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# DECOMPOSITIONS OF BOSONIC MODULES OF LIE ALGEBRAS $\mathcal{W}_{1+\infty}$ AND $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$***

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**Abstract**

A bosonic construction (with central charge $c = 2$) of Lie algebras $\mathcal{W}_{1+\infty}$ and $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$, as well as the decompositions into irreducible modules are described. And for $\mathcal{W}_{1+\infty}$, when restricted to its Virasoro subalgebra Vir, a bosonic construction and the same decomposition for Vir are obtained.

**Keywords** Bosonic representation, Virasoro algebra
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## § 1. Introduction

Spinor representations for the affine Lie algebras were first developed by Frenkel [2] and Kac-Peterson [7] independently. The idea is to use a Clifford algebra with infinitely many generators to construct certain quadratic elements, which, together with the identity element, span an orthogonal affine Lie algebra. Thereafter, Feingold-Frenkel [1] constructed the so-called fermionic or bosonic representations for all classical affine Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Gao [6] constructed fermionic and bosonic representations for the extended affine Lie algebra $\widehat{\mathrm{gl}_N(\mathbf{C}_q)}$.

As we know, the Lie algebra $\widehat{\mathcal{D}^-}$, as the universal central extension of the Lie algebra of differential operators on the circle (cf. [7]), has appeared in various models of two-dimension -al quantum field theory and integrable systems (see the references in [4, 8]). A systematic study of the quasifinite highest weight representation theory of the Lie algebra $\widehat{\mathcal{D}^-}$, which is often referred to as $\mathcal{W}_{1+\infty}$ algebra by physicists, has been investigated by Kac et al (cf. [8, 4]).

In this paper, motivated by [6], we construct a bosonic representation for the Lie algebra $\mathcal{W}_{1+\infty}$, as well as the Lie algebra $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$, and then decompose such bosonic modules

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both for $\mathcal{W}_{1+\infty}$ and $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$. In particular, we obtain the same decomposition of bosonic realization for the Virasoro algebra with central charge $c=2$.

Throughout this paper, $\mathbf{Z}$, $\mathbf{N}$ and $\mathbf{C}$ denote the set of integers, non-negative integers and complex numbers, respectively.

Let $\mathbf{C}[t,t^{-1}]$ be the algebra of Laurent polynomials over $\mathbf{C}$, and $\mathcal{D}=\mathrm{Diff}\,\mathbf{C}[t,t^{-1}]$ the associative algebra of all differential operators over $\mathbf{C}[t,t^{-1}]$, whose a $\mathbf{C}$-basis is $\{t^mD^n \mid m\in\mathbf{Z},\ n\in\mathbf{N}\}$ with multiplication:

$$(t^aD^b)\cdot(t^cD^d)=\sum_{i=0}^{b}\binom{b}{i}c^it^{a+c}D^{b+d-i},$$

where $D=t\partial,\ \partial=d/dt$.

Let $\mathcal{D}^-$ be the Lie algebra of $\mathcal{D}$ under Lie bracket given by

$$[t^{m_1}D^{n_1},t^{m_2}D^{n_2}]=\sum_{i=0}^{n_1}\binom{n_1}{i}m_2^i\,t^{m_1+m_2}D^{n_1+n_2-i}-\sum_{j=0}^{n_2}\binom{n_2}{j}m_1^j\,t^{m_1+m_2}D^{n_1+n_2-j}$$

for all $m_1,\ m_2\in\mathbf{Z},\ n_1,\ n_2\in\mathbf{N}$.

Li [11] proved $\dim H^2(\mathcal{D}^-,\mathbf{C})=1$ (also see [7, 8, 13]). In this paper, we will adopt a convenient form of a specific 2-cocycle on $\mathcal{D}^-$, which is due to Kac and Radul (see the formula (1.5.5) in [8]) up to a sign. More precisely, we take $f(D)=D^{n_2}$, $g(D)=D^{n_1}$ and $\phi(t^{m_1}D^{n_1},t^{m_2}D^{n_2}):=\frac{1}{2}\,\psi(t^{m_2}D^{n_2},t^{m_1}D^{n_1})$ in the notation of [8]. So we have the following

**Lemma 1.1.** (cf. [8]) *Any non-trivial* 2*-cocycle on* $\mathcal{D}^-$ *is equivalent to* $\phi$:

$$\phi(t^{m_1}D^{n_1},t^{m_2}D^{n_2})=\begin{cases}0, & \text{if } m_1=0,\\[2mm] (-1)^{n_1+1}\delta_{m_1+m_2,0}\,\dfrac{1}{2}\displaystyle\sum_{i=1}^{m_1}(m_1-i)^{n_1}i^{n_2}, & \text{if } m_1>0,\\[2mm] (-1)^{n_1}\delta_{m_1+m_2,0}\,\dfrac{1}{2}\displaystyle\sum_{i=m_1}^{-1}(m_1-i)^{n_1}i^{n_2}, & \text{if } m_1<0.\end{cases}$$

Let $\mathcal{W}_{1+\infty}$ denote the universal (one-dimensional) central extension $\widehat{\mathcal{D}^-}$ of the Lie algebra $\mathcal{D}^-$ by the above modified 2-cocycle $\phi$. With this modified 2-cocycle $\phi$, we particularly have $\mathrm{Vir}:=\mathrm{Span}_{\mathbf{C}}\{L_m=t^mD,c\mid m\in\mathbf{Z}\}$ as the (standard) Virasoro subalgebra of $\mathcal{W}_{1+\infty}$, where its Lie bracket is given as follows $\Big($since $\sum\limits_{i=1}^{m}(m-i)i=\frac{1}{6}(m-1)m(m+1)$ for $m>0\Big)$

$$[L_m,L_n]=(n-m)L_{m+n}+\frac{1}{12}(m-1)m(m+1)\delta_{m+n,0}c,\qquad [c,L_m]=0.$$

## § 2. Bosonic Module and Its Decomposition of $\mathcal{W}_{1+\infty}$

Define $\mathcal{S}$ to be the unital associative algebra with infinitely many generators: $a(n),\ a^*(n)$ ($n\in\mathbf{Z}$) with relations

$$[\,a(n),a(m)\,]=[\,a^*(n),a^*(m)\,]=0, \tag{2.1}$$

$$[\, a(n), a^*(m)\,] = -\delta_{n+m,0}. \tag{2.2}$$

We define the normal ordering as follows:

$$:a(n)a^*(m):= \begin{cases} a(n)a^*(m), & n \leq m, \\ a^*(m)a(n), & n > m \end{cases} \tag{2.3}$$

for $n$, $m \in \mathbf{Z}$. Set

$$\theta(n) = \begin{cases} 1, & n > 0, \\ 0, & n \leq 0. \end{cases} \tag{2.4}$$

Then

$$a(n)a^*(m) =: a(n)a^*(m) : -\, \delta_{n+m,0}\, \theta(n-m) \tag{2.5}$$

and

$$\begin{aligned} [\, a(m)a^*(n), a(p)\,] &= \delta_{n+p,0}\, a(m), \\ [\, a(m)a^*(n), a^*(p)\,] &= -\delta_{m+p,0}\, a^*(n) \end{aligned} \tag{2.6}$$

for $m$, $n$, $p \in \mathbf{Z}$.

Let $\mathcal{S}^+$ be the subalgebra generated by $a(n)$, $a^*(0)$, $a^*(m)$ for $n$, $m > 0$. Let $\mathcal{S}^-$ be the subalgebra generated by $a(0)$, $a(n)$, $a^*(m)$ for $n$, $m < 0$. Those generators in $\mathcal{S}^+$ are called annihilation operators while those in $\mathcal{S}^-$ are called creation operators. Let $V$ be a simple $\mathcal{S}$-module containing an element $v_0$, called a "vacuum vector", and satisfying

$$\mathcal{S}^+ v_0 = 0. \tag{2.7}$$

So all annihilation operators kill $v_0$ and

$$V = \mathcal{S}^- v_0. \tag{2.8}$$

Now we may construct a class of bosons on $V$. For any $m \in \mathbf{Z}$, $n \in \mathbf{N}$, set

$$f(m,n) = \sum_{i \in \mathbf{Z}} (-i)^n : a(m-i)a^*(i) : . \tag{2.9}$$

Although $f(m,n)$ are infinite sums, they are well-defined as operators on $V$. Indeed, for any vector $v \in V = \mathcal{S}^- v_0$, only finitely many terms in (2.9) can make a non-zero contribution to $f(m,n)v$.

**Lemma 2.1.** *For $m$, $p$, $s \in \mathbf{Z}$, $n \in \mathbf{N}$,*

$$[\, f(m,n), a(p)\,] = p^n a(m+p), \tag{2.10}$$

$$[\, f(m,n), a^*(p)\,] = -(-m-p)^n a^*(m+p), \tag{2.11}$$

$$[\, f(m,n), a(p)a^*(s)\,] = p^n a(m+p)a^*(s) - (-m-s)^n a(p)a^*(m+s). \tag{2.12}$$

**Proof.** Since

$$\begin{aligned} [\, f(m,n), a(p)\,] &= \sum_{i \in \mathbf{Z}} (-i)^n [\, : a(m-i)a^*(i) :, a(p)\,] \\ &= \sum_{i \in \mathbf{Z}} (-i)^n [\, a(m-i)a^*(i), a(p)\,] \\ &= p^n a(m+p), \end{aligned}$$

(2.10) is true. The proof of (2.11) is similar, and (2.12) follows from (2.10) and (2.11).

**Proposition 2.1.** *For $m_1$, $m_2 \in \mathbf{Z}$, $n_1$, $n_2 \in \mathbf{N}$, we have*

$$\begin{aligned}[\,f(m_1,n_1),f(m_2,n_2)\,] = & \sum_{i=0}^{n_1}\binom{n_1}{i} m_2^i\, f(m_1+m_2,n_1+n_2-i) \\ & -\sum_{j=0}^{n_2}\binom{n_2}{j} m_1^j\, f(m_1+m_2,n_1+n_2-j) \\ & +\varphi(f(m_1,n_1),f(m_2,n_2)),\end{aligned}$$

*where $\varphi$ is given by*

$$\varphi(f(m_1,n_1),f(m_2,n_2)) = \begin{cases} 0, & \text{if } m_1=0,\\ (-1)^{n_1+1}\delta_{m_1+m_2,0}\displaystyle\sum_{i=1}^{m_1}(m_1-i)^{n_1} i^{n_2}, & \text{if } m_1>0,\\ (-1)^{n_1}\delta_{m_1+m_2,0}\displaystyle\sum_{i=m_1}^{-1}(m_1-i)^{n_1} i^{n_2}, & \text{if } m_1<0. \end{cases}$$

**Proof.** By Lemma 2.1, we have

$$\begin{aligned} & [\,f(m_1,n_1),f(m_2,n_2)\,] \\ = & \Big[\,f(m_1,n_1),\sum_{t\in\mathbf{Z}}(-t)^{n_2} :a(m_2-t)a^*(t): \Big] = \Big[\,f(m_1,n_1),\sum_{t\in\mathbf{Z}}(-t)^{n_2} a(m_2-t)a^*(t)\Big] \\ = & \sum_{t\in\mathbf{Z}}(-t)^{n_2}(m_2-t)^{n_1}a(m_1+m_2-t)a^*(t) - \sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}a(m_2-t)a^*(m_1+t) \\ = & \sum_{t\in\mathbf{Z}}(-t)^{n_2}(m_2-t)^{n_1} :a(m_1+m_2-t)a^*(t): - \sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2} :a(m_2-t)a^*(m_1+t): \\ & -\ \delta_{m_1+m_2,0}\Big(\sum_{t\in\mathbf{Z}}(m_2-t)^{n_1}(-t)^{n_2}\theta(m_1+m_2-2t) - \sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(m_2-m_1-2t)\Big) \\ = & \sum_{i=0}^{n_1}\binom{n_1}{i} m_2^i\, f(m_1+m_2,n_1+n_2-i) - \sum_{j=0}^{n_2}\binom{n_2}{j} m_1^j\, f(m_1+m_2,n_1+n_2-j) \\ & -\ \delta_{m_1+m_2,0}\Big(\sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(-2t) - \sum_{t\in\mathbf{Z}}(-m_1-t)^{n_1}(-t)^{n_2}\theta(-2m_1-2t)\Big) \\ = & \sum_{i=0}^{n_1}\binom{n_1}{i} m_2^i\, f(m_1+m_2,n_1+n_2-i) - \sum_{j=0}^{n_2}\binom{n_2}{j} m_1^j\, f(m_1+m_2,n_1+n_2-j) \\ & +\ \delta_{m_1+m_2,0}\sum_{t\in\mathbf{Z}}(-\theta(-2t)+\theta(-2m_1-2t))(-m_1-t)^{n_1}(-t)^{n_2} \\ = & \sum_{i=0}^{n_1}\binom{n_1}{i} m_2^i\, f(m_1+m_2,n_1+n_2-i) - \sum_{j=0}^{n_2}\binom{n_2}{j} m_1^j\, f(m_1+m_2,n_1+n_2-j) \\ & +\ \varphi\big(f(m_1,n_1),f(m_2,n_2)\big), \end{aligned}$$

where the last equality is given by

$$\sum_{t\in\mathbf{Z}}(-\theta(-2t)+\theta(-2m_1-2t))(-m_1-t)^{n_1}(-t)^{n_2}=\begin{cases}0, & m_1=0,\\ -\sum_{t=1}^{m_1}(t-m_1)^{n_1}t^{n_2}, & m_1>0,\\ \sum_{t=m_1}^{-1}(t-m_1)^{n_1}t^{n_2}, & m_1<0.\end{cases}$$

The proof is completed.

Let $T=f(0,0)$. Then Lemma 2.1 gives

$$[\,T,a(n)\,]=a(n),\qquad [\,T,a^*(n)\,]=-a^*(n) \tag{2.13}$$

for all $n\in\mathbf{Z}$. For any $v=a(n_1)\cdots a(n_s)a^*(m_1)\cdots a^*(m_l)v_0\in V$, noting that $Tv_0=0$, one has

$$Tv=(s-l)v. \tag{2.14}$$

According to Proposition 2.1 and Lemma 2.1, we obtain

**Theorem 2.1.** *$V$ is a module for the Lie algebra $\mathcal{W}_{1+\infty}$ with central charge $c=2$ under the action given by*

$$\pi(t^mD^n)=f(m,n),\qquad \pi(c)=2\,\mathrm{id}$$

*for all $m\in\mathbf{Z}$, $n\in\mathbf{N}$. Moreover,*

$$V=\bigoplus_{k\in\mathbf{Z}}V_k$$

*is completely reducible, where $V_k$ is an eigenspace with eigenvalue $k$ of operator $T$, and each component $V_k$ is irreducible as a $\mathcal{W}_{1+\infty}$-module.*

**Proof.** Note that $\varphi(f(m_1,n_1),f(m_2,n_2))=2\,\phi(t^{m_1}D^{n_1},t^{m_2}D^{n_2})$. Hence, Proposition 2.1 shows that $V$ is a $\mathcal{W}_{1+\infty}$-module with central charge $c=2$. On the other hand, Lemma 2.1 indicates that each eigenspace $V_k$ of operator $T$ is $\mathcal{W}_{1+\infty}$-stable. In what follows, we shall prove that $V_k$ is also irreducible under the actions of all $f(m,n)$'s.

To this end, we need introduce some notation. Fix a $k\in\mathbf{Z}$, for any $s\in\mathbf{N}$ such that $s+k\geq 0$, set $v_k^{(s)}:=a(0)^s\,a^*(-1)^{s+k}.v_0$ and

$$V_k^{(s)}:=\mathrm{Span}_{\mathbf{C}}\{\,a(n_1)\cdots a(n_s)a^*(m_1)\cdots a^*(m_{s+k}).v_0\mid n_i\leq 0, m_j<0\,\}.$$

It is clear that $V_k=\bigoplus_{s\in\mathbf{N},s+k\geq 0}V_k^{(s)}$. On the other hand, if we define the weight by

$$\mathrm{wt}(t^mD^n)=m,$$

which induces a principle $\mathbf{Z}$-gradation of $\mathcal{W}_{1+\infty}$:

$$\mathcal{W}_{1+\infty}=\bigoplus_{j\in\mathbf{Z}}\mathcal{W}_{1+\infty}{}^{(j)},$$

then we have a triangular decomposition of $\mathcal{W}_{1+\infty}$ as follows

$$\mathcal{W}_{1+\infty} = \mathcal{W}_{1+\infty}^{(-)} \bigoplus \mathcal{W}_{1+\infty}^{(0)} \bigoplus \mathcal{W}_{1+\infty}^{(+)},$$

where $\mathcal{W}_{1+\infty}^{(-)}=\{t^m D^n \mid m<0, n\in \mathbf{N}\}$, $\mathcal{W}_{1+\infty}^{(0)}=\{D^n \mid n\in \mathbf{N}\}$, $\mathcal{W}_{1+\infty}^{(+)}=\{t^m D^n \mid m>0, n\in \mathbf{N}\}$.

Lemma 2.1 shows that $V_k$ is a weight module with respect to the abelian subalgebra $\mathcal{W}_{1+\infty}^{(0)}$. Since for a fixed $m\neq 0$,

$$F(k,m) := [\,f(0,k), f(m,0)\,] - m\, f(m,0) = \sum_{j=1}^{k-1} \binom{k}{j} m^j\, f(m,k-j),$$

the actions of $f(m,1),\cdots,f(m,n-1)$ on $V$ can be expressed as some combinations of $F(k,m)$'s for $k=2,\cdots,n$, where $F(k,m)$'s ($k=2,\cdots,n$) acting on the weight module $V_k$ essentially depend on the action of $f(m,0)$ owing to $f(0,k)\in\mathcal{W}_{1+\infty}^{(0)}$. Therefore, it suffices to consider the actions of $f(m,0)$'s in the analysis of irreducibility of $V_k$. By Lemma 2.1, it is easily seen that $V_k^{(s)}$ is $\mathcal{W}_{1+\infty}^{(+)}$-stable, and $\{\, v_k^{(s)} \mid s\in\mathbf{N}, s+k\geq 0\,\}$ is the complete set of singular vectors of $\mathcal{W}_{1+\infty}$-module $V_k$ ( here $v\in V$ is called singular if $\mathcal{W}_{1+\infty}^{(+)}.v=0$ ) ( since $v_k^{(s)}$ is a unique ($\mathbf{C}$-linear independent) singular vector in $V_k^{(s)}$ according to the acting rule of $f(m,0)$ for $m>0$ ).

Finally, noticing that

$$f(-1,0)(v_k^{(s)}) \equiv v_k^{(s+1)} \ (\mathrm{mod}\, V_k^{(s)}), \qquad f(m,0)(V_k^{(s)}) \subseteq V_k^{(s)} + V_k^{(s+1)} \quad \text{for } m<0,$$

we see that $V_k$ is irreducible owing to

$$f(-m,0).\, f(m,0).\, v_k^{(s)} = [\, f(-m,0), f(m,0)\,].\, v_k^{(s)} = m\, v_k^{(s)} \neq 0,$$

by Proposition 2.1.

**Corollary 2.1.** *$V=\bigoplus_{k\in\mathbf{Z}} V_k$ is a completely reducible module for the Virasoro algebra Vir with central charge $c=2$ under the action given by*

$$\pi(L_m) = f(m,1), \qquad \pi(c) = 2\,\mathrm{id}$$

*for all $m\in\mathbf{Z}$. Each component $V_k$ is also irreducible for Vir.*

**Proof.** Lemma 2.1 indicates that $L_0=f(0,1)$ acts diagonalizably on the weight $\mathcal{W}_{1+\infty}$-module $V$. The proof of irreducibility of the weight $\mathcal{W}_{1+\infty}$-module $V_k$ (see the proof of Theorem 2.1) is reduced to considering the actions of operators $f(m,0)$'s for $m\in\mathbf{Z}$. Now the same observation applies to the proof of irreducibility of the weight Vir-module $V_k$ provided that we note the formula:

$$f(m,1) = \frac{1}{2m}[\, f(0,2), f(m,0)\,] - \frac{m}{2} f(m,0) \qquad \text{for } m\neq 0$$

derived from Proposition 2.1.

**Remark 2.1.** In the Virasoro algebra Vir, the operator $L_0$ is usually called the energy operator by physicists (cf. [9]). In [9], only positive-energy representations (that is, $L_0$ is diagonal and all its eigenvalues are nonnegative) were discussed there and all irreducible positive-energy representations are proven to be of the form $V(c,h)$ with $h \geq 0$ (see Remark 3.5 in [9], here $h$ is the eigenvalue of $L_0$, the highest weight module $V(c,h)$ is the irreducible quotient of the Vema Vir-module $M(c,h)$). The negative-energy representations of Vir, which are related to the Dirac positron theory, was pointed out to be interesting but lack of investigation there (see [9, Section 4.2]). In our case, Corollary 2.1 affords some negative-energy representations for Vir.

On the other hand, [8] classified positive-energy representations with finite degeneracies of the Lie algebra $\mathcal{W}_{1+\infty}$, while our bosonic construction in Theorem 2.1 then gives some negative-energy representations for $\mathcal{W}_{1+\infty}$. We conclude this section with the following interesting question.

**Question 2.1.** For any $v = a(n_1)\cdots a(n_s)a^*(m_1)\cdots a^*(m_{s+k}).v_0 \in V_k^{(s)}$ with $n_1 \leq n_2 \leq \cdots \leq n_s \leq 0$ and $m_1 \leq m_2 \leq \cdots \leq m_{s+k} < 0$, if set

$$n(s) = -\sum_{i=1}^{s} n_i - \sum_{j=1}^{s+k} m_j,$$

then we get a natural $\mathbf{Z}$-gradation of the irreducible module $V_k$ as below

$$V_k = \bigoplus_{n\in\mathbf{Z}} \Big( \bigoplus_{n(s)=n, s\geq 0, s+k\geq 0} V_k\big(n(s)\big) \Big).$$

Therefore, we have a corresponding $q$-character $\mathrm{ch}_q(V_k)$:

$$\mathrm{ch}_q(V_k) = \sum_{n\in\mathbf{Z}} \dim V(n)_k \, q^n,$$

where $V(n)_k = \bigoplus_{n(s)=n, s\geq 0, s+k\geq 0} V_k(n(s))$. The question is how to deduce the explicit formula for $\mathrm{ch}_q(V_k)$ via using some kind of partition functions.

## § 3. Bosonic Module and Its Decomposition of $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$

Let $M_N(\mathbf{C})$ be the $N \times N$ matrix algebra, $\mathrm{gl}_N(\mathbf{C}) = M_N(\mathbf{C})^-$ the general linear Lie algebra over $\mathbf{C}$, then $\mathrm{gl}_N(\mathcal{D}) := \mathrm{gl}_N(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{D}$ is the general linear Lie algebra with coefficients in $\mathcal{D}$. Let $e_{ij}$ be the $N \times N$ matrix unit with 1 in the $(i,j)$-entry and 0 elsewhere, then $\mathrm{gl}_N(\mathcal{D})$ has a basis

$$\{e_{ij} \otimes t^m D^l \mid m \in \mathbf{Z},\ l \in \mathbf{N},\ 1 \leq i,j \leq n\}.$$

Consider the subsequent central extension $\widehat{\mathrm{gl}_N}(\mathcal{D})$ by $\mathbf{C}$ of the Lie algebra $\mathrm{gl}_N(\mathcal{D})$, also denoted by $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$ ( since $\mathcal{W}_{1+\infty}(\mathrm{gl}_N) = \mathcal{W}_{1+\infty}$ when $N = 1$ ).

$$
\begin{aligned}
[\, e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2} \,] = \delta_{jk} e_{il} \otimes \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i \, t^{m_1+m_2} D^{n_1+n_2-i} \\
- \ \delta_{il} e_{kj} \otimes \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j \, t^{m_1+m_2} D^{n_1+n_2-j} \\
+ \ \Phi\big(e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}\big)\, c
\end{aligned}
$$

for all $m_1, m_2 \in \mathbf{Z}$, $n_1, n_2 \in \mathbf{N}$ and $1 \le i, j, k, l \le N$, where $\Phi$ is given by

$$
\begin{aligned}
&\Phi(e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}) \\
&= \begin{cases} 0, & \text{if } m_1 = 0, \\ (-1)^{n_1+1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \, \dfrac{1}{2} \displaystyle\sum_{i=1}^{m_1} (m_1 - i)^{n_1} i^{n_2}, & \text{if } m_1 > 0, \\ (-1)^{n_1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \, \dfrac{1}{2} \displaystyle\sum_{i=m_1}^{-1} (m_1 - i)^{n_1} i^{n_2}, & \text{if } m_1 < 0. \end{cases}
\end{aligned}
$$

Now we give a representation of the Lie algebra $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$.

Define $\mathcal{S}(N)$ to be the unital associative algebra with infinite many generators: $a_i(n)$, $a_j^*(n)$ ($n \in \mathbf{Z}, 1 \le i,\, j \le N$) with the relations

$$[\, a_i(n), a_j(m) \,] = [\, a_i^*(n), a_j^*(m) \,] = 0, \tag{3.1}$$

$$[\, a_i(n), a_j^*(m) \,] = -\delta_{i,j} \delta_{n+m,0}. \tag{3.2}$$

We define the normal ordering as follows:

$$: a_i(n) a_j^*(m) := \begin{cases} a_i(n) a_j^*(m), & n \le m, \\ a_j^*(m) a_i(n), & n > m \end{cases} \tag{3.3}$$

for $n, m \in \mathbf{Z}, 1 \le i, j \le N$.

Similarly to (2.5)–(2.6), we have

$$
\begin{aligned}
&a_i(n) a_j^*(m) =: a_i(n) a_j^*(m) : - \, \delta_{i,j} \delta_{n+m,0} \theta(n - m), && (3.4) \\
&[\, a_i(m) a_j^*(n), a_k(p) \,] = \delta_{j,k} \delta_{n+p,0} a_i(m), \\
&[\, a_i(m) a_j^*(n), a_k^*(p) \,] = -\delta_{i,k} \delta_{m+p,0} a_j^*(n) && (3.5)
\end{aligned}
$$

for $m,\, n,\, p \in \mathbf{Z}$, $1 \le i, j, k \le N$.

Let $\mathcal{S}(N)^+$ be the subalgebra generated by $a_i(n)$, $a_i^*(n)$, $a_i^*(0)$ for $n > 0$ and $1 \le i \le N$. Let $\mathcal{S}(N)^-$ be the subalgebra generated by $a_i(0)$, $a_i(n)$, $a_i^*(n)$ for $n < 0$ and $1 \le i \le N$. Those generators in $\mathcal{S}(N)^+$ are called annihilation operators while those in $\mathcal{S}(N)^-$ are called creation operators. Let $V(N)$ be a simple $\mathcal{S}(N)$-module containing an element $v_0$, called a "vacuum vector", and satisfying

$$\mathcal{S}(N)^+ v_0 = 0. \tag{3.6}$$

So all annihilation operators kill $v_0$ and

$$V(N) = \mathcal{S}(N)^- v_0. \tag{3.7}$$

Now we may construct a class of bosons on $V(N)$. For any $m \in \mathbf{Z}, n \in \mathbf{N}$ and $1 \le i, j \le N$, set

$$f_{i,j}(m,n) = \sum_{k \in \mathbf{Z}} (-k)^n : a_i(m-k) a_j^*(k) : \tag{3.8}$$

Although $f_{i,j}(m,n)$ are infinite sums, they are well-defined as operators on $V(N)$. Since, for any vector $v \in V(N) = \mathcal{S}(N)^- v_0$, only finitely many terms in (3.8) can make a non-zero contribution to $f_{i,j}(m,n)v$.

**Lemma 3.1.** *For $m, p, s \in \mathbf{Z},\ n \in \mathbf{N}$ and $1 \le i, j, k \le N$,*

$$[\, f_{i,j}(m,n), a_k(p)\,] = \delta_{j,k} p^n a_i(m+p), \tag{3.9}$$

$$[\, f_{i,j}(m,n), a_k^*(p)\,] = -\delta_{i,k} (-m-p)^n a_j^*(m+p), \tag{3.10}$$

$$[\, f_{i,j}(m,n), a_k(p) a_l^*(s)\,] = \delta_{j,k} p^n a(m+p) a^*(s) - \delta_{i,l} (-m-s)^n a(p) a^*(m+s). \tag{3.11}$$

**Proof.** The proof is similar to that of Lemma 2.1.

**Proposition 3.1.** *For $m_1, m_2 \in \mathbf{Z},\ n_1, n_2 \in \mathbf{N}$ and $1 \le i, j, k, l \le N$, we have*

$$\begin{aligned}[\, f_{i,j}(m_1,n_1), f_{k,l}(m_2,n_2)\,] = {} & \delta_{j,k} \sum_{i=0}^{n_1} \binom{n_1}{i} m_2^i f_{i,l}(m_1+m_2, n_1+n_2-i) \\ & - \ \delta_{i,l} \sum_{j=0}^{n_2} \binom{n_2}{j} m_1^j f_{k,j}(m_1+m_2, n_1+n_2-j) \\ & + \ \Psi(f_{i,j}(m_1,n_1), f_{k,l}(m_2,n_2)),\end{aligned}$$

*where $\Psi$ is given by*

$$\begin{aligned}&\Psi(f_{i,j}(m_1,n_1), f_{k,l}(m_2,n_2)) \\ &= \begin{cases} 0, & \text{if } \ m_1 = 0, \\ (-1)^{n_1+1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \displaystyle\sum_{i=1}^{m_1} (m_1-i)^{n_1} i^{n_2}, & \text{if } \ m_1 > 0, \\ (-1)^{n_1} \delta_{j,k} \delta_{i,l} \delta_{m_1+m_2,0} \displaystyle\sum_{i=m_1}^{-1} (m_1-i)^{n_1} i^{n_2}, & \text{if } \ m_1 < 0. \end{cases}\end{aligned}$$

**Proof.** The proof is similar to that of Proposition 2.1.

Let $T = \sum_{i=1}^{N} f_{i,i}(0,0)$. Then one can easily show that

$$[\, T, a_j(n)\,] = a_j(n), \qquad [\, T, a_j^*(n)\,] = -a_j^*(n) \tag{3.12}$$

for all $n \in \mathbf{Z},\ 1 \le j \le N$. For any

$$v = a_{i_1}(n_1) \cdots a_{i_s}(n_s) a_{j_1}^*(m_1) \cdots a_{j_t}^*(m_t) v_0$$

from $V(N)$, noting that $Tv_0 = 0$, one has

$$Tv = (s - t)v. \tag{3.13}$$

Noting that

$$\Psi(f_{i,j}(m_1, n_1), f_{k,l}(m_2, n_2)) = 2\,\Phi(e_{ij} \otimes t^{m_1} D^{n_1}, e_{kl} \otimes t^{m_2} D^{n_2}),$$

we may prove similarly

**Theorem 3.1.** *$V(N)$ is a level 2 module for the Lie algebra $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$ under the action given by*

$$\pi(e_{ij} \otimes t^m D^n) = f_{i,j}(m, n), \qquad \pi(c) = 2\,\mathrm{id}$$

*for all $m \in \mathbf{Z}$, $n \in \mathbf{N}$ and $1 \le i,\ j \le N$. Moreover,*

$$V(N) = \bigoplus_{k \in \mathbf{Z}} V_k$$

*is completely reducible, where $V_k$ is an eigenspace with eigenvalue $k$ of operator $T$, and each component $V_k$ is irreducible as a $\mathcal{W}_{1+\infty}(\mathrm{gl}_N)$-module.*

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## References

[1] Feingold, A. J. & Frenkel, I. B., Classical affine Lie algebras, *Adv. in Math.*, **56**(1985), 117–172.

[2] Frenkel, I. B., Spinor representations of affine Lie algebras, *Proc. Natl. Acad. Sci. USA*, **77**(1980), 6303–6306.

[3] Frenkel, I. B., Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory, *J. Funct. Anal.*, **44**(1981), 259–327.

[4] Frenkel, E., Kac, V., Radul, A. & Wang, W., $\mathcal{W}_{1+\infty}$ and $\mathcal{W}(\mathrm{gl}_N)$ with central charge $N$, *Comm. Math. Phys.*, **170**(1995), 337–357.

[5] Gao, Y., Representations of the extended affine Lie algebra coordinatized by certain quantum tori, *Composito Mathematica*, **132**(2000), 1–25.

[6] Gao, Y., Fermionic and bosonic representations of the extended affine Lie algebra $\widetilde{\mathrm{gl}_N(\mathbf{C}_q)}$, *Canad. Math. Bull.*, **45:**4(2002), 623–633.

[7] Kac, V. G. & Peterson, D. H., Spin and wedge representations of infinfite dimensional Lie algebras and groups, *Natl. Acad. Sci. USA*, **78**(1981), 3308–3312.

[8] Kac, V. G. & Radul, A., Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Comm. Math. Phys.*, **157**(1993), 429–457.

[9] Kac, V. G. & Raina, A., Highest Weight Representations of Infinite Dimensional Lie Algebras, Advanced Series in Math. Physics, Vol. 2, 1987.

[10] Kassel, C., Cyclic homology of differential operators, the Virasoro algebra and a $q$-analogue, *Comm. Math. Phys.*, **164**(1992), 343–356.

[11] Li, W., 2-Cocycles on the algebra of differential operators, *J. Algebra*, **122**(1989), 64–80.

[12] Liu, D. & Hu, N., Derivation algebras and 2-cocycles of the algebras of $q$-differential operators, *Comm. Algebra*, **32:**11(2004), 4387–4413.

[13] Li, W. & Wilson, R. L., Central extensions of some Lie algebras, *Proc. Amer. Math. Soc.*, **126:**9(1998), 2569–2577.