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EMBEDDINGS OF SIMPLE TWO-FOLD BALANCED INCOMPLETE BLOCK DESIGNS WITH BLOCK SIZE FOUR***

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Abstract

The necessary and sufficient conditions for the existence of simple incomplete block design $(v, w; 4, 2)$-IPBDs are determined. As a consequence, the necessary and sufficient conditions for the embeddings of simple two-fold balanced incomplete block designs with block size 4 are also determined.

Keywords Block design, Simple, Embedding
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§ 1. Introduction

A balanced incomplete block design $\mathrm{B}(k, \lambda; v)$ is an ordered pair $(X, \mathcal{A})$ where X is a set of v points and $\mathcal{A}$ is a collection of subsets (called blocks) of X such that $|B| = k$ for each block $B \in \mathcal{A}$, and each pair of distinct points of X is contained in exactly λ blocks. A $\mathrm{B}(k, \lambda; v)$ is called simple and denoted by $\mathrm{NB}(k, \lambda; v)$ if it contains no repeated blocks.

Let $(X, \mathcal{A})$ be a $\mathrm{B}(k, \lambda; v)$, $Y \subset X, |Y| = w$ and $\mathcal{B} \subset \mathcal{A}$. If $(Y, \mathcal{B})$ is a $\mathrm{B}(k, \lambda; w)$, then it is called a subdesign of $(X, \mathcal{A})$, or it is embedded in $(X, \mathcal{A})$. An incomplete pairwise balanced design $(v, w; k, \lambda)$-IPBD is an ordered triple $(X, Y, \mathcal{A})$ where X is a v-set, Y is a w-subset (called a hole) of X and $\mathcal{A}$ is a collection of subsets (called blocks) of X such that $|B| = k$ and $|B \cap Y| \le 1$ for each $B \in \mathcal{A}$ and each pair of distinct points of X, not both in Y, is contained in exactly λ blocks. A $(v, w; k, \lambda)$-IPBD is called simple if it contains no repeated blocks. It is obvious that we can get an $\mathrm{NB}(k, \lambda, v)$ containing an $\mathrm{NB}(k, \lambda, w)$ as a subdesign by filling an $\mathrm{NB}(k, \lambda, w)$ in the hole of size w in a simple $(v, w; k, \lambda)$-IPBD.

By some simple counting argument, the following conditions are necessary for the embedding of an $\mathrm{NB}(k, \lambda; w)$ in an $\mathrm{NB}(k, \lambda; v)$:

$$v \ge (k-1)w + 1,$$
$$\lambda v(v-1) \equiv \lambda w(w-1) \equiv 0 \pmod{k(k-1)},$$

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$$\lambda(v-1) \equiv \lambda(w-1) \equiv 0 \pmod{(k-1)},$$
$$\lambda \leq \binom{w-2}{k-2}. \tag{1.1}$$

For given k and λ, any positive integers v, w satisfying the above conditions are called admissible.

The embeddings of simple triple systems for arbitrary λ was completely determined by Shen [6]: An NB$(3, \lambda; w)$ can be embedded in some NB$(3, \lambda; v)$ if and only if v and w are admissible. However, for arbitrary $\lambda \geq 2$, the embedding problem of NB$(4, \lambda; v)$s is still open. Rees and Rodger [3] proved that there exists a B$(4, 2; v)$ containing a B$(4, 2; w)$ as a subdesign if and only if $v \equiv w \equiv 1 \pmod 3$ and $v \geq 3w+1$, but the embedding may contain repeated blocks.

The main purpose of this paper is to give a complete solution to the existence problem of simple $(v, w; 4, 2)$-IPBDs. As a consequence, we have determined the necessary and sufficient conditions for the embeddings of NB$(4, 2; v)$s. Our techniques are a construction from self-orthogonal Latin squares with holes in Section 2 and a construction from incomplete pairwise balanced designs with index unity in Section 3.

§ 2. Simple $(v, w; 4, 2)$-IPBDs with $v - w \equiv 0 \pmod 6$

A group divisible design (k, λ)-GDD is an ordered triple $(X, \mathcal{G}, \mathcal{A})$ where $\mathcal{G}$ is a partition of a set X (of points) into subsets called groups, $\mathcal{B}$ is a set of subsets of X (called blocks) such that for each $B \in \mathcal{B}, |B| = k$ and a group and a block contain at most one common point, every pair of points from distinct groups occurs in exactly λ blocks. The type of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We also use an exponential notation to describe types: so type $t_1^{n_1} t_2^{n_2} \cdots t_k^{n_k}$ denotes n_i occurrences of t_i, $1 \leq i \leq k$. A $(k, 1)$-GDD of type m^k is called a transversal design and denoted by TD(k, m). It is well known that the existence of a TD(k, m) is equivalent to the existence of $k-2$ mutually orthogonal Latin squares of order m. A (k, λ)-GDD is called simple if it contains no repeated blocks. Obviously, a simple (k, λ)-GDD of type 1^v is just an NB$(k, \lambda; v)$.

We shall need the following construction for simple IPBDs from simple GDDs whose proof is clear.

Construction 2.1. *Let k and λ be positive integers and let d be a nonnegative integer. Suppose that the following designs exist:*

(1) *a simple (k, λ)-GDD of type $t_1 t_2 \cdots t_n$;*

(2) *a simple $(t_i + d, d; k, \lambda)$-IPBD for $1 \leq i \leq n-1$.*

Then there exists a simple $(v, w; k, \lambda)$-IPBD, where $v = \sum\limits_{1 \leq i \leq n} t_i + d$ and $w = t_n + d$.

To apply the above lemma, we shall also need holey self-orthogonal Latin squares to get the required GDDs. Let $\mathcal{H} = \{X_1, X_2, \cdots, X_n\}$ be a partition of a set X and $|X_i| = t_i, 1 \leq i \leq n$. A holey self-orthogonal Latin square of type $\prod\limits_{1 \leq i \leq n} t_i$, denoted by HSOLS$\big(\prod\limits_{1 \leq i \leq n} t_i\big)$, is an $|X| \times |X|$ array L, indexed by X, satisfying the following properties:

(1) Every cell of L either contains an element of X or is empty.

(2) Every element of X occurs at most once in any row or column of L.

(3) The subarrays indexed by $X_i \times X_i$ are empty for $1 \leq i \leq n$ (these subarrays are called holes).

(4) An element $x \in X$ occurs in row or column y if and only if $(x, y) \in (X \times X) \setminus \bigcup_{1 \leq i \leq n} (X_i \times X_i)$.

(5) The superposition of L and its transpose L^T yields all ordered pairs in $(X \times X) \setminus \bigcup_{1 \leq i \leq n} (X_i \times X_i)$.

It is clear that an HSOLS(1^n) is just an idempotent self-orthogonal Latin square of order n. For simple GDD, we have the following construction from HSOLS.

Lemma 2.1. *Suppose that there exists an* HSOLS$(t_1 t_2 \cdots t_n)$*. Then there exists a simple* $(4, 2)$-GDD *of type* $(3t_1)(3t_2) \cdots (3t_n)$.

Proof. Let L be an HSOLS$(t_1 t_2 \cdots t_n)$ on the set X with hole set $\mathcal{H} = \{X_1, X_2, \cdots, X_n\}$ where $|X_i| = t_i, 1 \leq i \leq n$, and L^T be its transpose. Furthermore, let us denote the (x, y)-entry in L by $x \circ y$ and the (x, y)-entry in L^T by $y \circ x$ for each ordered pair $(x, y) \in (X \times X) \setminus \bigcup_{1 \leq i \leq n} (X_i \times X_i)$. Let $\mathcal{G} = \{X_i \times Z_3 \mid 1 \leq i \leq n\}$ and $\mathcal{B} = \{(x, g), (y, g), (x \circ y, g + 1), (y \circ x, g + 1) \mid x \in X_i, y \in X_j, 1 \leq i < j \leq n, g \in Z_3\}$ (addition is mod 3 for the second coordinate). Then $(X \times Z_3, \mathcal{G}, \mathcal{B})$ is a simple $(4, 2)$-GDD of type $(3t_1)(3t_2) \cdots (3t_n)$.

Without loss of generality, let $x \in X_i, y \in X_j$ and $1 \leq i < j \leq n$. For each pair $\{(x, g), (y, g)\}$, by the self-orthogonality of L, we have a unique pair r, s such that $r \circ s = x$ and $s \circ r = y$, where $r \in X_{i_1}, s \in X_{j_1}, 1 \leq i_1 < j_1 \leq n$ and $(i, j) \neq (i_1, j_1)$. Therefore, the pair $\{(x, g), (y, g)\}$ occurs in exactly two blocks $\{(x, g), (y, g), (x \circ y, g + 1), (y \circ x, g + 1)\}$ and $\{(r, g - 1), (s, g - 1), (x, g), (y, g)\}$. Considering the pair $\{(x, g), (y, g + 1)\}$, by the definition of L, we suppose $x \circ r = y$ and $s \circ x = y$ where $r \in X_{i_1}, s \in X_{i_2}, i_1 > i_2 > i$. Then the pair $\{(x, g), (y, g + 1)\}$ occurs in exactly two blocks $\{(x, g), (r, g), (y, g + 1), (r \circ x, g + 1)\}$ and $\{(x, g), (s, g), (x \circ s, g + 1), (y, g + 1)\}$. This proves that $(X \times Z_3, \mathcal{G}, \mathcal{B})$ is a $(4, 2)$-GDD. Moreover, let $B_1 = \{(x_1, g_1), (y_1, g_1), (x_1 \circ y_1, g_1 + 1), (y_1 \circ x_1, g_1 + 1)\}$, $B_2 = \{(x_2, g_2), (y_2, g_2), (x_2 \circ y_2, g_2 + 1), (y_2 \circ x_2, g_2 + 1)\}$ and $B_1, B_2 \in \mathcal{B}$. If $B_1 = B_2$, then $g_1 \equiv g_2 \pmod 3$ or $g_1 \equiv g_2 + 1 \pmod 3$, but $g_1 \equiv g_2 + 1 \pmod 3$ and $B_1 = B_2$ imply $g_2 \equiv g_1 + 1 \pmod 3$, hence $2 \equiv 0 \pmod 3$, a contradiction. So if $B_1 = B_2$, then $g_1 \equiv g_2 \pmod 3$. This shows that there is no repeated blocks in $\mathcal{B}$. The proof is completed.

Corollary 2.1. *There exists a simple* $(4, 2)$-GDD *of type* $6^n (3u)^1$ *for* $n \geq 4, n \geq 1 + u$.

Proof. Applying Lemma 2.1, since there exists an HSOLS$(2^n u^1)$ for $n \geq 4, n \geq 1 + u$ from [7], we obtain a simple $(4, 2)$-GDD of type $6^n (3u)^1$.

The following result is from [2].

Lemma 2.2. *There exists an* NB$(4, 2; v)$ *if and only if* $v \equiv 1 \pmod 3), v \geq 7$.

Now we obtain the main result of this section.

Lemma 2.3. *Let v and w be positive integers, $v \geq 3w + 1$, and suppose that either* $v, w \equiv 1, 7 \pmod{12}$ *or* $v, w \equiv 4, 10 \pmod{12}$*. Then there exists a simple* $(v, w; 4, 2)$-IPBD *with 3 possible exceptions* $(v,w)=(16,4),\ (22,4)$ *and* $(25,7)$.

Proof. When $w=1$ and $v\equiv 1,7 \pmod{12}$, there exists a simple $(v,1;4,2)$-IPBD by Lemma 2.2. Now let $w=3u+1, u\ge 1$. By the hypotheses, we can always write $v=6n+3u+1, n\ge 1+u$ and $n\ge 4$. Hence there exists a simple $(4,2)$-GDD of type $6^n(3u)^1$ by Corollary 2.1. Further by applying Construction 2.1 with $d=1$ and Lemma 2.2, we get the required designs.

§ 3 . Simple $(v,w;4,2)$-IPBDs with $v-w\equiv 3 \pmod 6$

In this section, we shall use a technique of permuting IPBD with index unity to produce a simple IPBD with index two. The following result is needed.

Lemma 3.1. (cf. [4]) *There exists a* $(v,w;4,1)$-IPBD *if and only if* $v\ge 3w+1$ *and either* $v,w\equiv 1,4 \pmod{12}$, *or* $v,w\equiv 7,10 \pmod{12}$.

The following lemma is a generalization of Theorem 1 in [5], and it is a useful tool for constructing simple IPBDs. Let X,Y,Z be three disjoint sets, where $|X|=v-w, |Y|=w, |Z|=m$. Let S be the symmetric group on $X\cup Y\cup Z$ and $\pi\in S$ be a permutation. For each subset $M=\{x_1,x_2,\cdots,x_k\}$ of $X\cup Y\cup Z$, let $\pi(M)=\{\pi(x_1),\pi(x_2),\cdots,\pi(x_k)\}$. Further let G be the subgroup of S fixing Y and z for each $z\in Z$. Then we have the following lemma.

Lemma 3.2. *Suppose that* $(X\cup Y\cup Z, Y\cup Z,\mathcal{A})$ *is a simple* $(v+m,w+m;k,\lambda_1)$-IPBD*,* $(X\cup Y,Y,\mathcal{B})$ *is a simple* $(v,w;k,\lambda_2)$-IPBD *and* v,w,k,λ_1 *and* λ_2 *satisfy the following inequality:*

$$\begin{aligned}&\lambda_1\lambda_2(k-2)!(v-w)\{kw(v-w-k+1)+(v-(k-1)w-1)^2\}(v-w-k)!\\ &<k(k-1)(v-w-1)!.\end{aligned}\tag{3.1}$$

Then there exists a permutation $\pi\in G$ *such that* $\pi(\mathcal{A})\cap\mathcal{B}=\phi$*, where* $\pi(\mathcal{A})=\{\pi(A)\mid A\in\mathcal{A}\}$.

Proof. Let $\mathcal{A}=\mathcal{A}_0\cup\mathcal{A}_{11}\cup\mathcal{A}_{12}$, $\mathcal{B}=\mathcal{B}_0\cup\mathcal{B}_1$, where $\mathcal{A}_0=\{A\in\mathcal{A}\mid |A\cap(Y\cup Z)|=0\}$, $\mathcal{A}_{11}=\{A\in\mathcal{A}\mid |A\cap Y|=1\}$, $\mathcal{A}_{12}=\{A\in\mathcal{A}\mid |A\cap Z|=1\}$, $\mathcal{B}_0=\{B\in\mathcal{B}\mid |B\cap Y|=0\}$, and $\mathcal{B}_1=\{B\in\mathcal{B}\mid |B\cap Y|=1\}$. By simple counting argument, we have

$$|\mathcal{A}_0|=\frac{\lambda_1(v-w)[v-(k-1)w-1-(k-2)m]}{k(k-1)},$$
$$|\mathcal{A}_{11}|=\frac{\lambda_1 w(v-w)}{k-1},\qquad |\mathcal{A}_{12}|=\frac{\lambda_1 m(v-w)}{k-1},$$
$$|\mathcal{B}_0|=\frac{\lambda_2(v-w)[v-(k-1)w-1]}{k(k-1)},\qquad |\mathcal{B}_1|=\frac{\lambda_2 w(v-w)}{k-1}.$$

Since S is the symmetric group on $X\cup Y\cup Z$, and G is the subgroup of S fixing Y and z for each $z\in Z$, we see that $|G|=w!(v-w)!$ and for any $\pi\in G$, $(X\cup Y\cup Z,Y\cup Z,\pi(\mathcal{A}))$ is also a $(v+m,w+m;k,\lambda_1)$-IPBD.

Now for two given blocks $A \in \mathcal{A}$ and $B \in \mathcal{B}$, if $|A \cap Y| \neq |B \cap Y|$, then there does not exist $\pi \in G$ such that $\pi(A) = B$. If $|A \cap Y| = |B \cap Y| = 0$ and $|A \cap Z| \neq 0$, then there does not exist $\pi \in G$ such that $\pi(A) = B$. If $|A \cap Y| = |B \cap Y| = 0$ and $|A \cap Z| = 0$, then the number of such permutations π is $k!w!(v-w-k)!$. If $|A \cap Y| = |B \cap Y| = 1$, then the number of such permutations is $(k-1)!(w-1)!(v-w-k+1)!$.

Let n be the number of permutations $\pi \in G$ such that

$$|\pi(\mathcal{A}) \cap \mathcal{B}| \geq 1.$$

Then

$$\begin{aligned} n &\leq \lambda_1\lambda_2(v-w)^2[v-(k-1)w-1][v-(k-1)w-1-(k-2)m]\frac{k!w!(v-w-k)!}{k^2(k-1)^2} \\ &\quad + \lambda_1\lambda_2 w^2(v-w)^2\frac{(k-1)!(w-1)!(v-w-k+1)!}{(k-1)^2} \\ &= \lambda_1\lambda_2(k-2)!(v-w)^2\{kw(v-w-k+1) \\ &\quad + [v-(k-1)w-1][v-(k-1)w-1-(k-2)m]\}\frac{w!(v-w-k)!}{k(k-1)} \\ &\leq \lambda_1\lambda_2(k-2)!(v-w)^2\{kw(v-w-k+1)+(v-(k-1)w-1)^2\}\frac{w!(v-w-k)!}{k(k-1)} \\ &< w!(v-w)!. \end{aligned}$$

Thus there exists a permutation $\pi \in G$ such that $\pi(\mathcal{A})$ and $\mathcal{B}$ share no common blocks, that is, $\pi(\mathcal{A}) \cap \mathcal{B} = \phi$.

For $k = 4, \lambda_1 = \lambda_2 = 1$, (3.1) is just the following inequality:

$$(v-w)\{4w(v-w-3)+(v-3w-1)^2\} < 6(v-w-1)(v-w-2)(v-w-3). \tag{3.2}$$

Lemma 3.3. *Let v and w be positive integers, $v \geq 3w+1$ and $(v,w) \neq (4,1)$. Then v and w satisfy* (3.2).

Proof. Since $v, w > 0, v \geq 3w+1$, and $(v,w) \neq (4,1)$, we have

$$(v-3w-1) \geq 0, \qquad (v+w-5) \geq 0.$$

Therefore, we get

$$(v-3w-1)^2 \leq 2(v-3w-1)(v-w-3).$$

Adding $4w(v-w-3)$ to both sides of the above inequality, we have

$$0 < 4w(v-w-3)+(v-3w-1)^2 \leq 2(v-w-1)(v-w-3).$$

Since $0 < v-w < 3(v-w-2)$, we get the desired inequality.

Lemma 3.4. *Let v and w be positive integers, $(v,w) \neq (4,1)$ and $m = 0$ or 6. If there exist a $(v+m, w+m; 4, 1)$-IPBD and a $(v, w; 4, 1)$-IPBD, then there exists a simple $(v+\frac{m}{2}, w+\frac{m}{2}; 4, 2)$-IPBD.*

Proof. Let sets X, Y and $Z = \{\infty_1, \infty_2, \cdots, \infty_m\}$ satisfy $|X| = v - w, |Y| = w$ and $X \cap Y \cap Z = \phi$. Let S be the symmetric group on $X \cup Y \cup Z$, and let G be the subgroup of S fixing Y and z for each $z \in Z$. Further let $(X \cup Y \cup Z, Y \cup Z, \mathcal{A})$ be a $(v+m, w+m; 4, 1)$-IPBD and $(X \cup Y, Y, \mathcal{B})$ be a $(v, w; 4, 1)$-IPBD. By Lemma 3.1, we have $v \geq 3w + 1$, thus v and w satisfy (3.2). Applying Lemma 3.2 with $k = 4, \lambda_1 = \lambda_2 = 1$, we see that there exists a permutation $\pi \in G$ such that $\pi(\mathcal{A}) \cap \mathcal{B} = \phi$.

Let

$$\mathcal{A}_Z = \{A \in \pi(\mathcal{A}) \mid |A \cap Z| = 1\},$$
$$\mathcal{P}_i = \{A \in \mathcal{A}_Z \mid \infty_i \in A\}, \qquad 1 \leq i \leq m,$$

and set

$$\mathcal{Q}_j = \{\{a, b, c, \infty_j\} \mid \{a, b, c, \infty_{m/2+j}\} \in \mathcal{P}_{m/2+j}\}, \qquad 1 \leq j \leq \frac{m}{2}.$$

Then

$$\Big(X \cup Y \cup \{\infty_1, \infty_2, \cdots, \infty_{\frac{m}{2}}\}, Y \cup \{\infty_1, \infty_2, \cdots, \infty_{\frac{m}{2}}\}, \mathcal{B} \cup \pi(\mathcal{A}) \cup \Big(\bigcup_{1 \leq i \leq \frac{m}{2}} (\mathcal{P}_i \cup \mathcal{Q}_i)\Big) \Big\backslash \mathcal{A}_Z\Big)$$

is a simple $(v + \frac{m}{2}, w + \frac{m}{2}; 4, 2)$-IPBD. The proof is completed.

Now we are in a position to provide our main results of this section.

Lemma 3.5. *Let v and w be positive integers. If $v \geq 3w + 1, (v, w) \neq (4, 1)$, and either $v, w \equiv 1, 4 \pmod{12}$, or $v, w \equiv 7, 10 \pmod{12}$, then there exists a simple $(v, w; 4, 2)$-IPBD.*

Proof. Apply Lemma 3.1 and Lemma 3.4 with $m = 0$.

Lemma 3.6. *Let v and w be positive integers. If $v \geq 3w + 7$, and v, w satisfy one of the following conditions:*

(1) $v \equiv 1 \pmod{12}$, *and* $w \equiv 10 \pmod{12}$;
(2) $v \equiv 4 \pmod{12}$, *and* $w \equiv 7 \pmod{12}$;
(3) $v \equiv 7 \pmod{12}$, *and* $w \equiv 4 \pmod{12}$;
(4) $v \equiv 10 \pmod{12}$, *and* $w \equiv 1 \pmod{12}$.

Then there exists a simple $(v, w; 4, 2)$-IPBD.

Proof. From the hypotheses and Lemma 3.1, we have a $(v + 3, w + 3; 4, 1)$-IPBD and a $(v - 3, w - 3; 4, 1)$-IPBD. Applying Lemma 3.4 with $m = 6$, we obtain the desired design.

§ 4. Main Results

Before concluding this study, we first deal with the remaining cases in Lemma 2.3. Noting that the case with $(v, w) = (16, 4)$ is covered by Lemma 3.5, we can now restrict our attention to the cases with $(v, w) = (22, 4)$ and $(25, 7)$.

Lemma 4.1. *There exists a simple $(25, 7; 4, 2)$-IPBD.*

Proof. We first construct a simple $(4,2)$-GDD of type 2^4 on $X = \{1,2,\cdots,8\}$. The groups are $X_i = \{1+2i, 2+2i\}, 0 \le i \le 3$ and the blocks are listed as follows:

$$\{1,3,5,7\},\ \{2,3,6,7\},\ \{1,4,5,8\},\ \{2,4,6,8\},$$

$$\{1,3,6,8\},\ \{2,3,5,8\},\ \{1,4,6,7\},\ \{2,4,5,7\}.$$

Then give each point of this GDD weight 3. Since there exists a $\mathrm{TD}(4,3)$, this forms a simple $(4,2)$-GDD of type 6^4. Employing Construction 2.1 with $d = 1$ and Lemma 2.2, we obtain the required design.

In order to handle with the case $(v,w) = (22,4)$, we recall the concept of Kirkman triple system. A parallel class of a $\mathrm{B}(k,\lambda;v)$ $(X,\mathcal{A})$ is a subset $\mathcal{P}$ of $\mathcal{A}$ such that $\mathcal{P}$ is a partition of X. A $\mathrm{B}(k,\lambda;v)$ $(X,\mathcal{A})$ is said to be resolvable if $\mathcal{A}$ can be partitioned into parallel classes. It is well known that a resolvable $\mathrm{B}(3,1;v)$ is called a Kirkman triple system ($\mathrm{KTS}(v)$).

Lemma 4.2. *There exists a simple* $(22,4;4,2)$-IPBD.

Proof. From [1], there exist two KTS(15)s $(X,\mathcal{A})$ and $(X,\mathcal{B})$ such that $\mathcal{A}$ and $\mathcal{B}$ have exactly one common block $\{a,b,c\}$. Let $\mathcal{A} = \bigcup_{0\le i\le 6} \mathcal{P}_i$, $\mathcal{B} = \bigcup_{0\le i\le 6} \mathcal{Q}_i$ and $\mathcal{P}_0 \cap \mathcal{Q}_0 = \{a,b,c\}$, where $\mathcal{P}_i$ and $\mathcal{Q}_i$ are parallel classes of $(X,\mathcal{A})$ and $(X,\mathcal{B})$, respectively, and let $(Y,\mathcal{C})$ be an $\mathrm{NB}(4,2;7)$ from Lemma 2.2, where $Y = \{\infty_0,\infty_1,\cdots,\infty_6\}$. Furthermore, let

$$\mathcal{P}_i^{'} = \{\{x_1,x_2,x_3,\infty_i\} \mid \{x_1,x_2,x_3\} \in \mathcal{P}_i\}, \qquad 0 \le i \le 6,$$

$$\mathcal{Q}_i^{'} = \{\{x_1,x_2,x_3,\infty_i\} \mid \{x_1,x_2,x_3\} \in \mathcal{Q}_i\}, \qquad 0 \le i \le 6,$$

and set

$$\mathcal{A}^{'} = \bigcup_{0\le i\le 6} \mathcal{P}_i^{'} \backslash \{\{a,b,c,\infty_0\}\},$$

$$\mathcal{B}^{'} = \bigcup_{0\le i\le 6} \mathcal{Q}_i^{'} \backslash \{\{a,b,c,\infty_0\}\}.$$

Then $(X \cup Y, \{a,b,c,\infty_0\}, \mathcal{A}^{'} \cup \mathcal{B}^{'} \cup \mathcal{C})$ is a simple $(22,4;4,2)$-IPBD.

We are now in a position to give our conclusions.

Theorem 4.1. *There exists a simple* $(v,w;4,2)$-IPBD *if and only if* $v \ge 3w+1, v \ge 7$ *and* $v,w \equiv 1 \pmod 3$.

Proof. The necessity is obvious by simple counting argument. Now we prove the sufficiency. In fact, the necessary condition is equivalent to the following cases:

(1) $v \ge 3w+1, v,w \equiv 1 \pmod 3$ and $v - w \equiv 0 \pmod 6$,

(2) $v \ge 3w+1, v,w \equiv 1 \pmod 3, v - w \equiv 3 \pmod 6$ and $(v,w) \ne (4,1)$.

Combining Lemmas 2.3, 3.5, 3.6, 4.1 and 4.2, the conclusion then follows.

As an immediate consequence of Theorem 4.1, we have the main theorem of this paper.

Theorem 4.2. *There exists an* $\mathrm{NB}(4,2;v)$ *containing an* $\mathrm{NB}(4,2;w)$ *as a subdesign if and only if* $v \ge 3w+1, w \ge 7$ *and* $v,w \equiv 1 \pmod 3$.

Proof. Filling an $\mathrm{NB}(4,2;w)$ in the hole of size w in a simple $(v,w;4,2)$-IPBD from Theorem 4.1 gives the desired design.

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