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DIFFERENTIABILITY OF CONVEX FUNCTIONS ON SUBLINEAR TOPOLOGICAL SPACES AND VARIATIONAL PRINCIPLES IN LOCALLY CONVEX SPACES***

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Abstract

This paper presents a type of variational principles for real valued w^* lower semicontinuous functions on certain subsets in duals of locally convex spaces, and resolve a problem concerning differentiability of convex functions on general Banach spaces. They are done through discussing differentiability of convex functions on nonlinear topological spaces and convexification of nonconvex functions on topological linear spaces.

Keywords Convex function, β differentiability, Variational principle, Perturbed optimization, Banach spaces, Locally convex spaces
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§ 1 . Introduction

It is well known that various types of variational principles in infinitely dimensional Banach spaces have played an important role in nonlinear analysis and this fact has brought the theoretical research of variational questions to mathematians great attention (see, for instance, [1–6, 8, 13–18, 20, 22, 24, 25, 27, 29–35, 37]).

Generally speaking, variational principle can be interpreted as perturbed optimization: Given an extended-real-valued proper function and $\varepsilon > 0$, we claim that there is a Lipschitzian function h with the Lipschitz norm less than ε such that $f + h$ attains a minimum. Therefore, lower boundedness and lower semicontinuity-type conditions become natural assumptions. For every lower semicontinuous function f bounded below on a Banach space (or, more general, a complete metric space) E, the famous Ekeland's variational principle (see [23, 24]) can cast in this form, in which the perturbation h can be chosen as $\varepsilon\|\cdot - x_0\|$ for some point x_0 in E. If the Banach space E admits some kind of smoothness, say, it admits an equivalent norm or a bump function which is everywhere differentiable in its support set for some kind of differentiability β, due to smooth variational principles such as

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were developed by Borwein and Preiss [4] and Deville, Godefroy and Zizler [14], Li and Shi [29, 30], the perturbed version would be β-differentiable. Stegall [35] and Fabian [22] established a strong type variational version on certain subsets of Banach spaces. To be exact, they all focussed the class of lower semicontinuous proper functions f bounded below on a Banach space E, whose effective domains dom$f$ are bounded, and the closed convex hulls of dom$f$ have the Radon-Nikodým property. For each such function f, the perturbation h can be chosen as a linear functional x^* with $\|x^*\| < \varepsilon$ such that $f + x^*$ attains a strong minimum. This means that there exists $x_0 \in \mathrm{dom} f$ satisfying $(f + x^*)(x_0) = \min(f + x^*)$ and $(f + x^*)(x_n) \to \min(f + x^*)$ implies $x_n \to x_0$ whenever $\{x_n\}$ is a sequence in dom$f$. Ghoussoub and Maurey [24] considered variational problems of w^*-type. They found that if f is a w^* lower semicontinuous proper function bounded below on the dual E^* of a Banach space E with bounded dom$f$, and with some additional assumptions on dom$f$, then the perturbation h can be chosen as a w^* continuous functional $x \in E$ with $\|x\| < \varepsilon$ such that $f + x$ attains a strong or strict minimum. There are still many types of variational principles such as the Brøndsted-Rockafellar theorem and the Bishop-Phelps theorem (see [3]) for convex sets and convex functions, a geometric version of Ekeland's variational principle—Danes' drop theorem (see [17, 29]), have found applications in wide variety of topics in nonlinear analysis.

However, it seems that, at least to the authors' knowledge, there are few variational principles with linear perturbation can cast in a general class of locally convex spaces. Starting from this view point, the first aim of this paper, was motivated by Ghoussoub and Maurey [24], is to establish variational principles of a w^* type for w^* lower semicontinuous functions in the duals of locally convex spaces.

Theorem 1.1. *Suppose that f is a real-valued w^* lower semicontinuous function defined on a w^* closed bounded subset A^* of the dual E^* of a locally convex space E, which is bounded below on A^*, and suppose every w^* closed convex subset of $C^* \equiv w^*\text{-clco}A^*$ is the w^* closed convex hull of its w^*-β-exposed points. Then for every $\varepsilon > 0$, there exist $x^* \in A^*$ and $x \in E$ such that*

(i) $\sup_{A^*} |x| < \varepsilon$;

(ii) $(f + x)(y^*) > (f + x)(x^*)$ *for all* y^* *in* $A^* \setminus \{x^*\}$;

(iii) $(f + x)(x_\iota^*) \to (f + x)(x^*)$ *implies* $x_\iota^* \xrightarrow{\beta} x^*$ *whenever* $\{x_\iota^*\}$ *is a net in* A^*.

The letter β in the theorem above denotes a bornology, that is, a family of bounded subsets in E whose union is the whole space E and $x_\iota^* \xrightarrow{\beta} x^*$ means x_ι^* uniformly converges to x^*on each number S of β. We should note that there are many possibility for choosing β. For example, if we choose β to be the family of all bounded sets and all singletons in E, resp., then we immediately obtain the Fréchet bornology and the Gateaux bornology, respectively.

Suppose that A^* is a subset in the dual E^* of a locally convex space E. $x^* \in A^*$ is said to be a w^*-β-exposed point of A^* if there exists $x \in E$ such that $\langle x^*, x\rangle > \langle y^*, x\rangle$ for all y^* in

$A^*\backslash\{x^*\}$ and $\langle x_\iota^*, x\rangle \to \langle x^*, x\rangle$ implies $x_\iota^* \xrightarrow{\beta} x^*$ whenever $\{x_\iota^*\}$ is a net in A^*. Particularly, a w^*-β-exposed point is called a w^*-strongly-exposed point (w^*-exposed point, resp.) if the bornology β is taken to be the Fréchet bornology (the Gateaux bornology, resp.). Thus, the following theorems are immediate consequences of Theorem 1.1.

Theorem 1.2. *Suppose that f is a real-valued w^* lower semicontinuous function defined on a w^* closed bounded subset A^* of the dual E^* of a locally convex dual E, which is bounded below on A^*, and suppose every w^* closed convex subset of $C^* \equiv w^*$-clcoA^* is the w^* closed convex hull of its w^*-strongly-exposed points. Then for every $\varepsilon > 0$, there exist $x^* \in A^*$ and $x \in E$ such that*

(i) $\sup\limits_{A^*} |x| < \varepsilon$;

(ii) $(f + x)(y^*) > (f + x)(x^*)$ *for all* y^* *in* $A^* \setminus \{x^*\}$;

(iii) $(f + x)(x_\iota^*) \to (f + x)(x^*)$ *implies that* $x_\iota^* \to x^*$ *whenever* $\{x_\iota^*\}$ *is a net in* A^*.

Theorem 1.3. *Suppose that f is a real-valued w^* lower semicontinuous function defined on a w^* closed bounded subset A^* of the dual E^* of a locally convex space E, and suppose every w^* closed convex subset of $C^* \equiv w^*$-clcoA^* is the w^* closed convex hull of its w^*-exposed points. Then for every $\varepsilon > 0$, there exist $x^* \in A^*$ and $x \in E$ such that*

(i) $\sup\limits_{A^*} |x| < \varepsilon$;

(ii) $(f + x)(y^*) > (f + x)(x^*)$ *for all* y^* *in* $A^* \setminus \{x^*\}$.

If C^* is a w^* closed convex set in a dual Banach space E^*, then the assumption of Theorem 1.1 on C^* is equivalent to that C^* has the Radon-Nikodým property (RNP) (see, for instance, [7]). We should mention that Theorem 1.2 (Theorem 1.3, resp.) has been shown by Ghoussoub and Maurey [24] with the assumptions: (a) E^* is the dual of a Banach space E; (b) $C^* \equiv w^*$-clcoA^* has the RNP (C^* is w^* metrizable, resp.); (c) $C^*\backslash A^* = \bigcup\limits_{n=1}^{\infty} K_n^*$, where K_n^* are w^* compact convex sets with $d\,(A^*, K_n^*) > 0$ for all $n \in N$.

Suppose that C^* is a w^* closed convex set in the dual E^* of a locally convex space E; we say the w^*-β variational principle holds on C^* if Theorem 1.1 is valid for every w^* closed and bounded set $A^* \subset C^*$. We also have the following converse version of Theorem 1.1.

Theorem 1.4. *The w^*-β variational principle holds on a bounded w^* closed convex set C^* if and only if C^* has the w^*-β-exposed property, that is, every w^* compact convex set in C^* is the w^* closed convex hull of its w^*-β-exposed points.*

We should also remark that the proof of Theorem 1.1 is quite different from Ghoussoub-Maurey's. The proof of the theorem in this paper is divided into 3 parts. The first one (i.e., Section 2) shows "if the w^* closed convex set C^* in E^* has the w^*-β-exposed property, so dose $C^* \times R$" through discussing differentiability property of convex functions on nonlinear topological spaces; the second part (Section 3) presents some formulas of convexification of nonconvex functions, and the last part (Section 4) completes the proof.

The second target is to search for a characterization of a continuous convex function f

on a Banach space E satisfying the property that every convex function g with $g \le f$ on E must have some kind of differentiability points.

A continuous convex function f is said to be β-differentiable at $x \in E$ if

$$\lim_{h\to 0}\sup_{y\in S}\Big[\frac{f(x+hy)-f(x)}{h}-\langle x^*, y\rangle\Big]=0$$

for some $x^* \in E^*$ and for every S in β; f is said to have the βDP (for β-differentiability property) if every continuous real-valued convex function g with $g \le f$ is densely β-differentiable in E.

Recently, a number of articles investigated Fréchet differentiability property of convex functions on general Banach spaces (see, for instance, [9–11, 26, 34]). [9] introduced the notion of Fréchet differentiability property (FDP) of convex functions on a Banach space E : A continuous convex function f is said to have FDP provided every continuous convex function g with $g \le f$ is densely Fréchet differentiable in E. It showed that such a function f can be characterized by $\partial f(E)$ (the image of the subdifferential mapping ∂f of f) being seperable on each seperable subspace of E, also by $\partial f(E)$ having the RNP. [11] further proved that for some suitable locally convex topology τ all continuous convex functions with the FDP on the Banach space E are exactly all continuous convex functions on the locally convex space (E, τ).

Corresponding to the FDP, a continuous convex function f on a Banach space E is said to have Gateaux differentiability property (GDP) provided every continuous convex function g with $g \le f$ is densely Gateaux differentiable in E. Article [11] also arised the following question:

($*$) How to characterize a continuous convex function f with the GDP?

This paper proves the following

Theorem 1.5. *Suppose that f is a continuous convex function on a Banach space E and f^* is the conjugate of f on E^* (the dual of E). Then*

(i) *f has the* βDP *if and only if each level set of f^* in E^* has w^*-β-exposed property; In particular,*

(ii) *if p is a seminorm, then p has the* βDP *if and only if every p-continuous convex function is densely β-differentiable in E.*

We should note that if we take β to be the Gateaux bornology in Theorem 1.5(i), then the problem ($*$) is completely answered. It also arises some interesting questions, for example, whether the sum of two continuous convex functions again has the GDP if both the two functions have, and an affirmative answer to this question should completely resolve the long-standing open problem that whether the product of two Gateaux differentiability spaces is again a Gateaux differentiability space with a positive solution (see [30]).

The concept of GDS (MDS)—those Banach spaces on which every continuous convex function (Minkowski functional) is densely Gateaux differentiable in its open domain—was introduced by G. Larman and R. R. Phelps [28] in 1979, and they showed that a Banach space E is a GDS if and only if $E \times R$ is an MDS, or equivalently, $E^* \times R$ has the w^*-exposed property. Several years later, it was showen by Fabian [30] that if E is a GDS, then so is $E \times R$, which was also generalized to the product of a GDS E and a seperable Banach space [12]. However, the proof of Theorem 1.5 is more difficult than those of the FDP of a convex function in [9, 10] and the Fabian-Larman-Phelps theorem since we can neither take advantage of the RNP and its equivalents of sets in the dual, nor apply Bishop-Phelps' Parrallel-Hyperlane Lemma (see [8, 30]) and linearity of the topology on E. This is done through establishing a more general Fabian-Larman-Phelps theorem or a version of the FLP theorem in Minkowski topological spaces.

§ 2 . Differentiability of Convex Functions on Sublinear Topological Spaces

This section is a key for showing Theorems 1.1 and 1.5.

I. Convex functions on Minkowski topological spaces

Suppose that X is a real linear space and $\wp$ is a family of real-valued Minkowski functionals on X, that is, for every $p \in \wp$ there exists a convex absorbing set $V \subset X$ such that $p(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in V\}$. We put $U_{p,\alpha} = \{x \in X : p(x) < \alpha\}$ and $\mho = \{U_{p,\alpha} : p \in \wp, \alpha > 0\}$. Then we call the sublinear topology generated by the local base $\mho$, which we still denote by $\wp$, Minkowski topology on X and $(X, \wp)$ a Minkowski topological space, or simply, a Minkowski space. If the generating family is a singleton $\{p\}$, then the corresponding topology is also denoted by p instead of $\wp$.

Clearly, a Minkowski topological space $(X, \wp)$ is a kind of sublinear topological space, that is, both the operations:

$$(x, y) \to x + y, \qquad x, y \in X$$

and

$$(\alpha, x) \to \alpha x, \qquad \alpha \geq 0$$

are continuous.

Proposition 2.1. *A Minkowski topological space* $(X, \wp)$ *is a locally convex* (*not necessarily Hausdorff*) *space if and only if for each* $U \in \mho$, *there exists an absolutely convex absorbing set* $V \in \mho$ *such that* $V \subset U$.

For a Minkowski topology $\wp$ generated by the local base $\mho$ on X, we denote by $|\wp|$ the topology generated by the local base $|\mho| = \{(-U) \cap (U) : U \in \mho\}$. That is, the generating family $|\wp|$ of seminorms is $\{|p| : p \in \wp\}$, where $|p|$ is defined by $|p|(x) = \max\{p(-x), p(x)\}$ for all $x \in X$.

We denote by $(X, \wp)^*$ the cone of real valued functions in X which are bounded above on some $U \in \mho$, and also by $(X, |\wp|)^*$ the dual of the locally convex space $(X, |\wp|)$ in the usual sence. It is easy to observe $(X, \wp)^* \subset (X, |\wp||)^*$. Thus, for w^* topology on $(X, \wp)^*$ we mean the w^* topology on $(X, |\wp|)^*$ which is restricted to $(X, \wp)^*$.

An extended-real-valued function defined on a subset D of $(X, \wp)$ is said to be

(i) upper semicontinuous at $x \in D$ provided for every $\varepsilon > 0$, there exists $U \in \mho$ such that $f(x+u) - f(x) < \varepsilon$ for all $x \in U$;

(ii) p-upper semi-Lipschitzian around $x \in D$ if there exist $p \in \wp$, $U \in \mho$ and $L > 0$ such that

$$f(z) - f(y) \leq Lp(z-y) \qquad \text{whenever} \;\; y, z \in x + U.$$

Other notions such as continuity, lower semi-continuity, p-lower semi-Lipschitzian around x and p-locally semi-Lipschitzian on D are automatically meaningful.

In this paper, for any Minkowski space $(X, \wp)$ we can always assume that $(X, |\wp|)$ is a Hausdorff space; otherwise, we substitute the quotient space X/K for X, where $K = \cap\{\ker|p| : |p| \in |\wp|\}$.

Proposition 2.2. *Suppose that f is a real-valued convex functon defined on a nonempty open convex set D of a Minkowski space $(X, \wp)$. Then the following statements are equivalent:*

(i) *f is upper bounded on $x_0 + U$ for some $x_0 \in D$ and $U \in \mho$;*

(ii) *f is upper semicontinuous at some point $x_0 \in D$;*

(iii) *f is locally upper bounded on D;*

(iv) *f is upper semicontinuous on D;*

(v) $\mathrm{epi}_D f$ *(the epigraph of f on D) is solid in $X \times R$, where the topology $\wp$ on $X \times R$ means the usual product topology of X and R.*

Proof. We need only to show (i)⇒(iii) and (iii)⇒(iv). Without loss of generality we assume $0 \in D$ and $f(0) = 0$; otherwise, choosing any $x_0 \in D$ we substitute $g = f(\cdot + x_0) - f(x_0)$ for f. We can also assume $0 < \alpha = \sup\limits_V f < \infty$ for some $V \in \mho$ with $V \subset D$.

(i)⇒(iii) Given $z_0 \in D$, convexity and openness of D imply that there exists $\gamma > 1$ such that $\gamma z_0 \in D$ and further there exists $V_0 \subset V$, $V_0 \in \mho$ such that $\gamma z_0 + V_0 \subset D$. Let $\lambda = 1 - \gamma^{-1}$ and $U = \lambda V_0$. For any $z = z_0 + u \in z_0 + U$, there exists $v_0 \in V_0$ such that $z = z_0 + \lambda v_0$. So we have

$$f(z) = f(z_0+u) = f(z_0 + \lambda v_0) \leq \gamma^{-1} f(\gamma z_0) + (1-\gamma^{-1}) f(v_0) \leq \gamma^{-1} f(\gamma z_0) + (1-\gamma^{-1})\alpha. \quad (**)$$

This says that f is upper bounded by $\gamma^{-1} f(\gamma z_0) + (1-\gamma^{-1})\alpha$ on $z_0 + \lambda V_0$.

(iii)⇒(iv) Inequalities $(**)$ imply $f(z) - f(z_0) \leq \gamma^{-1} f(\gamma z_0) - f(z_0) + (1-\gamma^{-1})\alpha$ for all $z \in z_0 + U = z_0 + (1-\gamma^{-1})V_0$ for some $V_0 \in \mho$ and all $\gamma \in (1, 1+\delta)$ for some $\delta > 0$. Letting $\gamma \to 1^+$ we see that $\lim\limits_{z \to z_0} \sup f(z) - f(z_0) \leq 0$. So that f is upper semicontinuous at z_0.

Proposition 2.3. *Every upper semicontinuous convex function on a nonempty convex set D of $(X, \wp)$ is $|\wp|$-continuous on D.*

Proof. It suffices to note that $(X, |\wp|)$ is a linear topological (actually, a locally convex) space, and $\wp$-upper continuty implies $|\wp|$-continuity.

Proposition 2.4. *suppose that (X,τ) is a locally convex space and f is a real valued convex function on a nonempty open convex set $D \subset (X,\tau)$. Then the following assertions are equivalent:*

(i) *f is locally bounded above on D;*

(ii) *f is continuous on D;*

(iii) *there is a continuous seminorm p such that f is p-locally Lipschitzian on D.*

Proof. By Propositions 2.2 and 2.3, it suffices to show (ii)⇒(iii). We assume $0 \in D$ and $f(0) = 0$. By the assumption, f is locally bounded on D. Thus there exists an absorbing and absolutely convex set $U \in \mho$ with $U \subset D$ such that f is p-locally Lipschitzian and bounded by M on U.

First we show that f is p-locally bounded on D. For each $z_0 \in D$, there exist $\gamma > 1$ such that $\gamma z_0 \in D$. Both $\{U, \gamma z_0\} \subset D$ and convexity of D imply

$$\frac{1}{\gamma}(\gamma z_0) + (1-\gamma^{-1})U = z_0 + (1-\gamma^{-1})U \subset D.$$

Clearly, f is upper bounded by $\gamma^{-1}f(\gamma z_0) + (1-\gamma^{-1})M$ on $z_0 + (1-\gamma^{-1})U$. Hence f is p-locally bounded above on D and D is a p-open convex set. Note that (X,p) is itself a locally convex space, f must be p-continuous on D.

Next we show that for every $x \in D$ there exist $U \in \mho(p)$ and $L > 0$ such that $|f(z) - f(y)| \le Lp(z-y)$ whenever $z, y \in x + U$. For each fixed $z_0 \in D$, there exists $V \equiv \{x \in X : p(x) < 2\delta_0\}$ for some $\delta_0 > 0$ with $z_0 + V \subset D$ such that f is bounded by C on $z_0 + V$. Let $U = \frac{1}{2}V$. Given $x, y \in z_0 + U$, let $\alpha_n = p\,(y-x) + \frac{1}{n}, z_n = y + \frac{\delta_0}{\alpha_n}(y-x)$, we see

$$p\,(z_n - z_0) \le p\,(y - z_0) + \frac{\delta_0}{\alpha_n} p\,(y-x) < 2\delta_0,$$

that is, $z_n \in z_0 + V$ for all $n \ge 1$.

Since $y = \frac{\alpha_n}{\alpha_n+\delta_0} z_n + \frac{\delta_0}{\alpha_n+\delta_0} x \in V$, we have $f(y) \le \frac{\alpha_n}{\alpha_n+\delta_0} f(z_n) + \frac{\delta_0}{\alpha_n+\delta_0} f(x)$. So

$$f(y) - f(x) \le \frac{\alpha_n}{\alpha_n+\delta_0}[f(z_n) - f(x)] \le \frac{\alpha_n \cdot 2C}{\alpha_n+\delta_0} \le \frac{2C}{\delta_0}\alpha_n = \frac{2C}{\delta_0}\Big(p(y-x) + \frac{1}{n}\Big).$$

Letting $n \to \infty$ and interchanging x and y we observe $|f(y) - f(x)| \le \frac{2C}{\delta_0}(p(y-x))$. This says f is p-locally Lipschitzian on D.

Proposition 2.4 yields the following

Proposition 2.5. *Suppose that f is a continuous convex function defined on an open convex set D of a locally convex space (X,τ). Then there exists a continuous seminorm p on (X,τ) such that for each x there exists a p-open neighbourhood U of x and a p-Lipschitzian convex function g such that $g = f$ in U.*

Proof. Assume that p satisfies Proposition 2.4(iii), that is, for each $x \in D$ there exist a p-open convex neighbourhood U of x and $L > 0$ such that $|f(z) - f(y)| \leq Lp(z - y)$ whenever $z, y \in U$. Fix any $x \in D$. Let L and U be as above. Define $\bar{f}$ on X by $\bar{f}(z) = f(z)$ if $z \in U$; $= \infty$, otherwise, and let $g(x) = \inf\{\bar{f}(x) + Lp(y - x) : y \in X\}$. It is not difficult to check that g is convex on X with $g = \bar{f} = f$ in U and is p-Lipschitzian on X with the constant L.

Proposition 2.6. *Suppose that $(X, \wp)$ is a Minkowski space. If for each ray L starting from the origin of X' there exists $x^* \in (X, \wp)^*$ such that $x^* \notin L$, then there exists a hyperplane $H (\subset X)$ through the origin of X such that $(H \times R, \bar{\wp}) \cong (X, \wp)$, where $\bar{\wp}$ denotes the product topology of $\wp$ on H and the usual topology on R.*

Proof. Let $K = \{x \in X : p(x) = 0 \text{ for all } p \in \wp\}$. Then K is a cone of X. We observe that K is not a half-space of X; otherwise $(X, \wp)^*$ must lie in a ray. Therefore there exists $x \in X$ such that $\{\pm x\} \cap K = \emptyset$ and further there exists $p \in \wp$ such that $p(\pm x) > 0$. We can assume that $p(x) = 1$. Take $x^* \in (X, \wp)^*$ with $\langle x^*, x\rangle = 1$ and let $H = \ker x^*$. Then for any $z \in X$ it has the following unique decomposition $z = h + \alpha x$ for some $h \in H$ and $\alpha \in R$. Define $\bar{p} : H \times R \to R^+$ by $\bar{p}(h, \alpha) = p(x)$, and let $\bar{\wp} = \{\bar{p} : p \in \wp\}$. Clearly $(H \times R, \bar{\wp}) \cong (X, \wp)$ and the topology $\bar{\wp}$ restricted to R is equivalent to the usual product topology on R, and further the topology $\bar{\wp}$ on $H \times R$ is equivalent to the product topology of $\wp$ restricted to H and the usual topology on R.

II. Subdifferentiability of convex functions

It is well known that for a lower semicontinuous convex function f defined on a closed convex set C of a Banach space the Brøndsted-Rockafellar theorem (see [3]) shows that the set at each point of which subdifferential of f exists is always dense in C, but the theorem holds no longer in general locally convex spaces. This says that the subdifferential set of a lower semicontinuous proper convex function on a locally convex space may be empty. So we will deal with upper semicontinuous convex functions on Minkowski spaces.

Definition 2.1. *Suppose that $(X, \wp)$ is a Minkowski space. A bornology in X, denoted by β, is a family of bounded sets satisfying*

(i) *$S \in \beta$ implies $\lambda S \in \beta$ for all $\lambda \in R$;*

(ii) $X = \cup\{S : S \in \beta\}$.

It is not difficult to show that if (X, τ) is a locally convex space, and $(X, \tau)^*$ is its continuous dual, then $\mho^*_\beta$ forms a local base of a locally convex (Hausdorff) topology β on $(X, \tau)^*$, where $\mho^*_\beta$ is defined by

$$\mho^*_\beta = \{U^*_{S,\alpha} : S \in \beta,\ \alpha \in R^+\}$$

and

$$U^*_{S,\alpha} = \{x^* \in (X, \tau)^* : |\langle x^*, x\rangle| < \alpha,\ \text{for all } x \in S\}.$$

We call the linear topology on $(X,\tau)^*$ generated by the local base $\mho^*_\beta$ β-topology.

Definition 2.2. *Suppose that f is a convex function defined on a nonempty open convex set D of a Minkowski space $(X,\wp)$.*

(i) *The subdifferentiable mapping ∂f of $f: D \to 2^{(X,|\wp|)^*}$ is defined by*

$$\partial f(x) = \{x^* \in (X,|\wp|)^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \text{ for all } y \in D\}.$$

(ii) *f is said to be β-differentiable at $x \in D$ if there exists a (unique) $x^* \in (X,|\wp|)^*$ such that $\forall S \in \beta$,*

$$\lim_{t\to 0^+} \sup_{y\in S} \Big[\frac{f(x+ty) - f(x)}{t} - \langle x^*, y\rangle\Big] = 0.$$

In particular, we say f is Fréchet (Gateaux) differentiable at x provided the equality above holds for the Fréchet bornology (Gateaux bornology) β.

(iii) *We say the Minkowski space $(X,\wp)$ has β-differentiability property if every upper semicontinuous convex function f defined on $(X,\wp)$ is $\wp$-densely β-differentiable in it.*

Definition 2.3. *Suppose that $(X,\wp)$ is a Minkowski space and C^* is a w^* closed convex subset of $(X,\wp)^*$.*

(i) *A point $x^* \in C^*$ is said to be a w^*-β-exposed point of C^* if there exists $x \in X$, $x \neq 0$, such that $\langle x^*, x\rangle = \sup_{C^*}\langle \cdot, x\rangle$ and $\langle x^*_\alpha, x\rangle \to \langle x^*, x\rangle$ implies that $x^*_\alpha \xrightarrow{\beta} x^*$, whenever $\{x^*_\alpha\}$ is a net in C^*;*

(ii) *C^* has the w^*-β-exposed property if and only if every w^* compact convex subset of C^* is the w^* closed convex hull of its w^*-β-exposed points.*

Proposition 2.7. *Suppose that f is a real-valued upper semicontinuous convex function defined on a nonempty open convex set D of a Minkowski space X. Then*

(i) *∂f is nonempty, w^* compact valued everywhere in D and with $\partial f(D) \subset (X,\wp)^*$;*

(ii) *∂f is $|\wp|$-w^* upper semicontinuous on D;*

(iii) *∂f is $|\wp|$-locally upper bounded on D.*

Proof. Note that f is upper semicontinuous and convex on $D \subset (X,\wp)$, and that $(X,|\wp|)$ is a locally convex space, and note f is $|\wp|$-continuous on D. Thus, the subdifferential mapping ∂f is nonempty w^*-compact-convex-valued and w^* upper semicontinuous on D. It remains to show $\partial f(D) \subset (X,\wp)^*$.

Suppose $x_0 \in D$ and $x^*_0 \in \partial f(x_0)$. Since f is $\wp$-upper semicontinuous on D, it must be $\wp$-locally bounded above on D. Therefore there exist $U_0 \in \mho$ and $\alpha > 0$ such that f is bounded above by α on $x_0 + U_0 \subset D$. Consequently,

$$\sup_{U_0} x^*_0 \leq \sup_{U} f - f(x_0) \leq \alpha - f(x_0).$$

This means x^*_0 is bounded above on U_0 by $\alpha - f(x_0)$, and further says $x^*_0 \in (X,\wp)^*$.

Proposition 2.8. *Suppose that $(X, \wp)$ is a Minkowski space and f is an upper semicontinuous convex function on a nonempty open convex set $D \subset X$. Then the following statements are equivalent*:

(i) *f is β-differentiable at $x \in D$;*

(ii) *∂f is single-valued and $|\wp|$-β upper semicontinuous at x;*

(iii) *every selection for ∂f on D is $|\wp|$-β continuous at x;*

(iv) *there exists a selection ϕ for ∂f on D such that ϕ is $|\wp|$-β continuous at x.*

Proof. It suffices to note that $(X, |\wp|)$ is a locally convex space, that the convex function f is $|\wp|$-continuous on D, and that a bornology β in $(X, \wp)$ is again a bornology in $(X, |\wp|)$, since the corresponding property holds in locally convex spaces (see, for instance, [21, p.129]).

Proposition 2.9. *Suppose that p is a continuous sublinear functional on a Minkowski space $(X, \wp)$. Then*

(i) *$x^* \in \partial p(x)$ if and only if $x^* \in C^* \equiv \partial p(0)$ with $\langle x^*, x\rangle = p(x)$;*

(ii) *p is β-differentiable at x if and only if x is a w^*-β-exposing functional of C^* and exposing C^* at some point $x^* \in C^*$. In this case, we have $\partial p(x) = x^*$.*

Proof. (i) Suppose that $x^* \in \partial p(x)$. By definition,

$$p(y) - p(x) \geq \langle x^*, y - x\rangle \qquad \text{for all } y \in X.$$

Letting $y = 2x$ and $y = 0$, we see that $p(x) \geq \langle x^*, x\rangle$ and $-p(x) \geq -\langle x^*, x\rangle$ resp. This explains $p(x) = \langle x^*, x\rangle$. To see $x^* \in \partial p(0)$. It suffices to note that $p(z) \geq \langle x^*, z\rangle$ for all z in X by taking y to be $x + z$ since $p(x + z) \leq p(x) + p(z)$.

(ii) The proof is much like that of [30, Proposition 5.11] and just with the following changes: Substitute the continuous sublinear functional p, the locally convex space $(X, |\wp|)$, Proposition 2.8, the upper bound of $x_n^* - x^*$ on each given $S \in \beta$ and β-differentiability, successively, for the Minkowski functional p, the Banach space E, Proposition 5.10 of [6], the norm of $x_n^* - x^*$ and Fréchet differentiability.

Proposition 2.10. *Suppose that f is an upper semicontinuous convex function on a nonempty open convex set D of a Minkowski space $(X, \wp)$, that $0 \in D$ with $f(0) = -1$, and that p is the Minkowski functional generated by $\mathrm{epi}_D f$. Then*

(i) *$x^* \in \partial f(x)$ if and only if $(r^*)^{-1}(x^*, -1) \in \partial p\,(x, r)$;*

(ii) *f is β-differentiable at x if and only if p is β-differentiable at (x, r), where $r = f(x)$, $r^* = f^*(x^*) \geq 1$, and $f^*(x^*) = \sup\{\langle x^*, y\rangle - f(y) : y \in D\}$.*

Proof. Since f is upper semicontinuous on the nonempty convex set D, by Proposition 2.2, $\mathrm{epi}_D f$ is solid. Since $0 \in D$ with $f(0) = -1$, we have $(0, 0) \in \mathrm{int}(\mathrm{epi}_D f)$. Thus the Minkowski functional p generated by $\mathrm{epi}_D f$ is upper semicontinuous on $X \times R$, and further, it is continuous.

(i) it suffices to note $f^*(x^*) \geq \langle x^*, 0\rangle - f(0) = 1$ and note the following

$$\begin{aligned}
& x^* \in \partial f(x) \\
\Leftrightarrow\ & f(y) - f(x) \geq \langle x^*, y - x\rangle \quad \text{for all } y \in D \\
\Leftrightarrow\ & f^*(x^*) \equiv \sup\{\langle x^*, y\rangle - f(y) : y \in D\} = \langle x^*, x\rangle - f(x) \\
\Leftrightarrow\ & \langle x^*, x\rangle - f(x) \geq \langle x^*, y\rangle - s \quad \text{for all } y \in D \text{ and } s \geq f(y) \\
\Leftrightarrow\ & f^*(x^*) = \langle (x^*, -1), (x, f(x))\rangle \geq \langle (x^*, -1), (y, s)\rangle \quad \text{for all } (y, s) \in \mathrm{epi}_D f \\
\Leftrightarrow\ & [f^*(x^*)]^{-1}\langle (x^*, -1), (x, f(x))\rangle \geq [f^*(x^*)]^{-1}\langle (x^*, -1), (y, s)\rangle \quad \text{for all } (y, s) \in \mathrm{epi}_D f,
\end{aligned}$$

that is, $x^* \in \partial f(x) \Leftrightarrow [f^*(x^*)]^{-1}(x^*, -1) \in \partial p\,(x, f(x))$.

(ii) It immediately follows from (i) and Proposition 2.9.

III. $\boldsymbol{\beta}$-Differentiability and $\boldsymbol{w^*}$-$\boldsymbol{\beta}$-exposed property

Proposition 2.11. *If a Minkowski space $(X, \wp)$ does not have β-differentiability property, then there exists a real-valued upper semicontinuous convex function on $(X, \wp)$ which is nowhere β-differentiable in X.*

Proof. Suppose that f is an upper semicontinuous convex function on $(X, \wp)$ which is nowhere β-differentiable in a nonempty open convex set D of X.

Without loss of generality we assume $0 \in D$ with $f(0) = 0$, and further assume that there exist $U \in \mho$, $U \subset D$ and $\alpha > 0$ such that f is bounded above by α on U. $\forall\, n \in N$, let $f_n(x) = f(\frac{1}{n}x),\ x \in X$. Therefore for any $x \in X$, $\lim\limits_{n\to\infty} f_n(x) = 0$. Since for all $x \in X$ and $\lambda \in [0, 1]$ we have $f(\lambda x) \leq \lambda f(x),\ g(x) \equiv \sum\limits_{n=1}^{\infty} 2^{-n} f_n(x)$ is a real-valued convex function and is upper bounded by α on U. Thus by Proposition 2.2, g is upper semicontinuous on $(X, \wp)$. Since that g is β-differentiable at x implies that f_n are β-differentiable at x for all $n \in N$, and since for each $x \in U$ there exists $n \in N$ such that $\frac{1}{n}x \in U, f_n(x) = f(\frac{1}{n}x)$ and f is nowhere β-differentiable in U, g is nowhere β-differentiable in X.

Proposition 2.12. *Suppose that $(X, \wp)$ is a Minkowski space; if every continuous Min -kowski functional on X is densely β-differentiable in X, then there exists a $|\wp|$-closed hyperplane H such that $(H, \wp)$ has the β-differentiability property.*

Proof. Suppose that $(X, \wp)$ is a Minkowski space. If $(X, \wp)^*$ is a ray, that is, there exists $x^* \in (X, \wp)^*$ such that $(X, \wp)^* = \{\lambda x^* : \lambda \geq 0\}$. If $x^* = 0$, then $(X, \wp)^* = \{0\}$, this says $(X, \wp)$ itself is a (trival) space with the β-differentiability property. Thus for any hyperplane H through the origin, $(H, \wp)$ also has the β-differentiability property. If $x^* \neq 0$, let $H = \ker x^*$. Then $(H, \wp)^* = \{0\}$ and $(H, \wp)$ has the β-differentiability property. If $(X, \wp)^*$ does not lie in a ray, then, by Proposition 2.6, there exists a $|\wp|$-closed hyperplane H through the origin such that $(H \times R, \bar{\wp}) \cong (X, \wp)$, where the topology $\bar{\wp}$ on $H \times R$ is the product topology of $\wp$ on H and the usual topology on R.

We want to prove that $(H, \wp)$ has the β-differentiability property. Suppose that f is an upper semicontinuous convex function on H. Without loss of generality we assume

$f(0) = -1$. By Proposition 2.2, epi$f \subset (H \times R, \bar{\wp})$ is solid. Note $(0, 0) \in \operatorname{int} \operatorname{epi} f$. Let q be the Minkowski functional generated by epi$f$. Then q is upper seemicontinuous on $(H \times R, \bar{\wp})$, and further it is continuous. By the assumption of the lemma, every continuous Minkowski functional is densely β-differentiable in $(X, \wp)$, q must be $\bar{\wp}$-densely β-differentiable in $H \times R$ since $(H \times R, \bar{\wp}) \cong (X, \wp)$. By Proposition 2.9(ii), f is $\wp$-densely β-differentiable in H. This tells us that $(H, \wp)$ has the β-differentiability property.

The following lemma was motivated by [30, Proposition 6.5].

Lemma 2.1. *Suppose that $(X, \wp)$ is a Minkowski space, and suppose that f is an upper semicontinuous convex function on $X \times R$ and g is a continuous differentiable function on $I = (a, b) \subset R$ with $g \leq 0$ and $\lim\limits_{t \to a^+, b^-} g(t) = -\infty$. Let $h : X \to R$ be defined by $h(x) = \sup\{f(x, t) + g(t) : t \in I\}$. Then*

(i) *h is an upper semicontinuous convex function on X;*

(ii) *$F : X \to 2^I$ defined by $F(x) = \{t \in I : h(x) = f(x, t) + g(t)\}$ is an upper semicontinuous compact valued mapping;*

(iii) *if h is β-differentiable at x_0, then f is β-differentiable at (x_0, t_0), where $h(x_0) = f(x_0, t_0) + g(t_0)$, $t_0 \in F(x_0)$;*

(iv) *$\partial h(x) = w^*$-clco$\{\partial f(x, t)|_X : t \in F(x)\}$.*

Proof. (i) Convexity and upper semicontinuity of h are obvious.

(ii) Given $x \in X$, compactness of $F(x)$ is obvious. It remains to show that F is upper semicontinuous, that is, for each open interval V of R with $F(x) \subset V$, there is an open neighbourhood U of x such that $F(U) \subset V$. Otherwise, there exist $x_0 \in X$, an open set V with $V \supset F(x)$ and a net $\{x_\iota\}$ in X converging to x_0 and $\{t_\iota\} \in I$ with $t_\iota \in F(x_\iota)$ such that cl$\{t_\iota\} \cap V = \emptyset$. By definition of $F(x)$, we have $h(x_\iota) = f(x_\iota, t_\iota) + g(t_\iota)$. Compactness of $[a, b]$ and properties of f and g implies $\{t_\iota\}$ has an accumulating point $t_0 \in (a, b)$. Without loss of generality we assume $t_\iota \to t_0 \in I$. Upper semicontinuity of f, h and g imply

$$f(x_0, t_0) + g(t_0) \leq h(x_0) = \lim_\iota f(x_\iota, t_\iota) + \lim_\iota g(t_\iota) \leq f(x_0, t_0) + g(t_0).$$

This says that $t_0 \in F(x_0) \subset V$ and it is a contradiction.

(iii) Suppose that h is β-differentiable at x_0, and that $t_0 \in F(x_0)$. $\forall\, \varepsilon > 0$, $S \in \beta$, there exists $\delta > 0$ such that

$$0 \leq h(x_0 \pm ty) + h(x_0 \mp ty) - 2h(x_0) \leq t\varepsilon;$$

$$|g(t_0 \pm t) + g(t_0 \mp t) - 2g(t_0)| \leq t\varepsilon$$

whenever $0 < t < \delta$, $y \in S$. So

$$\begin{aligned} 0 &\leq f(x_0 \pm ty, t_0 \pm t) + f(x_0 \mp ty, t_0 \mp t) - 2f(x_0, t_0) \\ &\leq h(x_0 \pm ty) + h(x_0 \mp ty) - 2h(x_0) - [g(t_0 \pm t) + g(t_0 \mp t) - 2g(t_0)] \\ &\leq 2t\varepsilon. \end{aligned}$$

This says that f is β-differentiable at $(x_{0,}t_0)$.

(iv) To show $\partial h(x_0) \supset \partial f(x_{0,}t_0)|_X$ for all $t_0 \in F(X_0)$. Suppose that $(x^*, r) \in \partial f(x_{0,}t_0)$. Then for any $x \in X$,

$$\begin{aligned}\langle x^*, x\rangle &\leq f(x_0 + x, t_0) - f(x_{0,}t_0)\\ &= [f(x_0 + x) + g(t_0)] - [f(x_0, t_0) + g(t_0)]\\ &\leq h(x_0 + x) - h(x_0) \qquad \text{for all } x \in X.\end{aligned}$$

So $x^* \in \partial h(x)$.

Conversely, suppose that $x^* \in \partial h(x_0)$. Then for any $y \in X$, $\delta \in R^+$, there exists $t_\delta \in I$ such that $h(x_0 + \delta y) = f(x_0 + \delta y, t_\delta) + g(t_\delta)$. Thus

$$\begin{aligned}\delta\langle x^*, y\rangle &\leq h(x_0 + \delta y) - h(x_0) = [f(x_0 + \delta y, t_\delta) + g(t_\delta)] - h(x_0)\\ &= [f(x_0 + \delta y, t_\delta) - f(x_0, t_\delta)] + [f(x_0, t_\delta) + g(t_\delta) - h(x_0)]\\ &\leq f(x_0 + \delta y, t_\delta) - f(x_0, t_\delta) \leq \delta\langle x^*_\delta, y\rangle\end{aligned}$$

for any $x^*_\delta \in (X, \wp)^*$ with $(x^*_\delta, r_\delta) \in \partial f(x_0 + \delta y, t_\delta) \subset (X \times R, \wp)^*$ for some $r_\delta \in R$, where $\wp$ on $X \times R$ denotes the usual product topology. By (ii), there is a subsequence $\{t_n\}$ of the net $\{t_\delta\}$ $(\delta \to 0^+)$ such that $t_n \to t_0 \in F(x_0)$. Since ∂f is $|\wp|$-w^* upper semicontinuous on $X \times R$, for any selection ϕ of ∂f with $\phi(x_0 + \delta y, t_\delta) = (x^*_\delta, r_\delta)$ there is a w^*-accumulation point (x^*_0, r_0) $(\in \partial f(x_0, t_0))$ of the net $\{(x^*_n, r_n)\}$ corresponding to $\{t_n\}$. Therefore from the inequalities above we have

$$\langle x^*, y\rangle \leq \lim_{\delta\to 0^+} \frac{f(x_0 + \delta y, t_\delta) - f(x_0, t_\delta)}{\delta} \leq \langle x^*_0, y\rangle.$$

This says that for any $y \in X$ there exists $x^*_0 \in \partial f(x_0, t_0)|_X$ for some $t_0 \in F(x_0)$ such that $\langle x^*, y\rangle \leq \langle x^*_0, y\rangle$. Applying the separation theorem we see

$$x^* \in w^* - \text{clco}\{\partial f(x_0, t_0)|_X : t_0 \in F(x_0)\} \equiv C^*.$$

Thus $\partial h(x_0) \subset C^*$, and (iv) is proved.

Theorem 2.1. *If the Minkowski space $(X, \wp)$ has the β-differentiability property, so dose $X \times R$.*

Proof. Suppose that f is a real-valued upper semicontinuous convex function on $X \times R$, where the topology $\wp$ on $X \times R$ is the usual product topology. We claim that for any $\wp$-open convex set D in $X \times R$, there is a β-differentiability point of f in D. Without loss of generality, we assume that f is upper bounded by α on D. Thus there exist a nonempty open convex set U of X and an open interval $I = (a, b) \subset R$, with $0 < b - a < \infty$ such that $U \times I \subset D$. Choosing any differentiable non-positive valued function g on (a, b) such that $\lim\limits_{t\to a^+, b^-} g(t) = -\infty$ and such that $\max\limits_I g = 0$. Let

$$h(x) = \sup\{f(x, t) + g(t) : t \in I\}.$$

Clearly, h is a real-valued convex function and upper bounded on U. By Proposition 2.2, h is upper semicontinuous on U, and by the hypothesis, h is somewhere β-differentiable in U, say, at $x_0 \in U$. Let $h(x_0) = f(x_0, t_0) + g(t_0)$ for some $t_0 \in I$. Then by Lemma 2.1, we see that f is β-differentiable at $(x_0, t_0) \in U \times I \subset D$.

Theorem 2.2. *A Minkowski space $(X, \wp)$ has the β-differentiability property if and only if every continuous Minkowski functional has at least one β-differentiability point in X.*

Proof. It suffices to show the sufficiency. We can again assume that $(X, \wp)^*$ is not a ray. The proof of Proposition 2.12 and Proposition 2.6 explain that there exists a hyperplane $H \subset X$ such that $(H \times R, \bar{\wp}) \cong (X, \wp)$, where the topology $\wp$ on $H \times R$ is the product topology of $\wp$ on H and the usual topology on R. Therefore every continuous Minkowski functional on $(H \times R, \bar{\wp})$ has at least one β-differentiability point. This and Proposition 2.2(ii) inturn imply that every $\wp$-upper semicontinuous convex function on $(H, \wp) \cong (H \times R, \bar{\wp})$ has at least one β-differentiability point. Applying Proposition 2.11, we see $(H, \wp)$ has the β-differentiability property, and Theorem 2.1 further tells us that $(X, \wp) \cong (H \times R, \bar{\wp})$ again has the β-differentiability property.

Theorem 2.3. *A Minkowski space $(X, \wp)$ has the β-differentiability property if and only if for each $p \in \wp$, $\partial p(0) \equiv C^*$ has the w^*-β-exposed property.*

Proof. Sufficiency. Suppose that for each Minkowski functional $p \in \wp$, $\partial p(0) \equiv C^*$ has w^*-β-exposed property. We want to show that every continuous Minkowski functional q on $(X, \wp)$ has at least one β-differentiability point. Since q is continuous, there exists $U \in \mho$ such that $U \subset \{x \in X : q(x) \le 1\}$. Let q be the Minkowski functional generated by U. Then $q \le p$, and which implies $A^* \equiv \partial q(0) \subset \partial p(0) \equiv C^*$. Since C^* has the w^*-β-exposed property, the w^*-compact convex set A^* has at least one w^*-β-exposed point x^*. Proposition 2.9 explains that there is a point $x \in X$ such that q is β-differentiability at x and with $\partial q(x) = x^*$.

Necessity. Suppose that $(X, \wp)$ has β-differentiability property. Given $p \in \wp$, we claim that $C^* \equiv \partial p(0)$ has w^*-β-exposed property. For any nonempty w^*-compact convex set $A^* \subset C^*$, letting q be the support function of A^*, i.e., $q(x) = \sup_{A^*} \langle x^*, x \rangle$ for all $x \in X$; we see that q is upper semicontinuous on X, since $q \le p$ and since p is continuous. We observe that q is also lower semicontinuous. Indeed, given $x_0 \in X$, w^*-compactness of A^* implies that there is $x_0^* \in A^*$ such that $\langle x_0^*, x \rangle = q(x_0)$. Suppose that $\{x_\iota\}$ is a net in X with $x_\iota \to x_0$. We have

$$\liminf_{\iota} q(x_\iota) \ge \lim_{\iota} \langle x_0^*, x_\iota \rangle = \langle x_0^*, x_0 \rangle = q(x_0).$$

Therefore q is lower semicontinuous at x_0. This and the upper semicontinuity together imply that q is continuous.

Note $\partial q(0) = A^*$ and q is densely β-differentiable in X. By Proposition 2.8, the w^*-closed convex hull B^* of all w^*-β-exposed points of A^* is nonempty w^*-compact and convex. Since $B^* \subset A^*$, the support function r of B^* is continuous and with $r \le q$. If $B^* \ne A^*$, then

$G \equiv \{x \in X : (q-r)(x) > 0\}$ is a nonempty open set since both q and r are continuous. Thus there is a β-differentiability point $x_0 \in G$ of q. Again by Proposition 2.8, there is a w^*-β-exposed point x_0^* of A^* such that $q(x_0) = \langle x_0^*, x_0\rangle$. This says $x_0^* \in B^*$ and $\langle x_0^*, x_0\rangle = q(x_0) > r(x_0) \geq \langle x_0^*, x_0\rangle$, a contradiction.

Theorem 2.4. *Suppose that $(X, \wp)$ is a Minkowski space, that C^* is a nonempty w^*-compact convex set with $0 \in C^* \subset \partial p(0)$ for some $p \in \wp$, and suppose that C^* has w^*-β-exposed property. Then $C^* \times R \subset (X \times R, \wp)^*$ also has the w^*-β-exposed property.*

Proof. Let p be the support function of C^*. Then p is a continuous Minkowski functional with $\partial p(0) = C^*$. Note that (X, p) again is a Minkowski space. Since $\partial p(0) = C^*$ has the w^*-β-exposed property, by Theorem 2.3, (X, p) has the β-differentiability property. Applying Theorem 2.1, we observe that $(X \times R, p)$ has the w^*-β-exposed property, where the topology p on $X \times R$ is the usual product topology of p on X and τ on R. Note that for any bounded set $S \subset R$ the support function of $C^* \times S$ is continuous. We see that for any finite interval $I \subset R, C^* \times I$ has the w^*-β-exposed property and which says that $C^* \times R$ has the w^*-β-exposed property.

§ 3 . Convexification of Nonconvex Functions

We need to consider the convexification of nonconvex function when we deal with perturbed optimization of nonconvex functions. The idea is very useful and important in solving many problems, and we introduce it as follows.

Suppose that (E, τ) is a linear topological space and that $f : A \subset E \to R \cup \{+\infty\}$ is an extended-real-valued function. We denote

$$\begin{aligned}
&\text{(the epigraph of } f) \qquad && E(f) \equiv \{(x, r) \in A \times R; r \geq f(x)\},\\
&\text{(the hypergraph of } f) \qquad && H(f) \equiv \{(x, r) \in A \times R; r \leq f(x)\},\\
&\text{(the graph of } f) \qquad && G(f) \equiv \{(x, f(x)) \in A \times R\} = E(f) \cap H(f).
\end{aligned}$$

It is well known that f is convex if and only if $E(f)$ is convex in $E \times R$, and that f is concave (i.e., $-f$ is convex) if and only if $H(f)$ is convex. We also know that f is lower (upper) semicontinuous if and only if $E(f)$ $(H(f))$ is closed in $E \times R$.

Letting $E(g) = \mathrm{clco}[E(f)]$, and $H(h) = \mathrm{clco}[H(f)]$, we can obtain a lower semicontinuous convex function g and an upper semicontinuous concave function h. g and h are called the lower semicontinuous convexifiction and the upper semicontinuous concavication of f, respectively, which we denote by f_c and f_v, respectively. It is easy to check that for any extended-real-valued function f on E, $f_c \leq f \leq f_v$. Moreover, we have

Proposition 3.1. *Suppose that $f : A \subset E \to R \cup \{+\infty\}$ is an extended-real-valued function. Then for any $x \in E$, we have*

$$f_c(x) = \inf\{a : a = \liminf_{\iota} \alpha(x_\iota),\ \textit{whenever } \{x_\iota\} \textit{ is a net in } \mathrm{co}(A) \textit{ converging to } x\},$$

$$f_v(x) = \sup\{b : b = \limsup_{\iota} \beta(x_\iota),\ \textit{whenever } \{x_\iota\} \textit{ is a net in } \mathrm{co}(A) \textit{ converging to } x\},$$

where

$$\alpha(x) = \inf\Big\{\sum_{j=1}^{n}\lambda_j f(x_j) : n \in N, \lambda_j \geq 0 \textit{ with } \sum_{j=1}^{n}\lambda_j = 1; x_j \in A \textit{ with } \sum_{j=1}^{n}\lambda_j x_j = x\Big\},$$

$$\beta(x) = \sup\Big\{\sum_{j=1}^{n}\lambda_j f(x_j) : n \in N, \lambda_j \geq 0 \textit{ with } \sum_{j=1}^{n}\lambda_j = 1; x_j \in A \textit{ with } \sum_{j=1}^{n}\lambda_j x_j = x\Big\}.$$

This further implies $\sup f = \sup f_v$ *and* $\inf f = \inf f_c$.

Given an extended-real-valued function f on a locally convex space E, we define the conjugate f^* of f on E^* by

$$x^* \in E^*, \qquad f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},$$

and the bi-conjugate of f on E by

$$x \in E, \qquad f^{**}(x) = \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in E^*\}.$$

We should explain that if f is an extended-real-valued function defined on a locally convex space, then the convexification f_c of f on E is just the bi-conjugate of f. That is the following

Proposition 3.2. *Suppose that f is an extended-real-valued function defined on a locally convex space E. Then the convexification f_c of f on E is just the bi-conjugate f^{**}.*

§ 4 . On Variational Principles in Locally Convex Spaces

We are ready to prove Theorem 1.1. Now we restate the theorem as follows.

Theorem 4.1. *Suppose that f is a real-valued w^*-lower semicontinuous function defined on a w^* closed bounded subset A^* of the dual E^* of a locally convex space E, which is bounded below on A^*, and suppose every w^* closed convex subset of $C^* \equiv w^*\text{-clco}A^*$ is the w^* closed convex hull of its w^*-β-exposed points. Then for every $\varepsilon > 0$, there exist $x^* \in A^*, x \in E$ such that*

(i) $\sup_{A^*} |x| < \varepsilon$;

(ii) $(f + x)(y^*) > (f + x)(x^*)$ *for all* y^* *in* $A^* \setminus \{x^*\}$;

(iii) $(f + x)(x^*_\iota) \to (f + x)(x^*)$ *implies* $x^*_\iota \xrightarrow{\beta} x^*$, *whenever* $\{x^*_\iota\}$ *is a net in* A^*.

Or equivalently, we have

Theorem 4.1′. *Suppose that f is a real-valued w^*-upper semicontinuous function defined on a w^* closed bounded set A^* in the dual E^* of a locally convex space E, which is bounded above on A^*; and suppose that every w^* closed convex subset of $C^* \equiv w^*\text{-clco}A^*$*

is the w^* closed convex hull of its w^*-β-exposed points. Then for every $\varepsilon > 0$ there exist $x^* \in A^*$ and $x \in E$ such that

(i) $\sup_{A^*} |x| < \varepsilon$;

(ii) $(f + x)(x^*) > (f + x)(y^*)$ *for all* y^* $(\neq x^*)$ *in* A^*;

(iii) $(f + x)(x^*_\iota) \to (f + x)(x^*)$ *implies* $x^*_\iota \xrightarrow{\beta} x^*$, *whenever* $\{x^*_\iota\}$ *is a net in* A^*.

We should remark that Theorem 4.1 and Theorem 4.1$'$ are equivalent since f is w^* lower semicontinuous and bounded below if and only if $-f$ is w^* upper semicontinuous and bounded above. For example, if Theorem 4.1 holds and if f is a function satisfying the assumptions of Theorem 4.1$'$, then $-f$ satisfies the assumptions of Theorem 4.1. Thus there exist $x^* \in A^*, x \in E$ such that (i), (ii) and (iii) of Theorem 4.1 hold with $-f$. We substitute $-x$ for x and easy observe that Theorem 4.1$'$ holds with f.

Proof of Theorem 4.1. Without loss of generality we assume $0 \in A^*$; otherwise, choosing any $x^*_0 \in A^*$, letting $f_0 = f(x^*_0 + \cdot)$ and $A^*_0 = A^* - x^*_0$, we substitute f_0 and A^*_0 for f and A^*, respectively. Since A^* is bounded and E^* is also a locally convex space, we observe that $C^* \equiv w^*$-clcoA^* is w^*-compact and convex and that the support function p of C^* is a lower semicontinuous real-valued Minkowski functional on E. Note that (E, p) is a Minkowski space and $\partial p(0) = C^*$ has the w^*-β-exposed property. By Theorem 2.4, $C^* \times R$ again has the w^*-β-exposed property.

Let $M = \inf\{f(x^*) : x^* \in A^*\} > -\infty$. We can assume that for any $\mu > 0$,

$$\{x^* \in A^* : f(x^*) \leq M + \mu\}$$

is not a singleton. Given $\varepsilon > 0$, choose $r \in R$ with $r - M > 3\delta$ and $\varepsilon \geq \delta > 0$. We define $\bar{f} : A^* \to R$ by $\bar{f}(x^*) = f(x^*)$, if $f(x^*) \leq r$; $= r$, otherwise. Then $\bar{f}$ is w^* lower semicontinuous and bounded on A^*. Let $E(\bar{f}_c) = w^*$-clco$[E(\bar{f})]$ and $H(\bar{f}_v) = w^*$-clco$[H(\bar{f})]$. Then $\bar{f}_c$ $(\bar{f}_v)$ is a w^* lower (upper) semicontinuous convex (concave) function bounded on w^*-clcoA^*. Thus, $S^* \equiv E(\bar{f}_c) \cap H(\bar{f}_v)$ is a nonempty w^*-closed bounded and convex set in $C^* \times R$. Since $C^* \times R$ has the w^*-β-exposed property, S^* has the w^*-β-exposed property and w^*-β-exposing functionals of S^* are dense in $E \times R$, where the topology on $E \times R$ is the product topology of the topology generated by the Minkowski functional p on E and the usual topology on R.

We claim that all w^*-β-exposed points of S^* are in $G(\bar{f}_c) \cup G(\bar{f}_v)$. In fact, supposing that $(x^*, r^*) \in C^* \times R$ is an extreme point of S^* with $x^* \in w^*$-clco$A^* \equiv C^*_0$, and $\bar{f}_c(x^*) < r^* < \bar{f}_v(x^*)$, and choosing any δ' with $0 < \delta' < \min\{r^* - \bar{f}_c(x^*), \bar{f}_v(x^*) - r^*\}$, we have $(x^*, r^* \pm \delta') \in S^*$ and $\frac{1}{2}[(x^*, r^* + \delta') + (x^*, r^* - \delta')] = (x^*, r^*)$, a contradiction. Note that if x is a w^*-β-exposing functional, then λx is again a w^*-β-exposing functional and w^*-β-exposing the same point x for all $\lambda > 0$. We can choose a w^*-β-exposing functional $(\bar{x}, -1)$ in $E \times R$ with $\sup_{A^*} |\bar{x}| < \delta$, which w^*-β exposes some point (x^*, r^*) in S^*. Then

$$\langle (x^*, r^*), (\bar{x}, -1) \rangle > \langle (y^*, s^*), (\bar{x}, -1) \rangle \qquad \text{for all } (y^*, s^*) \ (\neq (x^*, r^*)) \text{ in } S^*$$

and

$$\langle (x_\iota^*, r_\iota^*), (\bar{x}, -1)\rangle \to \langle (x^*, r^*), (\bar{x}, -1)\rangle,$$

i.e.,

$$\langle x_\iota^*, \bar{x}\rangle - r_\iota^* \to \langle x^*, \bar{x}\rangle - r^*$$

implies

$$(x_\iota^*, r_\iota^*) \xrightarrow{\beta} (x^*, r^*), \qquad \text{whenever } \{(x_\iota^*, r_\iota^*)\} \text{ is a net in } S^*.$$

Let $x = -\bar{x}$. Then we also have $\sup\limits_{A^*}|x| < \delta$ and $\langle (x^*, r^*), (x, 1)\rangle < \langle (y^*, s^*), (x, 1)\rangle$ for all (y^*, s^*) $(\neq (x^*, r^*))$ and for any net $\{(x_\iota^*, r_\iota^*)\}$ in S^*, $\langle x_\iota^*, x\rangle + r_\iota^* \to \langle x^*, x\rangle + r^*$ implies $(x_\iota^*, r_\iota^*) \xrightarrow{\beta} (x^*, r^*)$.

We can easily observe $r^* = \bar{f}_c(x^*)$. So $\bar{f}_c + x$ attains its minimum at x^* and $(\bar{f}_c + x)(x_\iota^*) \to (\bar{f}_c + x)(\ x^*)$ implies $x_\iota^* \xrightarrow{\beta} x^*$ whenever $\{x_\iota^*\} \subset w^*\text{-clco}A^*$.

Noting that $f \geq \bar{f} \geq \bar{f}_c$. We need only to show $\bar{f}_c(x^*) = f(x^*)$. By Proposition 3.1, there exists a net $\{x_\iota^*\}$ in $\mathrm{co}(A^*)$ w^* converging to x^* such that $\bar{f}_c(x^*) = \lim\limits_\iota \alpha(x_\iota^*)$. By definition of α, for each x_ι^*, there exist $n_\iota \in N, \lambda_j^{(\iota)} \geq 0$ for $j = 1, 2, \cdots, n_\iota$ with $\sum\limits_{j=1}^{n_\iota} \lambda_j^{(\iota)} = 1$ and $x_j^{*(\iota)} \in A^*$ with $\sum\limits_{j=1}^{n_\iota} \lambda_j^{(\iota)} x_j^{*(\iota)} = x_\iota^*$ such that $\bar{f}_c(x^*) = \lim\limits_\iota \sum\limits_{j=1}^{n_\iota} \lambda_j^{(\iota)} f(x_j^{*(\iota)})$ and such that $\langle x^*, x\rangle = \lim\limits_\iota \sum\limits_{j=1}^{n_\iota} \lambda_j^{(\iota)} \langle x_j^{*(\iota)}, x\rangle$.

Let $\sigma' = \inf\limits_{S^*}\langle\cdot\,,\,\cdot\rangle$ on $E \times R$. Then

$$\begin{aligned}\sigma'(x, 1) &= \langle (x, 1), (x^*, \bar{f}_c(x^*))\rangle = \langle x^*, x\rangle + \bar{f}_c(x^*)\\ &= \lim_\iota \sum_{j=1}^{n_\iota} \lambda_j^{(\iota)}(\langle x_j^{*(\iota)}, x\rangle + \bar{f}(x_j^{*(\iota)})) = \lim_\iota \sum_{j=1}^{n_\iota} \lambda_j^{(\iota)} \xi_j^{(\iota)},\end{aligned}$$

where $\xi_j^{(\iota)} = \langle x_j^{*(\iota)}, x\rangle + \bar{f}(x_j^{*(\iota)})$. This further tells us that there exists a subsequence $\xi_{j_k}^{(\iota_k)}$ of $\{\xi_j^{(\iota)}\}$ such that $\xi_{j_k}^{(\iota_k)} \to \sigma'(x, 1)$. Therefore the corresponding net $\{(x_{j_k}^{*(\iota_k)}, \bar{f}(x_{j_k}^{*(\iota_k)}))\}$ in $G(\bar{f})$ w^* converging to $(x^*, \bar{f}_c(x^*))$. Since $\bar{f}$ is w^* lower semicontinuous on A^*, we have $\bar{f}(x^*) \leq \lim\limits_{k\to\infty} \bar{f}(x_{j_k}^{*(\iota_k)}) = \bar{f}_c(x^*)$ and this says $\bar{f}(x^*) = \bar{f}_c(x^*)$. It remains to show $\bar{f}(x^*) = f(x^*)$. Otherwise, we have $\bar{f}(x^*) = r < f(x^*)$. Thus

$$\bar{f}(x^*) + \langle x^*, x\rangle = r + \langle x^*, x\rangle \geq r - \delta > M + 2\delta \geq \delta + \bar{f}(y^*)$$

whenever $\bar{f}(y^*) \leq M + \delta \geq \bar{f}(y^*) + \langle y^*, x\rangle$. This is a contradiction.

The following result is Theorem 1.4, and it is a converse version of Theorem 4.1.

Theorem 4.2. *The w^*-β variational principle holds on a w^* closed convex set C^* if and only if C^* has the w^*-β-exposed property, that is, every w^*-compact convex set in C^* is the w^*-β closed convex hull of its w^*-β-exposed points.*

Proof. It suffices to show the necessity. We only show that every w^* compact convex set A^* in C^* has a w^*-β- exposed point. Given any $x_0 \in A$, x_0 is w^* continuous and bounded on A^*. By Theorem 4.1, there exist x and $x_0^* \in A^*$ such that

(i) $(x_0+x)(x_0^*) = \langle x_0^*, x_0+x\rangle > (x_0+x)(x^*)$ for all $x^* \neq x_0^*$ in A^*;

(ii) $(x_0+x)(x_\iota^*) \to (x_0+x)(x_0^*)$ implies that $x_\iota^* \xrightarrow{\beta} x_0^*$ whenever $\{x_\iota^*\} \subset A^*$,

that is, x_0^* is a w^*-β-exposed point of A^*.

§ 5. β Differentiability Property of Convex Functions on Banach Spaces

Suppose that f is a continuous convex function defined on a locally convex space E. We say f has β-differentiability property (βDP) if every continuous convex function g with $g \le f$ is densely β-differentiable in E. We call $C_r(f) = \{x \in E : f(x) \le r\}$ a level set and we denote by $S(A)$ the set $\mathrm{cl}A\backslash \mathrm{int}(A)$.

We restates Theorem 1.5(i) as follows.

Theorem 5.1. *Suppose that f is a continuous convex function on Banach space E and f^* is the conjugate of f on E^* (the dual of E). Then f has β-differentiability property if and only if each level set of f^* has w^*-β-exposed property.*

Proof. Without loss of generality, we assume that $f \ge 0$ with $f(0) = 0$. In this case we have $f^*(0) = \min f^* = 0$. Otherwise, choosing any $x_0 \in \mathrm{dom}f$ and $x_0^* \in \partial f(x_0)$, we substitute

$$g = f(x_0 + \cdot\,) - f(x_0) - \langle x_0^*, \cdot\,\rangle \ge 0, \quad g^* = f^*(x_0^* + \cdot\,) - f^*(x_0^*) - \langle\,\cdot\,, x_0\rangle \ge 0$$

for f and f^*. Note $\mathrm{dom}g^* = \mathrm{dom}f^* - x^*$.

Suppose that each level set $C_n(f^*)$ has w^*-β-exposed property. Let p be the Minkowski functional generated by $C(f,1) = \{x \in E : f(x) \le 1\} \equiv C$. Then f is p-upper semicontinuous since f is bounded by 1 on C. We want to prove that $\partial p(0) \equiv C^*$ has w^*-β-exposed property. By a simple convexity argument we see that $p - 1 \le f$. So $C^* = \mathrm{dom}p^* \subset \mathrm{dom}f^*$ and

$$\begin{aligned} C^* &= \{x^* \in E^* : p^*(x^*) = 0\} \\ &= \{x^* \in C^* : \sup[\langle x^*, x\rangle - p(x) : x \in S(C)] \le 0\} \\ &= \{x^* \in C^* : \sup[\langle x^*, x\rangle - f(x) : x \in S(C)] \le 0\} \\ &= \{x^* \in C^* : \sup[\langle x^*, x\rangle - f(x) : x \in E\backslash C] \le 0\} \\ &= \{x^* \in E^* : \sup[\langle x^*, x\rangle - f(x) : x \in E] \le 1\} \\ &= C_1(f^*) = \{x^* \in E^* : f^*(x^*) \le 1\}. \end{aligned}$$

So C^* has w^*-β-exposed property. By Theorem 2.2, the Minkowski space (X, p) has β-differentiability property. Since f is p-upper semicontinuous, every convex function g with

$f \geq g$ must be p-locally upper bounded and by Proposition 2.2, g is p-upper semicontinuous. Therefore g is p-densely β-differentiable in E. We claim that g is norm densely β-differentiable in E.

Suppose, to the contrary, that there is a norm open (convex) nonempty set in E such that g is nowhere β-differentiable in U. We can assume $0 \in U$. (Otherwise, as before, choosing $x_0 \in U,\ x_0^* \in \partial g(x_0)$, we substitute

$$g' = g(x_0 + \cdot\,) - g(x_0) - \langle x_0^*, \cdot\,\rangle \geq 0,$$
$$g'^* = g^*(x_0^* + \cdot\,) - g^*(x_0^*) - \langle\,\cdot\,, x_0\rangle \geq 0$$

and $U - x_0$, successively, for g, g^* and U.)

Let $g_n(x) = 2^{-n} g\big(\frac{1}{n}x\big)$ for all $x \in E$, and $h = \sum\limits_{n=1}^{\infty} g_n$. We have $h \leq g$, therefore h is p-upper semicontinuous. It is easy to check that h is nowhere β-differentiable in E, and this is a contradiction, since every $h \leq f$ is p-densely β-differentiable in E. Hence we have proved that f has the βDP.

Conversely, since $C_n(f^*) = \{x^* \in E^* : \sup[\langle x^*, x] - f(x) : x \in E] \leq n\}$, we have $\sigma_{C_n^*} \leq f+n$. By the hypothesis, $\sigma_{C_n^*}$ also has the βDP. Further we see that $\partial\sigma_{C_n^*}(0) = C_n(f^*)$ has w^*-β-exposed property, which completes the proof.

The following theorem is Theorem 1.5(ii).

Theorem 5.2. *If p is a seminorm on Banach space E, then p has the* βDP *if and only if every p-continuous convex function is densely β-differentiable in E.*

Proof. Sufficiency. Since (E,p) is a locally convex space (not necessarily Hausdorff) and since every p-continuous convex function has at least one β-differentiability point, (E,p) has the βDP. Suppose that f is a norm continuous convex function with $p \geq f$. Then f is p-upper semicontinuous. Therefore f is p-densely β-differentiable in E. By an argument similar to the last part of the proof of Theorem 5.1, f is norm densely β-differentiable in E. This says p has β-differentiability property.

Necessity. Suppose that p has the βDP. By Theorem 5.1, $\partial p(0) \equiv C^* = C_0(p)$ has the w^*-β-exposed property. So (E,p) has the βDP. Therefore every p-upper semicontinuous convex function is p-densely β-differentiable in E and further it must be norm densely β-differentiable in E.

Corollary 5.1. *Suppose that f is a continuous convex function on a Banach space E, and f^* is the conjugate of f on E^* (the dual of E). Then f has the* GDP *if and only if each level set of f^* has w^*-exposed property.*

Corollary 5.2. *Suppose that f is a continuous convex function on a Banach space E, and f^* is the conjugate of f on E^* (the dual of E). Then f has the* FDP *if and only if each level set of f^* has the w^*-strongly-exposed property.*

Remark 5.1. All the theorems and corollaries in this section hold in locally convex spaces.

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