \lim_{t\to\infty}\frac{\int_0^t Ce^{\int_0^s \alpha(\tau)d\tau}(R_0+BG(s))^{-n(\gamma-1)}ds}{m(R_0+BG(t))}\\ &= \lim_{t\to\infty}\frac{Ce^{\int_0^t \alpha(\tau)d\tau}(R_0+BG(t))^{-n(\gamma-1)}}{mBg(t)}\\ &= \lim_{t\to\infty}\frac{C(1+t)^{2A}}{mB(R_0+\frac{B}{1-A}((1+t)^{1-A}-1))^{n(\gamma-1)}}\\ &= \frac{C}{mB}\Big(\frac{1-A}{B}\Big)^{n(\gamma-1)}\lim_{t\to\infty}(1+t)^{2A-(1-A)n(\gamma-1)}.\end{aligned}$$

So if $2A-(1-A)n(\gamma-1)>0$, that is, $A>\frac{n(\gamma-1)}{2+n(\gamma-1)}$, we have

$$\lim_{t\to\infty}\frac{\int_0^t F(s)ds}{\frac{m}{2}(R_0+BG(t))^2}=+\infty. \tag{2.5}$$

From (2.3)–(2.5) we can draw the conclusion that t must be finite. So we have proved that if $1>A>\frac{n(\gamma-1)}{2+n(\gamma-1)}$ then the life span T of the regular solution must be finite.

When $\alpha(t)=\frac{1}{1+t}$, we have $g(t)=\frac{1}{(1+t)}, G(t)=\ln(1+t)$. The same deduction gives that the life span T of the regular solution must be finite, we omit the proof.

From the proof of Theorem 2.2, we can also conclude that if $A=\frac{n(\gamma-1)}{2+n(\gamma-1)}$ and

$$\frac{C}{mB}\Big(\frac{1-A}{B}\Big)^{n(\gamma-1)}=\frac{nk^2}{B}\Big(\frac{m}{\omega_n}\Big)^{(\gamma-1)}\Big(\frac{1-A}{B}\Big)^{n(\gamma-1)}>1,$$

then the life span T of the regular solution must be finite.

Remark 2.1. Suppose that (ρ,u) is a regular solution of (1.1) on $\mathbb{R}^n\times[0,T)$, $\alpha(t)=\frac{A}{1+t}$, $A=\frac{n(\gamma-1)}{2+n(\gamma-1)}$, $\mathrm{supp}\rho_0$ is compact. There exists a constant $m^*>0$ depending on n,γ and $\sup\limits_{x\in\partial\Omega_0}|u_0(x)|$, such that if the total mass $m>m^*$, the life span T of the regular solution must be finite.

§ 3 . Global Existence of the Regular Solution

3.1. Main results on global existence

Because the classical solution of the equations (1.3) corresponds to the regular solution of the equations (1.1), we will prove the existence of the classical solution of (1.3) to obtain the existence of the regular solution of (1.1). Now we study the Cauchy problem of (1.3) with initial data

$$\pi(x,0)=\pi_0(x)=\frac{2k}{\gamma-1}\sqrt{\gamma}\ \rho_0^{(\gamma-1)/2}(x),\qquad u(x,0)=u_0(x). \tag{3.1}$$

Since vacuum occurs, $(0, \bar{u})$ is considered as an approximate solution of the Cauchy problem (1.3), (3.1). Thus the approximate problem is as follows:

$$\begin{cases} \bar{u}_t + (\bar{u} \cdot \nabla)\bar{u} = -\alpha(t)\bar{u}, \\ \bar{u}(x, 0) = u_0(x). \end{cases} \tag{3.2}$$

Solving it, we obtain

$$\bar{u}(X(x_0, t), t) = u_0(x_0)g(t), \tag{3.3}$$

where

$$X(x_0, t) = x_0 + G(t)u_0(x_0), \tag{3.4}$$

$$g(t) = e^{\int_0^t -\alpha(s)ds}, \quad G(t) = \int_0^t g(s)ds. \tag{3.5}$$

We differentiate (3.3) with respect to x_0 and get

$$(I + G(t)Du_0(x_0))D\bar{u}(X(x_0, t), t) = Du_0(x_0)g(t).$$

In order to solve for $D\bar{u}$, the matrix $I + G(t)Du_0(x_0)$ must be invertible for all t. And Theorem 2.2 implies that $G(t) \to +\infty$ $(t \to \infty)$ is necessary for the existence of regular solution of (1.1), so we need the spectrum of Du_0 to be bounded away from real negative numbers. Thus we assume that u_0 satisfies

(H1) $Du_0 \in L^\infty(\mathbb{R}^n)$, $D^2u_0 \in H^{m-1}(\mathbb{R}^n)$,

(H2) There exists $\delta > 0$, such that for all $x \in \mathbb{R}^n$, $\mathrm{dist}(\mathrm{Sp}Du_0(x), \mathbb{R}_-) \geq \delta$,

where dist represents the distance and Sp the spectrum. From the assumptions (H1), (H2), we obtain the existence of the global classical solution of (3.2). But the hypothesis (H2) implies that u is not in the Sobolev space $H^m(\mathbb{R}^n)$. Also by (H1), we have $u_0 \in H^{m+1}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\bar{u} \in C([0, \infty), H^{m+1}_{\mathrm{loc}}(\mathbb{R}^n)) \cap C^1([0, \infty), H^m_{\mathrm{loc}}(\mathbb{R}^n)).$$

We denote $X = \{f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \mid Du_0 \in L^\infty(\mathbb{R}^n),\ D^2u_0 \in H^{m-1}(\mathbb{R}^n)\}$, its norm is denoted by $\|\cdot\|_X$. We also denote by $|\cdot|_q$ the norm of $L^q(\mathbb{R}^n)$, by $\|\cdot\|_0$ the norm of $L^2(\mathbb{R}^n)$, and by $\|\cdot\|_m$ the norm of $H^m(\mathbb{R}^n)$. Now we state the main results in this section as follows:

Theorem 3.1. *Let* $\alpha(t) = \frac{A}{1+t}$, $0 < A < 1 - \frac{1}{a}$ *or* $\alpha(t) = O\big(\frac{A}{(1+t)^\theta}\big)$ $(t \to \infty)$, $\theta > 1$, *where* $a = \min\{2, 1 + \frac{\gamma-1}{2}n\}$. *Suppose that* $m > 1 + \frac{n}{2}$, u_0 *satisfies the hypotheses* (H1), (H2), $\rho_0^{(\gamma-1)/2} \in H^m(\mathbb{R}^n)$ *and is sufficiently small,* $\mathrm{supp}\rho_0$ *is compact. Then the Cauchy problem* (1.1), (1.2) *has a global regular solution* (ρ, u) *satisfying*

$$(\rho^{(\gamma-1)/2}, u - \bar{u})^T \in C([0, \infty), H^m(\mathbb{R}^n)) \cap C^1([0, \infty), H^{m-1}(\mathbb{R}^n)).$$

Indeed we prove the existence of the classical solution of the Cauchy problem (1.3), (3.1). To prove Theorem 3.1, we adapt the proof of [2] with some improvements to our case. And the solution (π, u) we obtained here has the following estimate.

Theorem 3.2. *Under the same assumptions in Theorem* 3.1, *the solution in Theorem* 3.1 *satisfies*

(i) *for all* $1 \le k \le m, t \ge 0$, $\|D^k U(t)\|_0 \le C(1+G(t))^{-(k+r)}$,
(ii) *for all* $t \ge 0$, $|U|_\infty \le C(1+G(t))^{1-a}$,
(iii) *for all* $t \ge 0$, $|DU|_\infty \le C(1+G(t))^{-a}$,
where $U = (\pi, u-\bar u)^T$, $r = \min\{1, \frac{\gamma-1}{2}n\} - \frac{n}{2} = a - 1 - \frac{n}{2}$, C *depends on* δ, $\|u_0\|_X$, $\|\pi_0\|_m$ *and* $\int_0^\infty (1+G(t))^{-a}dt$. *If* $\alpha(t) = O(\frac{A}{(1+t)^\theta})$, $\theta > 1$, *then* $G(t) \sim t$; *and if* $\alpha(t) = \frac{A}{1+t}$, $0 < A < 1 - \frac{1}{a}$, *then* $G(t) \sim t^{1-A}$.

Remark 3.1. If $n(\gamma-1) \le 2$, then $a = \min\{2, 1+\frac{\gamma-1}{2}n\} = 1 + \frac{\gamma-1}{2}n, 1 - \frac{1}{a} = \frac{n(\gamma-1)}{2+n(\gamma-1)}$, it seems hardly possible to find other $\alpha(t)$ than those discussed in Theorem 2.2 and Theorem 3.1. For almost all the polytropic gases, $n(\gamma-1) \le 2$, so our conclusions show that $\alpha(t) = \frac{n(\gamma-1)}{(2+n(\gamma-1))(1+t)}$ is the critical degenerate damping coefficient.

3.2. Local existence

We now prove the local existence of the Cauchy problem (1.3), (3.1). It is convenient to write the equations (1.3) as

$$\partial_t V + \sum_{i=1}^n A^i(V)\partial_{x_i} V = -\alpha(t)\widetilde V,$$

where $V = (\pi, u^1, \cdots, u^n)^T, \widetilde V = (0, u^1, \cdots, u^n)^T, A^i(V)$ are $(n+1)\times(n+1)$ matrices:

$$A^i(V) = \begin{pmatrix} u^i & 0 & \cdots & \frac{\gamma-1}{2}\pi & \cdots & 0 \\ 0 & u^i & 0 & & & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ \frac{\gamma-1}{2}\pi & 0 & \cdots & u^i & \cdots & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & u^i \end{pmatrix}.$$

Note that every element in $A^i(V)$ is linear in V, which will simplify the calculation.

Since $u_0 \bar\in H^m(\mathbb{R}^n)$, we cannot apply directly the theory of symmetric hyperbolic system, as in [3, 6]. Let $R > 0$ be such that $\mathrm{supp}\rho_0 = \mathrm{supp}\pi_0 \subset B(O,R)$. We set $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\varphi = 1$ on $B(O, R+2\eta)$, where η is a positive number. Then we consider the Cauchy problem (1.3) with the initial data $(\pi_0^\varphi, u_0^\varphi)^T = (\pi_0, u_0\varphi)^T \in H^m(\mathbb{R}^n)$. There exists a solution $(\pi^\varphi, u^\varphi)^T \in C([0,T_{ex}), H^m(\mathbb{R}^n)) \cap C^1([0,T_{ex}), H^{m-1}(\mathbb{R}^n))$, that is, $(\pi^\varphi, u^\varphi)^T$ is a classical solution of (1.3) on $[0, T_{ex})$. Note that $(0, \bar u)^T$ is also a classical solution of (1.3) with initial data $(0, u_0)^T$.

For every $\varepsilon > 0$, we set $M = \sup\limits_{0\le t\le T_{ex}-\varepsilon} (\frac{\gamma-1}{2}|\pi^\varphi|_\infty + |u^\varphi|_\infty)$, $T = \min\{T_{ex}-\varepsilon, \frac{\eta}{2M} - \varepsilon\}$. Let

$$\Omega = \{(x,t) \mid 0 \le t \le T, x \in B(O, R+\eta+Mt)\},$$

we define

$$(\pi, u)^T = \begin{cases} (\pi^\varphi, u^\varphi)^T & \text{in } \Omega, \\ (0, \bar u)^T & \text{in } \Omega^c = (\mathbb{R}^n \times [0,T])\backslash\Omega. \end{cases}$$

Then we will prove that $(\pi, u)^T$ is a solution of Cauchy problem (1.3), (3.1) on $[0,T]$ with initial data $(\pi_0, u_0)^T$. In fact, $(\pi, u)^T$ is a solution of (1.3) in Ω and Ω^c. We need only to prove that $(\pi, u)^T$ is continuous across the lateral boundary $\mathcal{M}$ of Ω. Set

$$D = \{(x,t) \mid 0 \le t \le T, x \in B(x_0, \eta - Mt), x_0 \in S(O, R+\eta)\}.$$

We will prove that $(\pi^\varphi, u^\varphi)^T$ and $(0, \bar{u})^T$ are equal on domain D, and it is easily deduced that $\mathcal{M}$ lies in the interior of D. Since $(\pi^\varphi, u^\varphi)^T$ and $(0, \bar{u})^T$ belong to H^m on every horizontal section of D, so if they are equal on D, then $(\pi, u)^T$ is a solution of Cauchy problem of (1.3), (3.1) on $[0, T]$. We will utilize the local uniqueness of the equations (1.3) to prove this. See Proposition 1 in [2] and the proof after it, we omit the detail, since the term $-\alpha(t)u$ does not influence this property. And the solution $(\pi, u)^T$ satisfies

$$(\pi, u - \bar{u})^T \in C([0,T), H^m(\mathbb{R}^n)) \cap C^1([0,T), H^{m-1}(\mathbb{R}^n)).$$

3.3. Estimates for the approximate solution

Under the assumptions (H1), (H2), we have obtained the global classical solution of the approximate equations. Furthermore, the solution has the following property.

Proposition 3.1. *Suppose that u_0 satisfies* (H1), (H2). *Then the global solution $\bar{u}$ saitsfies*

(i) $D\bar{u}(x,t) = \left(\frac{I}{1+G(t)} + \frac{K(x,t)}{(1+G(t))^2}\right) g(t)$, *for all* $x \in \mathbb{R}^n, t \ge 0$,

(ii) $|D^2 \bar{u}|_\infty \le C(1+G(t))^{-3} g(t)$,

(iii) $\|D^l \bar{u}\|_0 \le C_l g(t)(1+G(t))^{n/2-l-1}, \quad 2 \le l \le m+1$,

where $K(x,t)$ is a function matrix of $n \times n$ and $|K(x,t)|_\infty \le \widetilde{C}$, constants $C, C_l, \widetilde{C}$ only depend on $n, m, \delta, \|u_0\|_X$.

Before the proof of this proposition, we give the following lemma.

Lemma 3.1. *Suppose that u_0 satisfies* (H1), (H2). *Then*

(i) *there exists a constant $K = K(\delta, \|Du_0\|_\infty)$, such that $|(Du_0)^{-1}|_\infty \le K$;*

(ii) *also there exists a constant $L = L(\delta, \|Du_0\|_\infty)$, such that $|(I + G(t)Du_0)^{-1}|_\infty \le \frac{L}{1+G(t)}$.*

We omit the proof of this lemma, see Remark 1 of [2]. The difference is that $G(t)$ replaces t.

Proof of Proposition 3.1. We differentiate (3.3) with respect to x_0 to obtain

$$(I + G(t)Du_0(x_0))D\bar{u}(X(x_0,t),t) = Du_0(x_0)g(t),$$

so

$$D\bar{u}(X(x_0,t),t) = (I + G(t)Du_0(x_0))^{-1}Du_0(x_0)g(t). \tag{3.6}$$

Let

$$D\bar{u}(X(x_0,t),t) = \Big(\frac{I}{1+G(t)} + \frac{K(x_0,t)}{(1+G(t))^2}\Big)g(t),$$

where $K(x_0,t) = (1+G(t))^2(I + G(t)Du_0(x_0))^{-1}Du_0(x_0) - (1+G(t))I$. Since $Du_0 \in L^\infty$, by Lemma 3.1(ii), $K(x_0,t)$ is bounded when $G(t)$ is bounded. And if $G(t)$ is large enough, we have $|(G(t))^{-1}(Du_0(x_0))^{-1}|_\infty \ll 1$, so

$$\begin{aligned}K(x_0,t) &= \frac{(1+G(t))^2}{G(t)}(I+(G(t))^{-1}(Du_0(x_0))^{-1})^{-1}-(1+G(t))I\\ &= \frac{(1+G(t))^2}{G(t)}\Big(I-\frac{(Du_0(x_0))^{-1}}{G(t)}+O\Big(\frac{1}{(G(t))^2}\Big)\Big)-(1+G(t))I\\ &= \frac{1+G(t)}{G(t)}I-\Big(\frac{1+G(t)}{G(t)}\Big)^2(Du_0(x_0))^{-1}+O\Big(\frac{1}{G(t)}\Big).\end{aligned}$$

That is, if $G(t)$ is large enough, then $K(x_0,t)$ is bounded. So $K(x_0,t)$ is also bounded for all x_0,t and $K(x,t)$ is bounded for all x,t. So we have proved (i).

Differentiate the equality (3.6) with respect to x_0. In a way similar to the proof of Proposition 2(iii) in [2], replacing $I+tD\bar{u}$ with $I+G(t)D\bar{u}$, we can obtain

$$|D^2\bar{u}|_\infty \le C(1+G(t))^{-3}g(t),$$

where $g(t)$ comes from $g(t)$ in the right-hand side of (3.6). Differentiating (3.6) with respect to $D^k_{x_0}$, also by the the same induction as the proof of Proposition 2(ii) in [2], we can prove (iii). $(1+G(t))^{n/2}$ in the conclusions is brought about by the Jacobi determinant of the transformation $x\to X(x_0,t)$. We omit the proof.

3.4. Energy estimates

Now we estimate $(\pi,u-\bar{u})\in H^m(\mathbb{R}^n)$ in order to obtain the global existence of classical solution. Let $w=u-\bar{u}$. By (1.3) and (3.2) we know that (π,w) satisfies

$$\begin{cases}(\partial_t+u\cdot\nabla)\pi+\dfrac{\gamma-1}{2}\pi\mathrm{div}w=-\bar{u}\cdot\nabla\pi-\dfrac{\gamma-1}{2}\pi\mathrm{div}\bar{u},\\ (\partial_t+(w\cdot\nabla))w+\dfrac{\gamma-1}{2}\pi\nabla\pi=-(\bar{u}\cdot\nabla)w-(w\cdot\nabla)\bar{u}-\alpha(t)w.\end{cases}\tag{3.7}$$

Let $U=(\pi,w)^T,\overline{U}=(0,\bar{u})^T,\widetilde{U}=(0,w)^T$. Then we write (3.7) into

$$\partial_tU+\sum_{i=1}^n A^i(U)\partial_{x_i}U=-B(D\overline{U},U)-\sum_{i=1}^n\bar{u}^i\partial_{x_i}U-\alpha(t)\widetilde{U},\tag{3.8}$$

where

$$B(D\overline{U},U)=\begin{pmatrix}\frac{\gamma-1}{2}\pi\mathrm{div}\bar{u}\\ (w\cdot\nabla)\bar{u}\end{pmatrix}.\tag{3.9}$$

Define

$$Y_k=\Big(\int_{\mathbb{R}^n}D^kU\cdot D^kU dx\Big)^{1/2},\qquad Z(t)=\sum_{k=0}^m(1+G(t))^kY_k,\tag{3.10}$$

where we introduce the powers $(1+G(t))^k$ in $Z(t)$ to balance the different decay rates of Y_k, which are suggested by $\bar{u}$. Since $U(t),Z(t)\in H^m(\mathbb{R}^n),m>1+\frac{n}{2}$, we have $D^lU\in L^\infty$, $l\le 1$. Before we give the energy estimates, we cite some known results on Gagliardo-Nirenberg type inequalities (see Theorem on page 125 in [8]) and write them as the following Lemmas 3.2 and 3.3.

Lemma 3.2. (Gagliardo-Nirenberg Inequality) *Suppose that $z \in H^i(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and integers i, j satisfy $0 \leq j \leq i$. Then $\partial^j z \in L^{2i/j}(\mathbb{R}^n)$ and there exists a constant C only depending on n, i, j such that*

$$|\partial^j z|_{2i/j} \leq C|z|_\infty^{1-j/i} \|D^i z\|_0^{j/i}. \tag{3.11}$$

Lemma 3.3. *Suppose that $z \in H^i(\mathbb{R}^n)$ and $i > \frac{n}{2}$. Let $\theta = \frac{n}{2i}$. Then*

$$|z|_\infty \leq C\|z\|_0^{1-\theta} \|D^i z\|_0^\theta. \tag{3.12}$$

From Lemma 3.3, we can deduce easily the following lemma.

Lemma 3.4. *Since $U(t) \in H^m(\mathbb{R}^n), m > 1 + \frac{n}{2}$, there exists a constant $C > 0$ depending on n, m only, such that*

$$|D^l U(t)|_\infty \leq C(1 + G(t))^{-l-n/2} Z(t), \qquad l = 0, 1.$$

We now estimate Y_k. Differentiate (3.8) with respect to D^k and make inner product with $D^k U$, then integrate it on $\mathbb{R}^n$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} D^k U \cdot D^k U(x,t)dx + \alpha(t)\int_{\mathbb{R}^n} D^k w \cdot D^k w(x,t)dx \\
&= \int_{\mathbb{R}^n} R_k(U)(x,t)dx + \int_{\mathbb{R}^n} S_k(\overline{U}, U)(x,t)dx.
\end{aligned} \tag{3.13}$$

Here we have utilized the symmetry of A^i to integrate by parts, and

$$\begin{aligned}
R_k(U) = {} & \frac{1}{2}\sum_{i=1}^n D^k U \cdot \partial_{x_i} A^i(U) D^k U \\
& - D^k U \cdot \sum_{i=1}^n (D^k(A^i(U)\partial_{x_i} U) - A^i(U)\partial_{x_i} D^k U),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
S_k(\overline{U}, U) = {} & -D^k U \cdot D^k B(D\overline{U}, U) + \frac{1}{2}\sum_{i=1}^n \partial_{x_i}\bar{u}^i D^k U \cdot D^k U \\
& - D^k U \cdot \sum_{i=1}^n (D^k(\bar{u}^i \partial_{x_i} U) - \bar{u}^i \partial_{x_i} D^k U).
\end{aligned} \tag{3.15}$$

We now devote ourselves to estimating some integrals of $R_k(U)$ and $S_k(\overline{U}, U)$. The conclusions are given as a series of lemmas.

Lemma 3.5. *There exist constants $C_1, C > 0$ depending on n, m only, such that*

$$\Big| \int_{\mathbb{R}^n} R_k(U)(x,t)dx \Big| \leq C_1 |DU|_\infty Y_k^2 \leq C(1 + G(t))^{-1-n/2} Z(t) Y_k^2.$$

The proof of Lemma 3.5 is the same as the proof of estimate of R_k in [2] by using Gagliardo-Nirenberg inequality (3.11) and Hölder inequality. We omit the proof.

We estimate the integral of $S_k(\overline{U}, U)$ similar to [2], we divide $S_k(\overline{U}, U)$ into two parts: $S_k^1(\overline{U}, U)$ containing the terms in the derivatives of $\bar{u}$ of first order, and the remaining part $S_k^2(\overline{U}, U)$. We first estimate the integral of $S_k^1(\overline{U}, U)$.

Lemma 3.6. *There exists a constant $C' > 0$ depending on $n, m, \gamma, \delta, \|u\|_X$ only, such that*

$$\int_{\mathbb{R}^n} S_k^1(\overline{U}, U)(x,t)dx + \frac{k+r}{1+G(t)} g(t) Y_k^2 \le C' Y_k Z (1+G(t))^{-k-2} g(t).$$

Proof. From (3.15) we know that $S_k^1(\overline{U}, U)$ has the following form

$$S_k^1(\overline{U}, U) = -D^k U \cdot B(D\overline{U}, D^k U) + \frac{1}{2}\sum_{i=1}^n \partial_{x_i} \bar{u}^i D^k U \cdot D^k U$$
$$- \sum_{\beta_1\beta_2\ldots\beta_k} \partial^k_{\beta_1\beta_2\cdots\beta_k} U \cdot \sum_{j=1}^k \sum_{i=1}^n \partial_{\beta_j} \bar{u}^i \partial^{k-1}_{\beta_1\cdots\beta_{j-1}\beta_{j+1}\cdots\beta_k} \partial_{x_i} U, \tag{3.16}$$

where ∂_{β_j} means the differentiation with respect to one component of x. By Proposition 3.1(i) we can estimate the integrals as follows

$$\int_{\mathbb{R}^n} -D^k U \cdot B(D\overline{U}, D^k U)dx = -\int_{\mathbb{R}^n} \Big(D^k \pi \frac{\gamma-1}{2} D^k \pi \mathrm{div} \bar{u} + \sum_{i,j=1}^n D^k w^i \partial_{x_i} \bar{u}^j D^k w^j \Big) dx$$
$$= -\int_{\mathbb{R}^n} \Big(\frac{\gamma-1}{2} |D^k \pi|^2 \frac{n g(t)}{1+G(t)} + \frac{g(t)}{1+G(t)} |D^k w|^2 + I_1 \Big) dx,$$

$$\int_{\mathbb{R}^n} \frac{1}{2} \sum_{i=1}^n \partial_{x_i} \bar{u}^i D^k U \cdot D^k U dx = \int_{\mathbb{R}^n} \Big(\frac{n g(t)}{2(1+G(t))} |D^k U|^2 + I_2 \Big) dx,$$

$$-\int_{\mathbb{R}^n} \sum_{\beta_1\beta_2\cdots\beta_k} \partial^k_{\beta_1\beta_2\cdots\beta_k} U \cdot \sum_{j=1}^k \sum_{i=1}^n \partial_{\beta_j} \bar{u}^i \partial^{k-1}_{\beta_1\cdots\beta_{j-1}\beta_{j+1}\cdots\beta_k} \partial_{x_i} U dxv = -\int_{\mathbb{R}^n} \Big(\frac{k g(t)}{1+G(t)} |D^k U|^2 + I_3 \Big) dx,$$

where the integrals of I_1, I_2, I_3 have the same estimate

$$\Big| \int_{\mathbb{R}^n} I_j dx \Big| \le C \frac{g(t)}{(1+G(t))^2} Y_k^2.$$

Combining the estimates above we have

$$\Big| \int_{\mathbb{R}^n} \Big(S_k^1(\overline{U}, U)(x,t) + \frac{g(t)}{1+G(t)} |D^k U|^2 \Big(k - \frac{n}{2}\Big) + \frac{g(t)}{1+G(t)} \Big(\frac{\gamma-1}{2} n |D^k \pi|^2 + |D^k w|^2 \Big) \Big) dx \Big|$$
$$\le C' \frac{g(t)}{(1+G(t))^2} Y_k^2 \le C' g(t) Y_k Z (1+G(t))^{-k-2}.$$

Let $r = \min\{\frac{\gamma-1}{2} n - \frac{n}{2}, 1 - \frac{n}{2}\}$. From the last inequality we can deduce easily the estimate needed.

Lemma 3.7. *There exists a constant $C'' > 0$ depending on $n, m, \gamma, \delta, \|u\|_X$ only such that*

$$\Big| \int_{\mathbb{R}^n} S_k^2(\overline{U}, U)(x,t)dx \Big| \le C'' Y_k Z (1+G(t))^{-k-2} g(t).$$

Proof. From (3.15) we know that $S_k^2(\overline{U}, U)$ is the sum of the following terms (neglecting $\frac{\gamma-1}{2}$) of the form $\partial^k U \partial^l U \partial^{k+1-l} \overline{U},\ l \le k-1$. We estimate it in two cases.

(1) $l = 0, 1$. By Lemma 3.4 and Proposition 3.1 we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \partial^k U \partial^l U \partial^{k+1-l}\overline{U}(x,t)dx\Big| &\le |\partial^l U|_\infty \|\partial^k U\|_0 \|\partial^{k+1-l}\overline{U}\|_0 \\ &\le CY_k(1+G(t))^{-l-n/2}Z(t)(1+G(t))^{n/2-(k+2-l)}g(t) \\ &\le CY_k Z g(t)(1+G(t))^{-(k+2)}.\end{aligned}$$

(2) $2 \le l \le k-1$. We use Gagliardo-Nirenberg inequality on ∂U and $\partial^2 \bar{u}$ to obtain

$$\begin{aligned}|\partial^l U|_{2\frac{k-2}{l-1}} &\le C|DU|_\infty^{1-\frac{l-1}{k-2}} \|D^{k-1}U\|_0^{\frac{l-1}{k-2}}, \\ |\partial^{k+1-l}\overline{U}|_{2\frac{k-2}{k-1-1}} &\le C|D^2\overline{U}|_\infty^{1-\frac{k-l-1}{k-2}} \|D^k\overline{U}\|_0^{\frac{k-l-1}{k-2}}.\end{aligned}$$

Since $\frac{1}{2} + \frac{l-1}{2(k-2)} + \frac{k-l-1}{2(k-2)} = 1$, by Proposition 3.1, Lemma 3.4 and Hölder inequality, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \partial^k U \partial^l U \partial^{k+1-l}\overline{U}(x,t)dx\Big| &\le \|\partial^k U\|_0 |\partial^l U|_{2\frac{k-2}{l-1}} |\partial^{k+1-l}\overline{U}|_{2\frac{k-2}{k-1-1}} \\ &\le CY_k |DU|_\infty^{1-\frac{l-1}{k-2}} \|D^{k-1}U\|_0^{\frac{l-1}{k-2}} |D^2\overline{U}|_\infty^{1-\frac{k-l-1}{k-2}} \|D^k\overline{U}\|_0^{\frac{k-l-1}{k-2}} \\ &\le C''Y_k((1+G(t))^{-1-n/2}Z)^{\frac{k-l-1}{k-2}}((1+G(t))^{-k+1}Z)^{\frac{l-1}{k-2}} \\ &\quad \cdot ((1+G(t))^{-3}g(t))^{\frac{l-1}{k-2}}((1+G(t))^{n/2-k-1}g(t))^{\frac{k-l-1}{k-2}} \\ &\le C''Y_k(t)Z(t)g(t)(1+G(t))^{-k-2}.\end{aligned}$$

Now we return to (3.13). Combining Lemmas 3.5–3.7 we have the following estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(Y_k(t)^2) + \alpha(t)\int_{\mathbb{R}^n} D^k w \cdot D^k w(x,t)dx + \frac{k+r}{1+G(t)}g(t)Y_k^2 \\ &\le C(1+G(t))^{-1-n/2}Z(t)Y_k^2 + C'Y_k Z(1+G(t))^{-k-2}g(t),\end{aligned} \tag{3.17}$$

where we combine the constants C', C'' into a new constant C'. Neglecting $\alpha(t)\int_{\mathbb{R}^n} D^k w \cdot D^k w dx$, we get

$$\frac{dY_k(t)}{dt} + \frac{k+r}{1+G(t)}g(t)Y_k \le C(1+G(t))^{-1-n/2}Z(t)Y_k + C'Z(1+G(t))^{-k-2}g(t).$$

Multiplying the above inequality by $(1+G(t))^k$, we have

$$\begin{aligned}&\frac{d((1+G(t))^k Y_k(t))}{dt} + \frac{r}{1+G(t)}g(t)(1+G(t))^k Y_k \\ &\le C(1+G(t))^{-1-n/2}Z(t)(1+G(t))^k Y_k + C'Z(1+G(t))^{-2}g(t).\end{aligned}$$

We sum the above inequalities from $k=0$ to $k=m$ to have

$$\frac{dZ(t)}{dt} + \frac{r}{1+G(t)}g(t)Z(t) \le C(1+G(t))^{-1-n/2}Z(t)^2 + C'Z(t)(1+G(t))^{-2}g(t), \tag{3.18}$$

where the constant C' contains the multiplier $m+1$.

Now we will prove that $Z(t)$ exists globally if $Z(0)$ is small enough. We first give the following lemma.

Lemma 3.8. *Let* $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$ *or* $\alpha(t)=O\big(\frac{A}{(1+t)^\theta}\big)$ $(t\to\infty)$, $\theta>1$, *where* $a=\min\{2,1+\frac{\gamma-1}{2}n\}$. *There exists a constant* $M>0$ *depending on* n,m,γ,δ, $\|u\|_X$ *and* $\displaystyle\int_0^\infty(1+G(t))^{-a}dt$ *only such that if* $0<y(0)<\frac{1}{M}$, *the solution of the following differential equation*

$$\begin{cases}\dfrac{dy}{dt}+\dfrac{r}{1+G(t)}g(t)y=C(1+G(t))^{-1-n/2}y^2+C'y(1+G(t))^{-2}g(t),\\ y(0)=Z(0)\end{cases}$$

exists on $[0,+\infty)$.

Proof. We solve the differential equation to obtain

$$y(t)=\frac{(1+G(t))^{-r}\ e^{\frac{C'G(t)}{1+G(t)}}}{\frac{1}{Z(0)}-\int_0^t C(1+G(s))^{-a}\ e^{\frac{C'G(s)}{1+G(s)}}ds}.$$

Since $1\le e^{\frac{C'G(t)}{1+G(t)}}\le e^{C'}$, the convergence of the integral $\int_0^\infty(1+G(s))^{-a}\ e^{\frac{C'G(s)}{1+G(s)}}ds$ is necessary for the global existence of $y(t)$, so $\int_0^\infty(1+G(s))^{-a}ds$ must converge. If $\int_0^\infty(1+G(s))^{-a}ds$ converges, then

$$\int_0^\infty C(1+G(s))^{-a}\ e^{\frac{C'G(s)}{1+G(s)}}ds\le Ce^{C'}\int_0^\infty(1+G(s))^{-a}ds=M.$$

When $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$ or $\alpha(t)=O(\frac{A}{(1+t)^\theta}),\ \theta>1$, by (3.5) it is easy to prove that $\int_0^\infty(1+G(s))^{-a}ds$ converges. So if $0<y(0)=Z(0)<\frac{1}{M}$, then $y(t)$ exists globally and there exists a constant $C>0$ depending on n, m, γ, δ, $\|u\|_X$, $\int_0^\infty(1+G(t))^{-a}dt$ and $Z(0)$ only, such that

$$0<y(t)\le C(1+G(t))^{-r},\qquad 0\le t<\infty.$$

When $Z(0)=\|\pi_0\|_m<\frac{1}{M}$, from (3.18) and Lemma 3.8 we have

$$Z(t)\le y(t)\le C(1+G(t))^{-r},\qquad 0\le t<\infty. \tag{3.19}$$

So $Z(t)$ exists globally, and so does $U=(\pi,u-\bar u)^T\in H^m(\mathbb{R}^n)$. Moreover we have the following estimate from the definition of $Z(t)$:

$$\|D^kU(t)\|_0=Y_k(t)\le C(1+G(t))^{-k-r},\qquad 0\le k\le m. \tag{3.20}$$

Thus, we have proved (i) of Theorem 3.2. Combining Lemma 3.4 and (3.19) we get

$$|D^lU(t)|_\infty\le C(1+G(t))^{-r-l-n/2}=C(1+G(t))^{-a+(1-l)},\qquad l=0,1. \tag{3.21}$$

These are (ii), (iii) of Theorem 3.2. From the proof we also have the following corollary.

Corollary 3.1. *Let* $\alpha(t)\ge0$ *such that*

$$\int_0^{+\infty}(1+G(t))^{-a}dt<\infty,$$

where $a=\min\{2,1+\frac{\gamma-1}{2}n\}>1$. *If the assumptions on* ρ_0, u_0 *are the same as Theorem* 3.1, *then the conclusions in Theorem* 3.1 *and Theorem* 3.2 *are valid.*

3.5. Further conclusions

We first give a property of $G(t)$, and then a uniqueness result of local in space and global in time. Finally we give an improvement of Theorem 3.1.

Lemma 3.9. *Let* $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$ *or* $\alpha(t)=O\big(\frac{A}{(1+t)^\theta}\big)$ $(t\to\infty),\ \theta>1$. *There exists* $0<\varepsilon<a-1$, *such that* $\lim\limits_{t\to\infty} g(t)(1+G(t))^\varepsilon=\infty$.

Proof. The proof is to check that when $\alpha(t)=O\big(\frac{A}{(1+t)^\theta}\big),\ \theta>1,\ 0<\lim\limits_{t\to\infty} g(t)\le 1,\ G(t)\sim g(\infty)t\ (t\to\infty)$, then any $\varepsilon\in(0,a-1)$ will do; when $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$, $g(t)=\frac{1}{(1+t)^A},\ G(t)=\frac{1}{1-A}((1+t)^{1-A}-1)$, then $\frac{A}{1-A}<a-1$, so any $\varepsilon\in\big(\frac{A}{1-A},a-1\big)$ will do.

With this lemma, we can prove similarly the following proposition as that in [2], we omit the proof.

Proposition 3.2. *Let* $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$ *or* $\alpha(t)=O(\frac{A}{(1+t)^\theta})\ (t\to\infty),\ \theta>1$. *Suppose that* u_0 *satisfies* (H1), (H2), $\rho_0^{(\gamma-1)/2}\in H^m(\mathbb{R}^n)$ *and is sufficiently small,* $\text{supp}\rho_0$ *is compact. Let* $U=(\pi,u)^T$ *be the global solution of Cauchy problem* (1.3), (3.1) *in Theorem* 3.1, $\bar u$ *is the solution of approximate problem* (3.2). *Suppose that* $V=(\tilde\pi,v)^T$ *is another solution of the equations* (1.3) *and* $DV\in L^\infty(\mathbb{R}^n\times\mathbb{R}_+)$. *For any* $\nu\in(2-a+\varepsilon,1)$ *and* $R_0>0$, *there exists* $T_0>0$, *such that if* $U(\,\cdot\,,T_0)=V(\,\cdot\,,T_0)$ *on* $B_{T_0}=B(O,R_0(1+G(T_0))^\nu)$, *then* $U\equiv V$ *in the domain* $\{(x,t)\mid |x-x(t)|\le R(t),t\ge T_0\}$. *Here* $x(t)$ *satisfies* $x'(t)=u(x(t),t),x(T_0)=O,R(t)=R_0(1+G(t))^\nu$.

We can remove the assumption that $\text{supp}\rho_0$ is compact to obtain the same conclusions in Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let* $\alpha(t)=\frac{A}{1+t},\ 0<A<1-\frac{1}{a}$ *or* $\alpha(t)=O\big(\frac{A}{(1+t)^\theta}\big)$ $(t\to\infty),\ \theta>1$. *Suppose that* u_0 *satisfies* (H1), (H2), $\rho_0^{(\gamma-1)/2}\in H^m(\mathbb{R}^n)$ *and is sufficiently small. Then Cauchy problem* (1.3), (3.1) *has a global regular solution, and*

$$(\rho^{(\gamma-1)/2},u-\bar u)^T\in C([0,\infty),H^m(\mathbb{R}^n))\cap C^1([0,\infty),H^{m-1}(\mathbb{R}^n)).$$

$(\rho^{(\gamma-1)/2},u-\bar u)$ *has the same estimates as in Theorem* 3.2.

The proof is similar to that of Corollary 1 in [2], we omit the details.

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