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STOCHASTIC HEAT EQUATIONS WITH RANDOM INITIAL CONDITIONS**

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Abstract

In this paper the author constructs a solution of parabolic stochastic partial differential equation with random initial conditions by Kolmogorov's criterion.

Keywords Stochastic partial differential equation, Green's function, Kolmogorov's criterion
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§ 1 . Introduction

Consider the following stochastic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u(t,x)) + \sigma(u(t,x))\frac{\partial^2 W}{\partial t \partial x}, \qquad (t,x) \in [0,T] \times [0,1], \tag{1.1}$$

with Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \tag{1.2}$$

or Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \tag{1.3}$$

and initial condition $u(0,x) = u_0(x)$. Here $\{\dot{W}(t,x),(t,x) \in [0,T] \times [0,1]\}$ is the space-time white noise, the coefficients $b(x)$ and $\sigma(x)$ are Lipschitz functions on $\mathbb{R}$, $u_0(x)$ is some real-valued function defined on $[0,1]$.

This kind of equations were investigated by many authors. The existence and uniqueness of the solution were discussed under different assumptions for the coefficients. These results can be found in [1, 5, 6, 8, 9, 19].

For ordinary stochastic differential equation with anticipative initial condition, several techniques such as Skorohod's integral (cf. [4]), Stratonovich's integral (cf. [13]) and forward integral (cf. [18]) are applied. It is worthy to say that Malliavin and Nualart in [11] gave a substitution formula using quasi sure analysis technique. Following this, in [21] a similar result was proved in two parameter case. Recently, Tindel [20] proved the existence of a solution for the equation (1.1) with a random initial condition

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$$u(0,x,w)=\sum_{k=1}^{d}\xi_k(w)\phi_k(x), \tag{1.4}$$

where $\phi_k(x)=\sqrt{2}\sin(k\pi x)$. His proof was based on forward integral and some flow properties for the stochastic heat equation. Here we define another anticipative stochastic integral borrowing the method of Malliavin and Nualart [11], which is different from Stratonovich's integral. Actually, the Stratonovich's integral does not make sense in this case because of an "infinite trace" phenomenon (cf. [20]). Then we construct a solution for the equation (1.1) with Dirichlet or Neumann boundary conditions and anticipating initial condition $u(0,x,w)=u_0(x,\xi(w))$, where $u_0(x,h)$ is a real function on $[0,1]\times\mathbb{R}^d$ satisfying some regularity assumptions, and ξ is any $\mathbb{R}^d$-valued random variable. Our argument is based on the Kolmogorov's continuity criterium (see [15, 11, 17]). This leads to a substitution formula (see Corollary 3.1 below) for the stochastic integral. The main difficulty in the proof (see Proposition 3.1) is to obtain the speed of the convergence. That will be overcome by using different scales to discretizing the time and space variables.

This paper is organized as follows: in Section 2, we will give some necessary notions and notations, in Section 3 we shall prove our main result.

§ 2 . Preliminary

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P)$ be a complete probability space. The space-time white noise is defined as a zero mean Gaussian random field $W=W(B);B\in\mathcal{B}([0,T]\times[0,1])$ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P)$ such that

(i) $\mathbb{E}[W(A)W(B)]=\lambda(A\cap B)$, $A,B\in\mathcal{B}([0,T]\times[0,1])$ and λ is the Lebesgue measure.

(ii) For every $C\in\mathcal{B}([0,1])$, the process $\{W([0,t]\times C);t\in[0,T]\}$ is an $\mathcal{F}_t$-Brownian motion.

We use $W(t,x)$ to denote $W([0,t]\times[0,x])$. A stochastic process $\{u(t,x);(t,x)\in[0,T]\times[0,1]\}$ is said $\mathcal{F}_t$-adapted if $u(t,x)$ is $\mathcal{F}_t$-measurable for any $(t,x)\in[0,T]\times[0,1]$.

In the following we only discuss the Neumann boundary condition. The conclusion for Dirichlet boundary still holds.

It is well-known that the equation (1.1) is formed because of the lack of $\frac{\partial^2 W}{\partial t\partial x}$. A rigorous meaning of this equation is given by means of weak solution. That is, we say that a continuous adapted process $\{u(t,x);(t,x)\in[0,T]\times[0,1]\}$ solves the equation (1.1) if for each $\phi\in C^2([0,1])$ with $\phi'(0)=\phi'(1)=0$, it holds

$$\begin{aligned}\int_0^1 u(t,x)\phi(x)dx=&\int_0^1 u_0(x)\phi(x)dx+\int_0^t\int_0^1 b(u(s,x))\phi(x)dxds\\&+\int_0^t\int_0^1\sigma(u(s,x))\phi(x)W(ds,dx)\qquad\text{for all } t\in[0,T],\end{aligned}$$

where the last integral is the stochastic Itô's integral (cf. [19]). It is shown in Wash [19] that the continuous adapted process u solves (1.1) if and only if u satisfies

$$\begin{aligned}u(t,x)=&\int_0^1 G_t(x,y)u_0(y)dy+\int_0^t\int_0^1 G_{t-s}(x,y)b(u(s,y))dyds\\&+\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy),\end{aligned}\tag{2.1}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation with Neumann boundary conditions (1.3). The kernel $G_t(x,y)$ has the following explicit formula:

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{+\infty} \Big\{ \exp\Big(-\frac{(y-x-2n)^2}{4t}\Big) + \exp\Big(-\frac{(y+x-2n)^2}{4t}\Big)\Big\}. \tag{2.2}$$

The existence and uniqueness of solution for this kind of equation were given by Walsh [19] under Lipschitz continuity assumptions on b and σ. Various assumptions on the coefficients are referred to [1, 5, 6, 8, 9].

The following result is taken from the paper [2, Lemma A2].

Proposition 2.1. *Let $(\mathcal{H}_\alpha, \|\cdot\|_{\mathcal{H}_\alpha})$ be the α-order Hölder continuous function space, i.e.,*

$$\|u\|_\alpha := \sup_{x\neq y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty.$$

For $0<\alpha<\frac{1}{2}$, if $u_0 \in \mathcal{H}_\alpha$, then we have

$$\Big|\int_0^1 [G_t(x,y)u_0(y) - G_s(x,y)u_0(y)]dy\Big| \leqslant C|t-s|^{\alpha/2}, \tag{2.3}$$

$$\Big|\int_0^1 [G_t(x,z)u_0(z) - G_t(y,z)u_0(z)]dz\Big| \leqslant C|x-y|^{\alpha}. \tag{2.4}$$

Here the constant C depends on $\|u_0\|_\alpha$ in such a way that if $\|u_0\|_\alpha$ remains bounded, so does $C(\|u_0\|_\alpha)$.

We also need the following property of Green's functions (cf. [2, 6, 12, 19]).

Proposition 2.2.

$$|G_t(x,y)| \leqslant \frac{C}{\sqrt{t}} \exp\Big(-\frac{(x-y)^2}{t}\Big), \tag{2.5}$$

$$\Big|\frac{\partial G_t(x,y)}{\partial x}\Big| = \Big|\frac{\partial G_t(x,y)}{\partial y}\Big| \leqslant \frac{C}{t} \exp\Big(-\frac{(x-y)^2}{t}\Big), \tag{2.6}$$

$$\Big|\frac{\partial G_t(x,y)}{\partial t}\Big| \leqslant \frac{C}{t^{3/2}} \exp\Big(-\frac{(x-y)^2}{t}\Big), \tag{2.7}$$

$$\int_0^t \int_0^1 |G_s(x,z) - G_s(y,z)|^\beta dz ds \leqslant C|x-y|^{3-\beta}, \tag{2.8}$$

$$\int_0^t \int_0^1 |G_{r+s}(x,y) - G_r(x,y)|^\beta dy dr \leqslant C \cdot s^{(3-\beta)/2}, \tag{2.9}$$

$$\int_0^t \int_0^1 |G_s(x,y)|^\beta dy ds \leqslant C \cdot t^{(3-\beta)/2}, \tag{2.10}$$

where $\beta \in (1,3)$, C is a universal constant.

Proof. These inequalities follow from the following observation:

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}} \Big\{ \exp\Big(-\frac{(x-y)^2}{4t}\Big) + \exp\Big(-\frac{(x+y)^2}{4t}\Big) + \exp\Big(-\frac{(x+y-2)^2}{4t}\Big)\Big\} + L(t,x,y),$$

where $L(t,x,y)$ is a smooth function on $[0,T]\times[0,1]\times[0,1]$.

§ 3 . Main Result and Its Proof

For the sake of simplicity, in this section we will take the parameter set $[0,1]^2$, and use the following notations:

$$\dot{W}_n(s,y) = 8^n \cdot 4^n [W(s_n^+, y_n^+) - W(s_n^+, y_n) - W(s_n, y_n^+) + W(s_n, y_n)],$$

and

$$s_n^+ = \frac{[8^n s]+1}{8^n}, \quad s_n = \frac{[8^n s]}{8^n}, \quad y_n^+ = \frac{[4^n y]+1}{4^n}, \quad y_n = \frac{[4^n y]}{4^n}.$$

Consider the equation (1.1) with Lipschitz coefficients b, σ and random initial condition $u(0,x) = u_0(x,\xi)$, where ξ is a $\mathbb{R}^d$ valued random variable, $u_0(x,h)$ is a real function on $[0,1] \times \mathbb{R}^d$ satisfying

(C1) $\|u_0(\cdot,h)\|_\alpha < c(h)$ for some $0 < \alpha < \frac{1}{2}$, where $c(h)$ is a locally bounded function on $\mathbb{R}^d$.

(C2) For any $R \in \mathbb{N}$ and $h_1, h_2 \in B(0,R)$, the ball in $\mathbb{R}^d$ with radius R,

$$|u_0(x,h_1) - u_0(x,h_2)| \leqslant M_R |h_1 - h_2|^\alpha$$

for some $M_R > 0$ and the same α as in (C1).

Remark 3.1. It is clear that these two conditions contain (1.4).

When the initial condition is random, the solution $u(t,x)$ (if it exists) must be non-adapted. Therefore it is necessary to give a definition for stochastic integral appearing in (2.1).

Definition 3.1. *We will say that a real valued measurable process $\{\varphi(s,x;w)\}$ is G-Itô integrable if the sequence*

$$\Lambda_n(\varphi) := \int_0^t \int_0^1 G_{t-s}(x,y)\varphi_n(s,y)\dot{W}_n(s,y)dyds$$

converges in probability, where

$$\varphi_n(s,y,w) := 8^n \int_{s_n^-}^{s_n} \varphi(r,y,w)dr \tag{3.1}$$

and $s_n^- = (s_n - 8^{-n}) \vee 0$. The limit is denoted by

$$\int_0^t \int_0^1 G_{t-s}(x,y)\varphi(s,y) \diamond W(ds,dy).$$

Now we can give the following definition.

Definition 3.2. *A measurable process $u(t,x)$ is called a solution for the equation* (1.1) *if it satisfies*

$$u(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)b(u(s,y))dyds$$
$$+ \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y)) \diamond W(ds,dy),$$

where the initial condition $u_0(x,w)$ is random.

We now show that if φ is adapted, this integral coincides with Itô's integral. Henceforth, we make a convention: C denotes a positive constant independent of n, whose value may change from one place to another one.

Proposition 3.1. *Let φ be a measurable adapted process. Assume that for some $p>3$,*

$$\int_0^1\int_0^1 \mathbb{E}|\varphi(s,y)|^{2p}dyds<\infty,$$

then

$$\begin{aligned}&\mathbb{E}\Big|\Lambda_n(\varphi)-\int_0^t\int_0^1 G_{t-s}(x,y)\varphi(s,y)W(ds,dy)\Big|^{2p}\\ &\leqslant C2^{-n(p-3)/2}+C\int_0^1\int_0^1\mathbb{E}\Big|32^n\int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+}\varphi(r,z)dzdr-\varphi(s,y)\Big|^{2p}dyds. \end{aligned}\tag{3.2}$$

In particular,

$$\int_0^t\int_0^1 G_{t-s}(x,y)\varphi(s,y)\diamond W(ds,dy)=\int_0^t\int_0^1 G_{t-s}(x,y)\varphi(s,y)W(ds,dy).$$

Proof. Define

$$\alpha_n(s,y)=32^n\int_{s_n}^{s_n^+\wedge t}\int_{y_n}^{y_n^+}G_{t-r}(x,z)\varphi_n(r,z)dzdr.$$

Then $\alpha_n(s,y)$ is a measurable adapted process, and

$$\Lambda_n(\varphi)=\int_0^{t_n^+}\int_0^1\alpha_n(s,y)W(ds,dy).$$

Set $t^*=\frac{[8^n(t-2^{-n})]}{8^n}$. Then $2^{-n}\leqslant t-t^*\leqslant 2^{-n}+8^{-n}\leqslant 2^{-n+1}$. We make the following decomposition:

$$\begin{aligned}&\Lambda_n(\varphi)-\int_0^t\int_0^1 G_{t-s}(x,y)\varphi(s,y)W(ds,dy)\\ &=\int_{t^*}^{t_n^+}\int_0^1\alpha_n(s,y)W(ds,dy)-\int_{t^*}^t\int_0^1 G_{t-s}(x,y)\varphi(s,y)W(ds,dy)\\ &\quad+\int_0^{t^*}\int_0^1[\alpha_n(s,y)-G_{t-s}(x,y)\varphi(s,y)]W(ds,dy)\\ &=:\mathrm{I}_1-\mathrm{I}_2+\mathrm{I}_3.\end{aligned}$$

By Burkhölder's inequality, Hölder's inequality and (2.10), we have

$$\begin{aligned}\mathbb{E}|\mathrm{I}_1|^{2p}&\leqslant C\mathbb{E}\Big(\int_{t^*}^{t_n^+}\int_0^1|\alpha_n(s,y)|^2dyds\Big)^p\\ &\leqslant C\mathbb{E}\Big(\int_{t^*}^{t_n^+}\int_0^1\Big(32^n\int_{s_n}^{s_n^+\wedge t}\int_{y_n}^{y_n^+}|G_{t-r}(x,z)\varphi_n(r,z)|^2dzdr\Big)dyds\Big)^p\end{aligned}$$

$$= C\mathbb{E}\Big(\int_{t^*}^{t}\int_0^1 |G_{t-s}(x,y)\varphi_n(s,y)|^2 dyds\Big)^p$$
$$\leqslant C\Big(\int_{t^*}^{t}\int_0^1 |G_{t-s}(x,y)|^{2p/(p-1)} dyds\Big)^{p-1}\Big(\int_{t^*}^{t}\int_0^1 \mathbb{E}|\varphi_n(s,y)|^{2p} dyds\Big)$$
$$\leqslant C|t-t^*|^{(p-3)/2}\Big(\int_0^1\int_0^1 \mathbb{E}|\varphi(s,y)|^{2p} dyds\Big) \leqslant C2^{-n(p-3)/2}.$$

Similarly, we also have

$$\mathbb{E}|\mathrm{I}_2|^{2p} \leqslant C2^{-n(p-3)/2}.$$

For I_3, by Burkhölder's inequality we have

$$\mathbb{E}|\mathrm{I}_3|^{2p} \leqslant C\mathbb{E}\Big(\int_0^{t^*}\int_0^1 |\alpha_n(s,y) - G_{t-s}(x,y)\varphi(s,y)|^2 dyds\Big)^p.$$

Note that for $0 < s < t^*$,

$$\begin{aligned}
&\alpha_n(s,y) - G_{t-s}(x,y)\varphi(s,y)\\
&= 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} [G_{t-r}(x,y) - G_{t-s}(x,y)]\varphi_n(r,z)dzdr\\
&\quad + 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} [G_{t-r}(x,z) - G_{t-r}(x,y)]\varphi_n(r,z)dzdr\\
&\quad + G_{t-s}(x,y)\Big[32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} \varphi_n(r,z)dzdr - \varphi(s,y)\Big]\\
&:= \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3.
\end{aligned}$$

By the inequality (2.7), we obtain

$$\begin{aligned}
|\mathrm{J}_1| &\leqslant 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |G_{t-r}(x,y) - G_{t-s}(x,y)| \cdot |\varphi_n(r,z)|dzdr\\
&= 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |s-r| \cdot \Big|\frac{\partial G_{t-s}(x,y)}{\partial s}\Big|_{s=r(\theta)}\Big| \cdot |\varphi_n(r,z)|dzdr\\
&\leqslant 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} \frac{|s-r|}{|t-r(\theta)|^{3/2}} \cdot |\varphi_n(r,z)|dzdr\\
&\leqslant 2^{-3n/2} \cdot 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |\varphi_n(r,z)|dzdr,
\end{aligned}$$

where in the second step the use of Taylor's formula yields $r(\theta) \in [s_n, s_n^+]$. The last step is due to $r \in [s_n, s_n^+]$, and $|t - r(\theta)| > 2^{-n}$.

Similarly, by the inequality (2.6) we have

$$\begin{aligned}
|\mathrm{J}_2| &\leqslant 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |G_{t-r}(x,y) - G_{t-r}(x,z)| \cdot |\varphi_n(r,z)|dzdr\\
&= 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |y-z| \cdot \Big|\frac{\partial G_{t-r}(x,y)}{\partial y}\Big|_{y=z(\theta)}\Big| \cdot |\varphi_n(r,z)|dzdr
\end{aligned}$$

$$\leqslant 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} \frac{|y-z|}{|t-r|}\cdot|\varphi_n(r,z)|dzdr$$
$$\leqslant 2^{-n}\cdot 32^n \int_{s_n}^{s_n^+}\int_{y_n}^{y_n^+} |\varphi_n(r,z)|dzdr,$$

where $z(\theta)\in[y_n,y_n^+]$.

Moreover,

$$\mathrm{J}_3 = G_{t-s}(x,y)\Big[32^n \int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+} \varphi(r,z)dzdr - \varphi(s,y)\Big].$$

Hence

$$\begin{aligned}\mathbb{E}|\mathrm{J}_3|^{2p} &\leqslant C2^{-2np}\mathbb{E}\Big(\int_0^{t^*}\int_0^1 |\varphi_n(r,z)|^2 dyds\Big)^p + C\mathbb{E}\Big(\int_0^{t^*}\int_0^1 |J_3|^2 dyds\Big)^p\\ &\leqslant C2^{-2np} + C\Big(\int_0^{t^*}\int_0^1 |G_{t-s}(x,y)|^{2p/(p-1)}dyds\Big)^{p-1}\\ &\quad\cdot\int_0^{t^*}\int_0^1 \mathbb{E}\Big|32^n\int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+}\varphi(r,z)dzdr-\varphi(s,y)\Big|^{2p}dyds\\ &\leqslant C2^{-2np} + C\int_0^1\int_0^1 \mathbb{E}\Big|32^n\int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+}\varphi(r,z)dzdr-\varphi(s,y)\Big|^{2p}dyds,\end{aligned}$$

which gives the desired estimate.

For fixed $h\in\mathbb{R}^d$, we denote by $u(t,x;u_0(\,\cdot\,,h))$ (or simply $u(t,x;h)$) the unique solution of (2.1) with initial condition $u(0,x)=u_0(x,h)$ (cf. [19]). Let

$$\sigma_n^u(s,y;h) := 8^n\int_{s_n^-}^{s_n}\sigma(u(r,y;u_0(\cdot,h)))dr.$$

Define

$$v_n(t,x;h) := \int_0^t\int_0^1 G_{t-s}(x,y)\sigma_n^u(s,y;h)\dot{W}_n(s,y)dyds, \tag{3.3}$$
$$v(t,x;h) := \int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(s,y;u_0(\cdot,h)))W(ds,dy). \tag{3.4}$$

Then this paper is mainly devoted to proving the following result.

Theorem 3.1. *Under the assumptions* (C1) *and* (C2)*, for any* $R\in\mathbb{N}$*, define*

$$A_R := \Big\{w : \lim_{n\to\infty}\sup_{(t,x,h)\in[0,1]^2\times B(0,R)}|v_n(t,x;h)-v(t,x;h)|=0\Big\}.$$

Then the set $A=\bigcup_{R\in\mathbb{N}}A_R^c$ *is a null set, where* $B(0,R)$ *denotes the open ball in* $\mathbb{R}^d$ *with radius* R*.*

From this theorem, we can deduce that

Corollary 3.1. *We have the following substitution formula:*

$$\int_0^t\int_0^1 G_{t-s}(x,y)\sigma(u(s,y;u_0(\cdot,h)))W(ds,dy)\Big|_{h=\xi}$$

$$= \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y;u_0(\cdot,\xi))) \diamond W(ds,dy).$$

Hence $u(t,x;u_0(\,\cdot\,,\xi))$ is a solution of the stochastic equation (1.1) *in the sense of Definition* 3.2*, where $u_0(x,h)$ satisfies* (C1) *and* (C2)*, ξ is any $\mathbb{R}^d$-valued random variable.*

Proof. For any $\epsilon > 0$ and $R > 0$, we have

$$\begin{aligned}
&\lim_{n\to\infty} P\{|v_n(t,x,\xi) - v(t,x,\xi)| \geqslant \epsilon\}\\
&\leqslant \lim_{n\to\infty} P\{|v_n(t,x,\xi) - v(t,x,\xi)| \cdot 1_{\{\xi<R\}} \geqslant \epsilon\} + P\{\xi \geqslant R\}\\
&\leqslant \lim_{n\to\infty} P\Big\{ \sup_{h\in B(0,R)} |v_n(t,x,h) - v(t,x,h)| \geqslant \epsilon\Big\} + P\{\xi \geqslant R\}\\
&= P\{\xi \geqslant R\} \to 0 \qquad \text{as } R \to \infty.
\end{aligned}$$

We begin our proof with the following lemma.

Lemma 3.1. *For any $p > \frac{2}{1-2\alpha}$ and $0 < s < t \leqslant 1, x, y \in [0,1], h, h' \in B(0,R)$, we have*

$$\begin{aligned}
\sup_{x\in[0,1],\, h\in B(0,R)} \mathbb{E}|u(t,x;h) - u(s,x;h)|^{2p} &\leqslant C|t-s|^{p\alpha},\\
\sup_{t\in[0,1],\, h\in B(0,R)} \mathbb{E}|u(t,x;h) - u(t,y;h)|^{2p} &\leqslant C|x-y|^{2p\alpha},\\
\sup_{(t,x)\in[0,1]^2} \mathbb{E}|u(t,x;h) - u(t,x;h')|^{2p} &\leqslant C|h-h'|^{2p\alpha}.
\end{aligned}$$

Proof. First of all, by a standard argument we have (cf. [19, 12])

$$\sup_{(t,x)\in[0,1]^2, h\in B(0,R)} \mathbb{E}|u(t,x;h)|^{2p} \leqslant C_R. \tag{3.5}$$

Let us look at the first inequality. From (2.1) we have

$$\begin{aligned}
u(t,x;h)-u(s,x;h) &= \int_0^1 [G_t(x,y)-G_s(x,y)]u_0(y,h)dy + \int_s^t\int_0^1 G_{t-r}(x,y)b(u(r,y;h))dydr\\
&\quad + \int_0^s \int_0^1 [G_{t-r}(x,y) - G_{s-r}(x,y)]b(u(r,y;h))dydr\\
&\quad + \int_s^t \int_0^1 G_{t-r}(x,y)\sigma(u(r,y;h))W(dr,dy)\\
&\quad + \int_0^s \int_0^1 [G_{t-r}(x,y) - G_{s-r}(x,y)]\sigma(u(r,y;h))W(dr,dy)\\
&:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5.
\end{aligned}$$

Since $c(h)$ is bounded on $B(0,R)$, the estimation (2.3) gives

$$|\mathrm{I}_1| \leqslant C|t-s|^{\alpha/2}. \tag{3.6}$$

Obviously, by (2.10), (3.5) and the linear growth of b we have

$$\mathbb{E}|\mathrm{I}_2|^{2p} \leqslant C\Big(\int_s^t \int_0^1 |G_{t-r}(x,y)|^{2p/(2p-1)} dydr\Big)^{2p-1} \cdot \Big(\int_s^t \int_0^1 \mathbb{E}|b(u(r,y;h))|^{2p} dydr\Big)$$

$$\leqslant C|t-s|^{2p-1/2}. \tag{3.7}$$

Similarly, by (2.9) we have

$$\mathbb{E}|\mathrm{I}_3|^{2p} \leqslant C|t-s|^{2p-1/2}. \tag{3.8}$$

By Burkhölder's inequality we have

$$\begin{aligned}\mathbb{E}|\mathrm{I}_4|^{2p} &\leqslant C\mathbb{E}\Big(\int_s^t\int_0^1 |G_{t-r}(x,y)|^2|\sigma(u(r,y;h))|^2 dydr\Big)^p\\ &\leqslant C\Big(\int_s^t\int_0^1 |G_{t-r}(x,y)|^{2p/(p-1)}dydr\Big)^{p-1}\cdot\Big(\int_s^t\int_0^1 \mathbb{E}|\sigma(u(r,y;h))|^{2p}dydr\Big)\\ &\leqslant C|t-s|^{(p-1)/2}.\end{aligned} \tag{3.9}$$

By the same reason we have

$$\mathbb{E}|\mathrm{I}_5|^{2p} \leqslant C|t-s|^{(p-1)/2}. \tag{3.10}$$

Combining (3.6)–(3.10) gives the first estimates.

The second one can be proved by the same method as the first one. Finally, we deal with the third one. We also have

$$\begin{aligned}u(t,x;h)-u(t,x;h') &= \int_0^1 G_t(x,y)[u_0(y,h)-u_0(y,h')]dy\\ &\quad+\int_0^t\int_0^1 G_{t-s}(x,y)[b(u(s,y;h))-b(u(s,y;h'))]dyds\\ &\quad+\int_0^t\int_0^1 G_{t-s}(x,y)[\sigma(u(s,y;h))-\sigma(u(s,y;h'))]W(ds,dy).\end{aligned}$$

Applying the condition (C2), we can deduce that

$$\mathbb{E}|u(t,x;h)-u(t,x;h')|^{2p} \leqslant C|h-h'|^{2p\alpha} + C\int_0^t\int_0^1 \mathbb{E}|u(s,y;h)-u(s,y;h')|^{2p}dyds.$$

Set $g(t) := \sup\limits_{s\leqslant t, x\in[0,1]} \mathbb{E}|u(s,x;h)-u(s,x;h')|^{2p}$. Then we have

$$g(t) \leqslant C|h-h'|^{2p\alpha} + C\int_0^t g(s)ds.$$

Gronwall's inequality produces the desired estimate.

Basing on this lemma and the estimate in Proposition 3.1, we immediately have

Theorem 3.2. *For any $p>3$, we have the following estimate for the speed of the convergence*

$$\sup_{(t,x)\in[0,1]^2, h\in B(0,R)} \mathbb{E}|v_n(t,x;h)-v(t,x;h)|^{2p} \leqslant C2^{-2np\beta}, \tag{3.11}$$

where $\beta := (\frac{1}{4}-\frac{3}{4p})\wedge(\frac{3\alpha}{2})$.

Proof. By the estimate (3.2) and Lemma 3.1, we have

$$\begin{aligned}
&\mathbb{E}|v_n(t,x;h)-v(t,x;h)|^{2p}\\
&\leqslant C2^{-n(p-3)/2}+C\int_0^1\int_0^1\mathbb{E}\Big|32^n\int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+}[\sigma(u(r,z;h))-\sigma(u(s,y;h))]dzdr\Big|^{2p}dyds\\
&\leqslant C2^{-n(p-3)/2}+C\int_0^1\int_0^1\Big(32^n\int_{s_n^-}^{s_n}\int_{y_n}^{y_n^+}\mathbb{E}|u(r,z;h)-u(s,y;h)|^{2p}dzdr\Big)dyds\\
&\leqslant C2^{-n(p-3)/2}+C2^{-3np\alpha}+C2^{-4np\alpha}\leqslant C2^{-2np\beta},
\end{aligned}$$

where β is given in the theorem.

We also need the following estimates.

Lemma 3.2. *For any $p>3$ and $0<s<t\leqslant 1,(x,y)\in[0,1]^2,h,h'\in B(0,R)$, we have*

$$\sup_n\mathbb{E}|v_n(t,x;h)-v_n(s,y;h')|^{2p}\leqslant C(|t-s|^{(p-1)/2}+|x-y|^{p-3}+|h-h'|^{2p\alpha}),\tag{3.12}$$

$$\mathbb{E}|v(t,x;h)-v(s,y;h')|^{2p}\leqslant C(|t-s|^{(p-1)/2}+|x-y|^{p-3}+|h-h'|^{2p\alpha}).\tag{3.13}$$

Proof. It is sufficient to prove the following three estimates for the first inequality:

$$\begin{aligned}
&\sup_n\sup_{x\in[0,1],h\in B(0,R)}\mathbb{E}|v_n(t,x;h)-v_n(s,x;h)|^{2p}\leqslant C|t-s|^{(p-1)/2},\\
&\sup_n\sup_{t\in[0,1],h\in B(0,R)}\mathbb{E}|v_n(t,x;h)-v_n(t,y;h)|^{2p}\leqslant C|x-y|^{p-3},\\
&\sup_n\sup_{(t,x)\in[0,1]^2}\mathbb{E}|v_n(t,x;h)-v_n(t,x;h')|^{2p}\leqslant C|h-h'|^{2p\alpha}.
\end{aligned}$$

We only prove the first one. The others are analogous. By the definition (3.3) we have

$$\begin{aligned}
&v_n(t,x;h)-v_n(s,x;h)\\
&=\int_s^t\int_0^1 G_{t-r}(x,y)\sigma_n^u(r,y;h)\dot W_n(r,y)dydr\\
&\quad+\int_0^s\int_0^1(G_{t-r}(x,y)-G_{s-r}(x,y))\sigma_n^u(r,y;h)\dot W_n(r,y)dydr\\
&=\mathrm{I}_1+\mathrm{I}_2.
\end{aligned}$$

By Burkhölder's inequality for discrete martingale, we obtain

$$\begin{aligned}
\mathbb{E}|\mathrm{I}_1|^{2p}&=\mathbb{E}\Big|\sum_{i,j}\Big[\int_{s\vee i8^{-n}}^{t\wedge(i+1)8^{-n}}\int_{j4^{-n}}^{(j+1)4^{-n}}G_{t-r}(x,y)\sigma_n^u(r,y;h)dydr\Big]\dot W_n(i8^{-n},j4^{-n})\Big|^{2p}\\
&\leqslant C\mathbb{E}\Big[\sum_{i,j}32^n\Big|\int_{s\vee i8^{-n}}^{t\wedge(i+1)8^{-n}}\int_{j4^{-n}}^{(j+1)4^{-n}}G_{t-r}(x,y)\sigma_n^u(r,y;h)dydr\Big|^2\Big]^p\\
&\leqslant C\mathbb{E}\Big[\int_s^t\int_0^1|G_{t-r}(x,y)\sigma_n^u(r,y;h)|^2dydr\Big]^p\\
&\leqslant C\Big(\int_s^t\int_0^1|G_{t-r}(x,y)|^{2p/(p-1)}dydr\Big)^{p-1}\Big(\int_s^t\int_0^1\mathbb{E}|\sigma_n^u(r,y;h)|^{2p}dydr\Big)
\end{aligned}$$

$$\leqslant C|t-s|^{(p-1)/2}.$$

The last step is due to the linear growth of σ and Lemma 3.1.

Similarly, we can get

$$\mathbb{E}|\mathrm{I}_2|^{2p} \leqslant C|t-s|^{(p-1)/2}.$$

The second one (3.13) is proved in a similar way.

Proof of Theorem 3.1. We introduce the following process:

$$Z(r,t,x,h) = \begin{cases} v(t,x;h), \qquad r=0; \\ v_n(t,x;h) + \Big(r-\dfrac{1}{n}\Big)\Big(\dfrac{1}{n+1}-\dfrac{1}{n}\Big)^{-1}[v_{n+1}(t,x;h) - v_n(t,x;h)]; \\ \dfrac{1}{n+1} < r \leqslant \dfrac{1}{n}, \qquad n\in\mathbb{N}. \end{cases}$$

By Lemma 3.2 we obtain

$$\mathbb{E}|Z(r,t,x,h) - Z(r,s,y,h')|^{2p} \leqslant C(|s-t|^{(p-1)/2} + |x-y|^{p-3} + |h-h'|^{2p\alpha}). \tag{3.14}$$

Now we can check by Theorem 3.2 that

$$\mathbb{E}|Z(r,t,x,h) - Z(r',t,x,h)|^{2p} \leqslant C|r-r'|^{2p}. \tag{3.15}$$

In fact, assuming that $m \geqslant n$ and $\frac{1}{n+1} < r \leqslant \frac{1}{n}$, $\frac{1}{m+1} < r' \leqslant \frac{1}{m}$, by Minkowski's inequality we have

$$\begin{aligned}
&\mathbb{E}|Z(r,t,x,h) - Z(r',t,x,h)|^{2p} \\
&\leqslant \Big(\Big\|Z(r,t,x,h) - Z\Big(\frac{1}{n+1},t,x,h\Big)\Big\|_{2p} + \sum_{k=n+1}^{m-1}\Big\|Z\Big(\frac{1}{k+1},t,x,h\Big) - Z\Big(\frac{1}{k},t,x,h\Big)\Big\|_{2p} \\
&\quad + \Big\|Z\Big(\frac{1}{m},t,x,h\Big) - Z(r',t,x,h)\Big\|_{2p}\Big)^{2p} \\
&\leqslant C\Big(\Big(r-\frac{1}{n+1}\Big)(n(n+1))2^{-n\beta} + \sum_{k=n+1}^{m-1} 2^{-k\beta} + \Big(\frac{1}{m}-r'\Big)(m(m+1))2^{-m\beta}\Big)^{2p} \\
&\leqslant C\cdot\sup_n\{n(n+1)2^{-n\beta}\}\Big(\Big(r-\frac{1}{n+1}\Big) + \sum_{k=n+1}^{m-1}\Big(\frac{1}{k}-\frac{1}{k+1}\Big) + \Big(\frac{1}{m}-r'\Big)\Big)^{2p} \\
&\leqslant C|r-r'|^{2p}.
\end{aligned}$$

Hence we can take p large enough such that

$$\begin{aligned}
&\mathbb{E}|(Z(r,t,x,h) - Z(r',s,y,h'))|^{2p} \\
&\leqslant C(|r-r'|^{d+3+\epsilon} + |t-s|^{d+3+\epsilon} + |x-y|^{d+3+\epsilon} + |h-h'|^{d+3+\epsilon})
\end{aligned}$$

for some $\epsilon > 0$ and $(s,t)\in[0,1]^2, (x,y)\in[0,1]^2, h,h'\in B(0,R)$.

Lastly, by Kolmogorov's continuity criterium, there is an integrable random variable $C(w)$ such that

$$\sup_{(t,x,h)\in[0,1]^2\times B(0,R)} |Z(r,t,x,h) - Z(r',t,x,h)| \leqslant C(w)|r-r'|^{\delta}, \qquad \text{a.s.,}$$

where $\delta \in (0, \frac{\epsilon}{2p})$. In particular, taking $r' = 0, r = \frac{1}{n}$ we have

$$\sup_{(t,x,h)\in[0,1]^2\times B(0,R)} |v_n(t,x,h) - v(t,x,h)| \leqslant C(w)n^{-\delta}, \qquad \text{a.s.}$$

Therefore A_R^c is a null set and the proof is completed.

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