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SOBOLEV INEQUALITY ON RIEMANNIAN MANIFOLDS**

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Abstract

Let M be an n dimensional complete Riemannian manifold satisfying the doubling volume property and an on-diagonal heat kernel estimate. The necessary-sufficient condition for the Sobolev inequality $\|f\|_q \le C_{n,,\nu,p,q}(\|\nabla f\|_p + \|f\|_p)$ $(2 \le p < q < \infty)$ is given.

Keywords Sobolev inequality, Complete manifold, Riesz transform, Potential, Heat kernel

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§ 1 . Introduction

Let M be an n dimensional complete Riemannian manifold, ρ be the geodesic distance on M, and $d\mu$ be Riemannian measure. Denote by $B(x,r)$ the geodesic ball of center $x \in M$ and radius $r > 0$, and by $V(x,r)$ its Riemannian volume.

One says that M satisfies the doubling volume property if there exists a constant D_0 such that

$$V(x,2r) \le D_0 V(x,r), \qquad \forall\, x \in M,\ r > 0. \tag{1.1}$$

We denote $\nu = \log_2 D_0$.

Let Δ be the Laplace-Beltrami operator on M, $H(x,y,t)$ be the heat kernel on M.

Using the boundedness of the potential $(I + (-\Delta)^{\frac{1}{2}})^{-1}$ and the Riesz transform, Li [1] obtained that for $1 < p < q < \infty$, the Sobolev inequality

$$\|f\|_q \le C_{n,p,q}(\|\nabla f\|_p + \|f\|_p), \qquad \forall\, f \in H_1^p(M) \tag{1.2}$$

holds on a complete manifold with non-negative Ricci curvature if and only if

$$\begin{cases} \inf\limits_{x\in M} V(x,1) > 0, \\ \frac{1}{p} - \frac{1}{n} \le \frac{1}{q} < \frac{1}{p}. \end{cases}$$

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In this note, we replace the condition Ricci $M \geq 0$ by the doubling volume property and an upper bounds for the heat kernel. We obtain the following result:

Theorem 1.1. *Let M be an n dimensional complete Riemannian manifold satisfying the doubling volume property* (1.1) *and the heat kernel $H(x,y,t)$ at M satisfies*

$$H(x,x,t) \leq \frac{C}{V(x,\sqrt{t}\,)}, \qquad \forall\, x \in M,\ t > 0 \tag{1.3}$$

for some $C > 0$. Then for $2 \leq p < q < \infty$, the Sobolev inequality

$$\|f\|_q \leq C_{n,\nu,p,q}(\|\nabla f\|_p + \|f\|_p), \qquad \forall\, f \in H_1^p(M) \tag{1.4}$$

holds for some constant $C_{n,\nu,p,q} > 0$ if and only if

$$\begin{cases} \inf\limits_{x \in M} V(x,1) > 0, \\ \frac{1}{p} - \frac{1}{n} \leq \frac{1}{q} < \frac{1}{p}. \end{cases}$$

Our assumption on M, apart from the doubling volume property, is the heat kernel estimate (1.3). From this on-diagonal estimate, the corresponding off-diagonal estimate automatically follows (see [5, Theorem 1.1]): for any $\alpha \in (0, \frac{1}{4})$,

$$H(x,y,t) \leq \frac{C_\alpha}{\sqrt{V(x,\sqrt{t}\,)V(y,\sqrt{t}\,)}} e^{-\alpha \frac{\rho^2(x,y)}{t}}, \qquad \forall\, x,y \in M,\ t > 0 \tag{1.5}$$

for some $C_\alpha > 0$.

With the doubling volume property, this implies that for any $\alpha \in (0, \frac{1}{4})$,

$$H(x,y,t) \leq \frac{C'_\alpha}{V(x,\sqrt{t}\,)} e^{-\alpha \frac{\rho^2(x,y)}{t}} \tag{1.6}$$

for some $C'_\alpha > 0$. Indeed $B(y,\sqrt{t}\,) \subset B(x, \sqrt{t} + \rho(x,y))$. Now an obvious consequence of the doubling volume property is

$$V(x,\theta r) \leq D_0 \theta^\nu V(x,r), \qquad \forall\, \theta > 1. \tag{1.7}$$

Therefore

$$V(y,\sqrt{t}\,) \leq V(x, \sqrt{t} + \rho(x,y)) \leq c\Big(1 + \frac{\rho(x,y)}{\sqrt{t}}\Big)^\nu V(x,\sqrt{t}\,),$$

and the estimate follows.

In fact, we should have $\nu \geq n$. Since $\lim\limits_{r \to 0} \frac{V(x,r)}{r^n} = \Omega_n > 0$, we have (see (1.7))

$$\frac{V(x,1)r^\nu}{D_0} \leq V(x,r) \leq C_n r^n, \qquad r \ll 1.$$

Example 1.1. The assumptions of Theorem 1.1 are satisfied on manifolds where a parabolic Harnack principle holds (see [6, Chapter 5]).

It is easy to construct manifolds that satisfy the assumptions of Theorem 1.1, but where the parabolic Harnack principle is false. A typical example is the following (see [3]) : take two copies of $\mathbb{R}^2 \backslash B(0,1)$, and glue them smoothly along the unit circles.

By [6, Theorem 5.6.4 and Theorem 5.6.5], we conclude that the parabolic Harnack principle is satisfied on manifolds with non-negative Ricci curvature. Thus Theorem 1.1 is an extension of the result in [1] when $2 \leq p < +\infty$.

§ 2. Proof of Theorem 1.1

We consider the potential $(I+(-\Delta)^{\frac{1}{2}})^{-1}$. For $1 \le p \le \infty$ and $\forall f \in L^p(M)$, we have

$$Tf \triangleq (I+(-\Delta)^{\frac{1}{2}})^{-1} f(x) = \int_0^\infty e^{-t} \int_M P(x,y,t) f(y) dy dt, \tag{2.1}$$

where $P(x,y,t)$ is the Poisson kernel on M, i.e.,

$$P(x,y,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s} s^{-\frac{1}{2}} H\Big(x,y,\frac{t^2}{4s}\Big) ds = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x,y,s) ds.$$

Proposition 2.1. *There exists some constant $C_p > 0$ such that*

$$\|Tf\|_{L^p(M)} \le C_p \|f\|_{L^p(M)}, \qquad \forall\, f \in L^p(M).$$

Proof. By Theorem 3.5 in [2], $\int_M H(x,y,t) \le 1$, so $\int_M P(x,y,t) \le 1$. Therefore T is L^p $(1 \le p \le \infty)$ bounded.

Next, we will show that T is also bounded from $L^p(M)$ to $L^q(M)$ for $\frac{1}{p} - \frac{1}{n} = \frac{1}{q}$. Let

$$k(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t e^{-t} \int_0^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x,y,s) ds dt, \tag{2.2}$$

$$k_1(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t e^{-t} \int_0^1 e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x,y,s) ds dt, \tag{2.3}$$

$$k_2(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t e^{-t} \int_1^\infty e^{-\frac{t^2}{4s}} s^{-\frac{3}{2}} H(x,y,s) ds dt, \tag{2.4}$$

$$T_1 f(x) \triangleq (I+(-\Delta)^{\frac{1}{2}})_1^{-1} f(x) = \int_M k_1(x,y) f(y) d\mu(y),$$

$$T_2 f(x) \triangleq (I+(-\Delta)^{\frac{1}{2}})_2^{-1} f(x) = \int_M k_2(x,y) f(y) d\mu(y).$$

Clearly

$$k(x,y) = k_1(x,y) + k_2(x,y), \tag{2.5}$$

$$Tf(x) = \int_M k(x,y) f(y) d\mu(y) = T_1 f(x) + T_2(x). \tag{2.6}$$

First, we state the following two lemmas.

Lemma 2.1. *Let M be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let $\inf\limits_{x \in M} V(x,1) = \delta > 0$, $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then T_1 is of type (p,q), i.e. T_1 is bounded from $L^p(M)$ to $L^q(M)$.*

Lemma 2.2. *Let M be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let $\inf\limits_{x \in M} V(x,1) = \delta > 0$, $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then T_2 is of type (p,q).*

Next, once we established Lemmas 2.1–2.2, with the (p,p) boundedness of T (see Proposition 2.1) and Marcinkiewicz interpolation theorem (see [4] for example), we obtain

Theorem 2.1. *Let M be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Let $1 < p, q < \infty$, and* $\inf_{x\in M} V(x,1) > 0$, $\frac{1}{p} - \frac{1}{n} \le \frac{1}{q} \le \frac{1}{p}$. *Then T is of type (p,q).*

Furthermore, we can get

Theorem 2.2. *Let M be a complete Riemannian manifold satisfying the same assumptions of Theorem 1.1. Then for $2 \le p < \infty$, there exists a constant $C_{n,p} > 0$, such that*

$$\|(-\Delta)^{\frac{1}{2}} f\|_p \le C_{n,p} \|\nabla f\|_p, \qquad \forall f \in C_0^\infty(M).$$

Proof. Let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for all $g \in C_0^\infty(M)$,

$$((-\Delta)^{\frac{1}{2}} f, g) = (f, (-\Delta)^{\frac{1}{2}} g) = (f, (-\Delta)(-\Delta)^{-\frac{1}{2}} g) = (\nabla f, \nabla(-\Delta)^{-\frac{1}{2}} g).$$

Since the Riesz transform $\nabla(-\Delta)^{-\frac{1}{2}}$ is L^r bounded for $1 < r \le 2$ (see [3, Theorem 1.1]),

$$|((-\Delta)^{\frac{1}{2}} f, g)| \le \|\nabla f\|_p \|\nabla(-\Delta)^{-\frac{1}{2}} g\|_{p'} \le C_{n,p} \|\nabla f\|_p \|g\|_{p'}.$$

Thus $\|(-\Delta)^{\frac{1}{2}} f\|_p \le C_{n,p} \|\nabla f\|_p$.

Now we prove Lemmas 2.1–2.2. To this end, we give the following two lemmas which are needed in the proof of Lemma 2.1.

Lemma 2.3. *For any $a > 0$, let* $g_a(\theta) = \begin{cases} \theta^n, & 0 < \theta \le 1, \\ \theta^a, & \theta > 1. \end{cases}$ *Then there exists a constant $C_{n,a} > 0$, such that*

$$\sum_{k=-\infty}^{h} e^{-\frac{2^k}{5}} g_a(2^{\frac{k}{2}}) \le C_{n,a} \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t}\,)}{t} dt. \tag{2.7}$$

Proof. One can easily see that $\lambda(t) = e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}})$ is a continuous and piecewise monotone function. Its maximum is obtained at

$$m = \begin{cases} 0, & a < \frac{2}{5}, \\ \log_2 \frac{5}{2} a, & a \ge \frac{2}{5}. \end{cases}$$

For $h \ge m - 1$, we have

$$\begin{aligned} \sum_{k=-\infty}^{h} e^{-\frac{2^k}{5}} g_a(2^{\frac{k}{2}}) &\le \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt + e^{-\frac{2^m}{5}} g_a(2^{\frac{m}{2}}) \\ &\le \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt + \frac{C'_a}{\int_{-\infty}^0 e^{-\frac{2^t}{5}} 2^{\frac{nt}{2}} dt} \int_{-\infty}^0 e^{-\frac{2^t}{5}} 2^{\frac{nt}{2}} dt \\ &\le C_{n,a} \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a(2^{\frac{t}{2}}) dt \le C_{n,a} \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t}\,)}{t} dt. \end{aligned} \tag{2.8}$$

While for $h < m - 1$,

$$\sum_{k=-\infty}^{h} e^{-\frac{2^k}{5}} g_a\big(2^{\frac{k}{2}}\big) \le \int_{-\infty}^{h+1} e^{-\frac{2^t}{5}} g_a\big(2^{\frac{t}{2}}\big) dt \le \int_0^{2^{h+1}} \frac{e^{-\frac{t}{5}} g_a(\sqrt{t}\,)}{t} dt. \tag{2.9}$$

Combining (2.8) and (2.9), we get the conclusion.

For simplicity, we denote $g(\theta) = g_\nu(\theta)$.

Lemma 2.4. *Let M be a complete Riemannian manifold satisfying condition* (1.1). *Then for any $x \in M$, $0 < s < 1$, and $\kappa > 0$,*

$$\int_{\rho(x,y)\le\kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \le C_{n,\nu} V(x,\sqrt{s}\,) \int_0^{\sqrt{2}\kappa} \frac{e^{-\frac{\rho^2}{5s}} g\big(\frac{\rho}{\sqrt{s}}\big)}{\rho} d\rho.$$

Proof. Since

$$\lim_{t\to 0} \frac{V(x,t)}{t^n} = \Omega_n > 0, \tag{2.10}$$

there exists an $r_0 > 0$, such that $C_1 \le \frac{V(x,s)}{s^n} \le C_2$ for $s \in (0, r_0)$. Therefore for any $r \in (0, r_0]$ and $\theta \in (0,1)$,

$$\frac{V(x,\theta r)}{V(x,r)} \le C_n \theta^n. \tag{2.11}$$

On the other hand, for any $r \in (r_0, 1]$, $\theta \in (0,1)$,

$$\frac{V(x,\theta r)}{V(x,r)} = \frac{V(x,\theta r)}{V(x,\theta r_0)} \cdot \frac{V(x,r_0)}{V(x,r)} \cdot \frac{V(x,\theta r_0)}{V(x,r_0)} \le D_0 \Big(\frac{r}{r_0}\Big)^\nu \theta^n \le C_{n,\nu}\theta^n. \tag{2.12}$$

For $\theta \in (0,\infty)$, we denote $\chi(\theta) = \sup\limits_{x\in M, 0<r\le 1} \frac{V(x,\theta r)}{V(x,r)}$. Then it follows from (1.7), (2.11) and (2.12) that $\chi(\theta) \le C_{n,\nu} g(\theta)$.

Let h be selected such that $2^h\sqrt{s} \le \kappa < 2^{h+1}\sqrt{s}$. Thus

$$\begin{aligned} \int_{\rho(x,y)\le\kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) &\le \sum_{k=-\infty}^{h} \int_{2^{\frac{k}{2}}\sqrt{s}<\rho(x,y)\le 2^{\frac{k+1}{2}}\sqrt{s}} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \\ &\le \sum_{k=-\infty}^{h} e^{-\frac{2^k}{5}} (V(x, 2^{\frac{k+1}{2}}\sqrt{s}\,) - V(x, 2^{\frac{k}{2}}\sqrt{s}\,)) \\ &\le C_{n,\nu} V(x,\sqrt{s}\,) \sum_{k=-\infty}^{h} e^{-\frac{2^k}{5}} g(2^{\frac{k}{2}}). \end{aligned} \tag{2.13}$$

By Lemma 2.3,

$$\int_{\rho(x,y)\le\kappa} e^{-\frac{\rho^2(x,y)}{5s}} d\mu(y) \le C_{n,\nu} V(x,\sqrt{s}\,) \int_0^{\sqrt{2}\kappa} \frac{e^{-\frac{\rho^2}{5s}} g\big(\frac{\rho}{\sqrt{s}}\big)}{\rho} d\rho.$$

And the lemma is proved.

Proof of Lemma 2.1. Let

$$k_{1,1}(x,y) = \begin{cases} k_1(x,y), & \rho(x,y) \le \kappa, \\ 0, & \rho(x,y) > \kappa, \end{cases} \qquad k_{1,\infty}(x,y) = \begin{cases} 0, & \rho(x,y) \le \kappa, \\ k_1(x,y), & \rho(x,y) > \kappa, \end{cases}$$

where κ is to be determined.

Let

$$T_{1,1}f(x)=\int_M k_{1,1}(x,y)f(y)d\mu(y),\qquad T_{1,\infty}f(x)=\int_M k_{1,\infty}(x,y)f(y)d\mu(y).$$

As long as we can show that T_1 is of weak type (p,q), it is also of (p,q) by Marcinkiewicz interpolation. To this aim, we need to prove

$$\mu\{x\in M:\|T_1f(x)\|>2\lambda\}\le C_{n,p,q,\nu}\frac{1}{\lambda^q},\tag{2.14}$$

where $\|f\|_p=1$.

Clearly

$$\mu\{x\in M:|T_1f(x)|>2\lambda\}\le\mu\{x\in M:|T_{1,1}f(x)|>\lambda\}+\mu\{x\in M:|T_{1,\infty}f(x)|>\lambda\}.\tag{2.15}$$

By interpolation,

$$\|T_{1,1}f(x)\|_p\le\sup_{x\in M}\int_M k_{1,1}(x,y)d\mu(y).\tag{2.16}$$

By Lemma 2.4 and (1.6),

$$\begin{aligned}\int_M k_{1,1}(x,y)d\mu(y)&\le\int_0^\infty te^{-t}\int_0^1 e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}\int_{\rho(x,y)\le\kappa}\frac{e^{-\frac{\rho^2(x,y)}{5s}}}{V(x,\sqrt{s})}d\mu(y)dsdt\\&\le C_{n,\nu}\int_0^{+\infty}te^{-t}\int_0^1 e^{-}\frac{t^2}{4s}s^{-\frac{3}{2}}\int_0^{\sqrt{2}\kappa}\frac{e^{-\frac{\rho^2}{5s}}g\big(\frac{\rho}{\sqrt{s}}\big)}{\rho}d\rho dsdt\\&\le C_{n,\nu}\int_0^{\sqrt{2}\kappa}\frac{1}{\rho}\int_0^\infty te^{-t}\int_0^1 e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}e^{-\frac{\rho^2}{5s}}g\Big(\frac{\rho}{\sqrt{s}}\Big)dsdtd\rho\\&\le C_{n,\nu}\int_0^{\sqrt{2}\kappa}\frac{1}{\rho}\int_0^\infty e^{-t}\int_{t^2}^\infty e^{-\frac{s}{4}}s^{-\frac{1}{2}}e^{-\frac{s\rho^2}{5t^2}}g\Big(\frac{\rho\sqrt{s}}{t}\Big)dsdstd\rho\\&\le C_{n,\nu}\int_0^{\sqrt{2}\kappa}\frac{1}{\rho}\int_0^\infty e^{-\frac{s}{4}}s^{-\frac{1}{2}}\int_0^{\sqrt{s}}e^{-t}e^{-\frac{s\rho^2}{5t^2}}g\Big(\frac{\rho\sqrt{s}}{t}\Big)dtdsd\rho\\&\le C_{n,\nu}\int_0^{\sqrt{2}\kappa}\frac{1}{\rho}\int_0^\infty e^{-\frac{s}{4}}\int_0^\infty e^{-\frac{\sqrt{s}\rho}{l}}e^{-\frac{l^2}{5}}g(l)\frac{\rho}{l^2}dldsd\rho.\end{aligned}$$

Since $g(l)\sim l^n$ for $l\to 0^+$, and $g(l)\sim l^\nu$ for $l\to+\infty$,

$$\int_M k_{1,1}(x,y)d\mu(y)\le C_{n,\nu}\kappa.\tag{2.17}$$

By (2.16) and (2.17),

$$\mu\{x\in M,|T_{1,1}f(x)|>\lambda\}\le C_{n,\nu}\frac{\kappa^p}{\lambda^p}.\tag{2.18}$$

On the other hand,

$$|T_{1,\infty}f(x)|\le\sup_{x\in M}\Big(\int_M k_{1,\infty}^{p'}(x,y)d\mu(y)\Big)^{\frac{1}{p'}},\tag{2.19}$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$.

By (2.10), there exists an $r_0 > 0$, such that for any $s \in (0, r_0)$, $V(x, \sqrt{s}\,) \geq c_n s^{\frac{n}{2}}$. By (1.7) and $\nu \geq n$, for $s \in (r_0, 1]$,

$$\frac{V(x,1)}{V(x,\sqrt{s}\,)} \leq D_0\Big(\frac{1}{\sqrt{s}}\Big)^n\Big(\frac{1}{\sqrt{s}}\Big)^{\nu-n} \leq D_0\Big(\frac{1}{\sqrt{s}}\Big)^n\Big(\frac{1}{r_0}\Big)^{\nu-n} \leq C_{n,\nu}s^{-\frac{n}{2}}.$$

Therefore

$$V(x,\sqrt{s}) \geq C_{n,\nu}\min(\delta,1)s^{\frac{n}{2}}, \qquad \forall\, 0 < s \leq 1. \tag{2.20}$$

It follows from (2.20) and [3, Lemma 2.1] that

$$\begin{aligned}\Big(\int_M k_{1,\infty}^{p'}(x,y)d\mu(y)\Big)^{\frac{1}{p'}} &\leq C\int_0^{+\infty} te^{-t}\int_0^1 e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}\Big(\int_{\rho(x,y)\geq\kappa} H^{p'}(x,y,s)d\mu(y)\Big)^{\frac{1}{p'}}dsdt\\ &\leq C\int_0^{+\infty} te^{-t}\int_0^1 \frac{e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}}{V(x,\sqrt{s}\,)}\Big(\int_{\rho(x,y)\geq\kappa} e^{-\frac{p'\rho^2(x,y)}{5s}}d\mu(y)\Big)^{\frac{1}{p'}}dsdt\\ &\leq C\int_0^{+\infty} te^{-t}\int_0^1 \frac{e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}}{V(x,\sqrt{s}\,)}e^{-\frac{\kappa^2}{10s}}dsdt\\ &\leq C_{n,\nu}\frac{1}{\min\{\delta,1\}}\int_0^{+\infty} te^{-t}\int_0^1 e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}-\frac{n}{2p}}e^{-\frac{\kappa^2}{10s}}dsdt.\end{aligned}$$

In a way Similar to the estimate of (2.17), we have

$$\Big(\int_M k_{1,\infty}^{p'}(x,y)d\mu(y)\Big)^{\frac{1}{p'}} \leq C_{p,n,\nu}\frac{1}{\min\{\delta,1\}}\kappa^{1-\frac{n}{p}}.$$

By (2.19),

$$|T_{1,\infty}f(x)| \leq C_{p,n,\nu}\frac{1}{\min\{\delta,1\}}\kappa^{1-\frac{n}{p}}. \tag{2.21}$$

Let $\kappa > 0$ be selected such that

$$\lambda = C_{p,n,\nu}\frac{1}{\min\{\delta,1\}}\kappa^{1-\frac{n}{p}} = C_{p,n}\frac{1}{\min\{\delta,1\}}\kappa^{-\frac{n}{q}}.$$

Then

$$\kappa = C_{p,n,,\nu,\delta}\lambda^{-\frac{q}{n}} \tag{2.22}$$

and (2.14) is obtained from (2.15), (2.18), (2.21) and (2.22). The lemma is proved.

Proof of Lemma 2.2. Write $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} = 1 - \frac{1}{n} = \frac{n-1}{n}$. By Hölder's inequality,

$$\begin{aligned}&|T_2f(x)| \leq \int_M k_2(x,y)|f(y)|dy\\ &\leq \Big\{\int_M k_2^r(x,y)|f(y)|^p dy\Big\}^{\frac{1}{q}}\cdot\Big\{\int_M k_2^r(x,y)dy\Big\}^{1-\frac{1}{p}}\cdot\Big\{\int_M |f(y)|^p dy\Big\}^{\frac{1}{n}}.\end{aligned}$$

Thus

$$\|T_2 f\|_q \le \sup_{x\in M}\Big\{\int_M k_2^r(x,y)dy\Big\}^{\frac{1}{r}}\|f\|_p. \tag{2.23}$$

By Mincowski's inequality, we have

$$\Big(\int_M k_2^r(x,y)dy\Big)^{\frac{1}{r}} \le \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}\Big(\int_M H^r(x,y,s)dy\Big)^{\frac{1}{r}}dsdt.$$

Since $V(x,\sqrt{s}\,) \ge V(x,1) \ge \delta$ for $s \ge 1$,

$$\Big(\int_M H^r(x,y,s)d\mu(y)\Big)^{\frac{1}{r}} \le C_\delta.$$

Therefore

$$\Big(\int_M k_2^r(x,y)d\mu(y)\Big)^{\frac{1}{r}} \le C_\delta\int_0^\infty te^{-t}\int_1^\infty e^{-\frac{t^2}{4s}}s^{-\frac{3}{2}}dsdt \le C_\delta.$$

Combining the above with (2.23), we get the lemma.

Proof of Theorem 1.1. By Theorem 2.1 and Theorem 2.2, we get the sufficient part of Theorem 1.1. Now let $f(y) = \max\{t-\rho(x,y),0\}$. Then

$$\|f\|_q \ge \Big(\frac{t}{2}\Big)V\Big(x,\frac{t}{2}\Big)^{\frac{1}{q}},\quad \|f\|_p \le t\,V(x,t)^{\frac{1}{p}},\quad \|\nabla f\|_p \le V(x,t)^{\frac{1}{p}}.$$

Hence

$$V(x,t)^{\frac{1}{p}-\frac{1}{q}} \ge C_{n,p,q}\frac{t}{1+t},\qquad \forall\, t>0. \tag{2.24}$$

Thus it follows from (2.10) and (2.24) that

$$t^{n(\frac{1}{p}-\frac{1}{q})} \ge C_{n,p,q}\frac{t}{1+t},\qquad \forall\, t \ll 1.$$

This means that $\frac{1}{p}-\frac{1}{q} \le \frac{1}{n}$.

Choosing $t=1$ in (2.24), we get $V(x,1) \ge C_{n,p,q}$, $\forall\, x \in M$. Therefore $\inf\limits_{x\in M} V(x,1) > 0$. And we get the necessary part of Theorem 1.1.

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