Chin. Ann. Math.
27B(1), 2006, 53–66
DOI: 10.1007/s11401-005-0297-3

Chinese Annals of Mathematics, Series B

Blow-up Mechanism of Classical Solutions to Quasilinear Hyperbolic Systems in the Critical Case**

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(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract This paper deals with the blow-up phenomenon, particularly, the geometric blow-up mechanism, of classical solutions to the Cauchy problem for quasilinear hyperbolic systems in the critical case. We prove that it is still the envelope of the same family of characteristics which yields the blowup of classical solutions to the Cauchy problem in the critical case.

Keywords Quasilinear hyperbolic system, Classical solution, Blowup, Critical case
2000 MR Subject Classification 35L45, 35L60

1 Introduction and Main Results

Consider the following Cauchy problem for the first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=0 \tag{1.1}$$

with the initial data

$$t=0:\ u=\phi(x), \tag{1.2}$$

where $u=(u_1,\cdots,u_n)^T$ is the unknown vector function of (t,x), $A(u)$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j=1,\cdots,n)$ and $\phi(x)=(\phi_1(x),\cdots,\phi_n(x))^T$ is a C^1 vector function of x.

We assume that in a neighbourhood of $u=0$, system (1.1) is strictly hyperbolic: $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u). \tag{1.3}$$

For $i=1,\cdots,n$, let $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))$ (resp. $r_i(u)=(r_{i1}(u),\cdots,r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$, namely,

$$l_i(u)A(u)=\lambda_i(u)l_i(u)\quad (\text{resp. } A(u)r_i(u)=\lambda_i(u)r_i(u)). \tag{1.4}$$

Without loss of generality, we assume that on the domain under consideration,

$$l_i(u)r_j(u)\equiv\delta_{ij}\quad (i,j=1,\cdots,n), \tag{1.5}$$

Manuscript received July 14, 2005.
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**Project supported by the Special Funds for Major State Basic Research Project of China and Specialized Research Fund for the Doctoral Program of Higher Education (No.20020246002).

where δ_{ij} stands for the Kronecker's symbol.

By means of the concept of weak linear degeneracy, in [4, 8, 11, 12], Li et al. studied the global existence and the blow-up phenomenon of C^1 solution to the Cauchy problem (1.1)-(1.2) under the assumption that the initial data (1.2) satisfy

$$\theta \triangleq \sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\} \ll 1 \tag{1.6}$$

for some constant $\mu > 0$ (some related results can be also found in [13]–[15]).

For $i \in \{1, \cdots, n\}$, the i-th characteristic $\lambda_i(u)$ is called to be weakly linearly degenerate, if, along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0 : u = 0, \end{cases} \tag{1.7}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \quad \forall\, |u| \text{ small}, \tag{1.8}$$

namely

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall\, |s| \text{ small}. \tag{1.9}$$

If all characteristics are weakly linearly degenerate, system (1.1) is said to be weakly linearly degenerate (see [5, 11, 12]).

On the other hand, if system (1.1) is not weakly linearly degenerate, then there exists a nonempty set $J \subseteq \{1, \cdots, n\}$ such that $\lambda_i(u)$ is not weakly linearly degenerate if and only if $i \in J$. For each $i \in J$, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^k\lambda_i(u^{(i)}(s))}{ds^k}\right|_{s=0} = 0 \ (k = 1, \cdots, \alpha_i) \ \text{ but } \ \left.\frac{d^{\alpha_i+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{1.10}$$

or

$$\left.\frac{d^k\lambda_i(u^{(i)}(s))}{ds^k}\right|_{s=0} = 0 \ (k = 1, 2, \cdots) \ \text{ but } \ \lambda_i(u^{(i)}(s)) \not\equiv \lambda_i(0), \ \text{ denoted by } \ \alpha_i = +\infty, \tag{1.11}$$

where $u = u^{(i)}(s)$ is defined by (1.7). Moreover, let

$$\alpha = \min\{\alpha_i \mid i \in J\}. \tag{1.12}$$

By [6] and [7], if system (1.1) is not weakly linearly degenerate, for any given $\theta_0 > 0$ suitably small, we can always find some initial data (1.2) with $\theta \in (0, \theta_0]$, where θ is defined by (1.6), such that the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1)-(1.2) blows up in a finite time. Thus, to find the blow-up mechanism is an interesting problem. The previous results mainly focus on the noncritical case that $\alpha < +\infty$, however, for the critical case $\alpha = +\infty$, only a few results are known (see [2, 12]). In this paper, we study the blow-up phenomenon, particularly, the geometric blow-up mechanism in the critical case.

Our main results are the following theorems which show that, although it is impossible to get a sharp estimate on the life-span in the critical case, the blow-up mechanism in the critical case is almost the same as in the noncritical case. However, we point out that the method used in previous papers can not be directly applied to the critical case and then some significant changes or improvements should be made in the proof.

Theorem 1.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^\infty$ *and system* (1.1) *is strictly hyperbolic. Suppose furthermore that system* (1.1) *is not weakly linearly degenerate and*

$$\alpha = +\infty. \tag{1.13}$$

Suppose finally that the initial data (1.2) *satisfy* (1.6)*. Then, for any given integer* $N \geq 1$*, there exists* $\theta_0 = \theta_0(N) > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$*, the life-span* $\widetilde{T}(\theta)$ *of* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (1.1)-(1.2) *satisfies*

$$\widetilde{T}(\theta) > \theta^{-N}. \tag{1.14}$$

Moreover, when $u = u(t, x)$ *blows up in a finite time,* $u = u(t, x)$ *itself is bounded on the domain* $[0, \widetilde{T}(\theta)) \times \mathbb{R}$*, while the first order partial derivatives of* $u = u(t, x)$ *tend to the infinity as* $t \nearrow \widetilde{T}(\theta)$*.*

Theorem 1.2 *Under the assumptions of Theorem* 1.1*, for any given integer* $N \geq 1$*, there exists* $\theta_0 = \theta_0(N) > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$*, the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (1.1)-(1.2) *blows up in a finite time if and only if at least one family of characteristics forms an envelope in the finite time.*

Theorem 1.3 *For each* $i \overline{\in} J$*, the family of the* i*-th characteristics never forms any envelope on the domain* $[0, \widetilde{T}(\theta)] \times \mathbb{R}$.

Theorem 1.4 *Under the assumptions of Theorem* 1.1*, on the line* $t = \widetilde{T}(\theta)$*, the set of blow-up points can not possess a positive measure.*

This paper is organized as follows: In Section 2 and Section 3 we give some preliminaries and some uniform a priori estimates respectively, then, the main results are proved in Section 4.

2 Preliminaries

By Lemma 2.5 in [11], when system (1.1) is strictly hyperbolic, there exists a suitably smooth invertible transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in the $\tilde{u}$-space, for each $i = 1, \cdots, n$, the i-th characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for $|\tilde{u}_i|$ small, namely

$$\tilde{r}_i(\tilde{u}_i e_i) // e_i, \quad \forall\, |\tilde{u}_i| \text{ small} \quad (i = 1, \cdots, n), \tag{2.1}$$

where $\tilde{r}_i(\tilde{u})$ denotes the i-th right eigenvector corresponding to $r_i(u)$ and

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{2.2}$$

This transformation is called a *normalized transformation*, and the unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)^T$ are called *normalized variables* or *normalized coordinates.*

Let

$$w_i = l_i(u) u_x \quad (i = 1, \cdots, n). \tag{2.3}$$

By (1.5), it is easy to see that

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

denote the directional derivative with respect to t along the i-th characteristic. We have

$$\frac{du}{d_i t} = \sum_{k=1}^{n}(\lambda_i(u) - \lambda_k(u)) w_k r_k(u) \quad (i = 1, \cdots, n). \tag{2.6}$$

Then, in normalized coordinates, it is easy to see that (see [9])

$$\frac{du_i}{d_i t} = \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k \quad (i = 1, \cdots, n), \tag{2.7}$$

where

$$\rho_{ijj}(u) \equiv 0, \quad \forall\, i, j \tag{2.8}$$

and

$$\rho_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) \int_0^1 \frac{\partial r_{ki}}{\partial u_j}(\tau u_1, \cdots, \tau u_{k-1}, u_k, \tau u_{k+1}, \cdots, \tau u_n) d\tau, \quad \forall\, j \neq k. \tag{2.9}$$

Obviously

$$\rho_{iji}(u) \equiv 0, \quad \forall\, i, j. \tag{2.10}$$

Moreover, noting (2.4) and (2.7), we have

$$\begin{aligned} d[u_i(dx - \lambda_i(u)dt)] &= \Big[\frac{du_i}{d_i t} + \sum_{k=1}^{n} \nabla\lambda_i(u) r_k(u) u_i w_k\Big] dt \wedge dx \\ &= \sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k dt \wedge dx, \end{aligned} \tag{2.11}$$

where

$$F_{ijk}(u) = \rho_{ijk}(u) + \nabla\lambda_j(u) r_k(u) \delta_{ij}. \tag{2.12}$$

Noting (2.8) and (2.10), it is easy to see that

$$F_{ijj}(u) \equiv 0, \qquad \forall\, j \neq i, \tag{2.13}$$

$$F_{iji}(u) \equiv 0, \qquad \forall\, j \neq i, \tag{2.14}$$

$$F_{iii}(u) = \nabla\lambda_i(u) r_i(u), \quad \forall\, i. \tag{2.15}$$

On the other hand, we have (see [1, 3, 5])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \cdots, n), \tag{2.16}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u) \delta_{ik} + (j|k)\}, \tag{2.17}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\, j \neq i, \tag{2.18}$$

$$\gamma_{iii}(u) = -\nabla\lambda_i(u) r_i(u), \quad \forall\, i. \tag{2.19}$$

Noting (2.4), by (2.16) we have (see [1])

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k dt \wedge dx, \tag{2.20}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u)) l_i(u)[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)]. \tag{2.21}$$

Hence

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall\, i, j. \tag{2.22}$$

3 Some Uniform a priori Estimates

Noting (1.3), we have

$$\lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{3.1}$$

Without loss of generality, we may assume that

$$\lambda_1(0) > 0. \tag{3.2}$$

Then, by continuity, there exist positive constant $\delta_0 (< \lambda_1(0))$ and δ so small that

$$\lambda_{i+1}(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall\, |u|, |u'| \leq \delta \quad (i = 1, \cdots, n-1), \tag{3.3}$$

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall\, |u|, |u'| \leq \delta \quad (i = 1, \cdots, n). \tag{3.4}$$

For any given $T \geq 0$, let

$$D_i^T = \begin{cases} \{(t,x) \mid 0 \leq t \leq T,\ x \leq (\lambda_1(0) + \delta_0)t\} & (i = 1), \\ \{(t,x) \mid 0 \leq t \leq T,\ (\lambda_i(0) - \delta_0)t \leq x \leq (\lambda_i(0) + \delta_0)t\} & (i = 2, \cdots, n-1), \\ \{(t,x) \mid 0 \leq t \leq T,\ x \geq (\lambda_n(0) - \delta_0)t\} & (i = n). \end{cases} \tag{3.5}$$

Obviously

$$\bigcup_{i=1}^{n} D_i^T \subset D(T) = \{(t,x) \mid 0 \leq t \leq T,\ -\infty < x < \infty\}. \tag{3.6}$$

On any given existence domain $D(T)$ of C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1)-(1.2), let

$$W_\infty^c(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D(T)\setminus D_i^T} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i(t,x)|\}, \tag{3.7}$$

$$\widetilde{W}_1(T) = \max_{i=1,\cdots,n} \max_{j\neq i} \sup_{c_j} \int_{c_j} |w_i(t,x)| dt, \tag{3.8}$$

where c_j denotes any given j-th characteristic on $D(T)$,

$$W_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{-\infty}^{\infty} |w_i(t,x)|dx, \tag{3.9}$$

$$U_\infty(T) = \|u(t,x)\|_{L^\infty(D(T))}. \tag{3.10}$$

For the time being, we assume that on any given existence domain $D(T)$,

$$|u(t,x)| \le \delta. \tag{3.11}$$

At the end of the proof of Lemma 3.2, we will explain that this hypothesis is reasonable.

Lemma 3.1 (See [9]) *For each $i = 1,\cdots,n$ and any given point $(t,x)\overline{\in} D_i^T$, let $c_i : \xi = \xi_i(\tau)$ $(0 \le \tau \le t)$ be the i-th characteristic passing through (t,x) and intersecting the x-axis at $(0,x_{i0})$. Then c_i never enters in D_i^T and there exist positive constants d_k $(k = 1,2,3)$ independent of (t,x) and i, such that*

$$|x - \lambda_i(0)t| \ge \delta_0 t, \tag{3.12}$$

$$d_1|x| \le |x - \lambda_i(0)t| \le d_2|x_{i0}|, \tag{3.13}$$

and, if $(\tau,\xi_i(\tau))\overline{\in} D_j^T$ for some j, then

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge d_3|x_{i0}|, \quad \forall\, 0 \le \tau \le t. \tag{3.14}$$

Lemma 3.2 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system (1.1) is strictly hyperbolic, i.e., (1.3) holds. Suppose furthermore that the initial data (1.2) satisfy (1.6). Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0,\theta_0]$, on any given existence domain $D(T)$ of C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1)-(1.2), we have the following uniform a priori estimates*

$$W_\infty^c(T) \le \kappa_1\theta, \tag{3.15}$$

$$\widetilde{W}_1(T),\ W_1(T) \le \kappa_2\theta \tag{3.16}$$

and

$$U_\infty(T) \le \kappa_3\theta, \tag{3.17}$$

where κ_i $(i = 1,2,3)$ are positive constants independent of θ and T.

Proof Noting (1.6), (2.3) and (3.11), we have

$$(1+|x|)^{1+\mu}|w_i(0,x)| \le C\theta, \tag{3.18}$$

here and henceforth C denotes a positive constant independent of θ and T.

We first estimate $W_\infty^c(T)$.

For any given $i \in \{1,\cdots,n\}$, passing through any fixed point $(t,x) \in D(T)\backslash D_i^T$, we draw the i-th characteristic c_i: $\xi = \xi_i(\tau)$ $(\tau \le t)$ which intersects the x-axis at a point $(0,x_{i0})$. Integrating (2.16) along c_i from 0 to t yields

$$w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u)w_jw_k(\tau,\xi_i(\tau))d\tau. \tag{3.19}$$

Noting (3.11) and (2.18) and using Lemma 3.1, it is easy to see that

$$\begin{aligned}&(1+|x-\lambda_i(0)t|)^{1+\mu}|w_i(t,x)| \le C(1+|x_{i0}|)^{1+\mu}\Big\{|w_i(0,x_{i0})|\\&+(W_\infty^c(T))^2\sum_{j,k=1}^n\int_{\xi_i(\tau)\not\subseteq(D_j^T\cup D_k^T)}[(1+|\xi_i(\tau)-\lambda_j(0)\tau|)(1+|\xi_i(\tau)-\lambda_k(0)\tau|)]^{-(1+\mu)}d\tau\\&+W_\infty^c(T)\sum_{j,k=1}^n\int_{\substack{\xi_i(\tau)\subseteq D_j^T\\ \xi_i(\tau)\not\subseteq D_k^T}}(1+|\xi_i(\tau)-\lambda_k(0)\tau|)^{-(1+\mu)}|w_j(\tau,\xi_i(\tau))|d\tau\Big\}\\&\le C\{(1+|x_{i0}|)^{1+\mu}|w_i(0,x_{i0})|+(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\}.\end{aligned}\tag{3.20}$$

Then, noting (3.18), it turns out that

$$W_\infty^c(T)\le C\{\theta+(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\}.\tag{3.21}$$

We next estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

For $i\in\{1,\cdots,n\}$, passing through two end points $A(t_A,x_A)$ and $B(t_B,x_B)$ of any given j-th characteristic $c_j:\xi=\xi_j(\tau)$ $(0\le t_A\le\tau\le t_B)$ on $D(T)$ $(j\ne i)$, we respectively draw the i-th characteristics which intersects the x-axis at point $C(0,x_C)$ and point $D(0,x_D)$ with $x_C\le x_D$. By (2.20), using Stokes' formula on the domain $ACDB$, we get

$$\begin{aligned}&\int_{t_A}^{t_B}|w_i(\lambda_j(u)-\lambda_i(u))(\tau,\xi_j(\tau))|d\tau\\&\le\int_{x_C}^{x_D}|w_i(0,x)|dx+\iint_{ACDB}\Big|\sum_{j,k=1}^n\Gamma_{ijk}(u)w_jw_k(t,x)\Big|dtdx.\end{aligned}\tag{3.22}$$

Then, noting (3.11), (3.18) and (2.22), we have

$$\begin{aligned}&\int_{t_A}^{t_B}|w_i(\lambda_j(u)-\lambda_i(u))(\tau,\xi_j(\tau))|d\tau\le C\Big\{\theta\int_{x_C}^{x_D}(1+|x|)^{-(1+\mu)}dx\\&+(W_\infty^c(T))^2\sum_{j,k=1}^n\iint_{(t,x)\,\overline{\in}\,(D_j^T\cup D_k^T)}[(1+|x-\lambda_j(0)t|)(1+|x-\lambda_k(0)t|)]^{-(1+\mu)}dtdx\\&+W_\infty^c(T)\sum_{j,k=1}^n\iint_{\substack{(t,x)\in D_j^T\\(t,x)\,\overline{\in}\,D_k^T}}(1+|x-\lambda_k(0)t|)^{-(1+\mu)}|w_j(t,x)|\,dtdx\Big\}.\end{aligned}\tag{3.23}$$

Then, noting (3.3) and using Lemma 3.1, we get

$$\int_{t_A}^{t_B}|w_i(\tau,\xi_j(\tau))|d\tau\le C\{\theta+(W_\infty^c(T))^2+W_\infty^c(T)W_1(T)\},\tag{3.24}$$

then

$$\widetilde{W}_1(T)\le C\{\theta+(W_\infty^c(T))^2+W_\infty^c(T)W_1(T)\}.\tag{3.25}$$

Similarly to (3.24), for any given positive constant r, we have

$$\int_{-r}^{r}|w_i(t,x)|dx\le C\{\theta+(W_\infty^c(T))^2+W_\infty^c(T)W_1(T)\},\tag{3.26}$$

where C is a positive constant independent of r. Taking $r \to +\infty$, we finally get

$$W_1(T) \le C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T)\}. \tag{3.27}$$

By (3.21), (3.25) and (3.27), we can obtain (3.15) and (3.16) by means of the method in [12].

Finally, we estimate $U_\infty(T)$.

Passing through any given point $(t,x) \in D(T)$, we draw the i-th characteristic $c_i : \xi = \xi_i(\tau)$ $(0 \le \tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6) along c_i from 0 to t gives

$$u(t,x) = u(0,x_{i0}) + \int_0^t \sum_{k=1}^n (\lambda_i(u) - \lambda_k(u)) w_k r_k(u)(\tau, \xi_i(\tau)) d\tau. \tag{3.28}$$

Then, noting (1.6) and using (3.16), we get

$$|u(t,x)| \le C\{\theta + \widetilde{W}_1(T)\} \le C\theta, \tag{3.29}$$

which leads to (3.17) immediately. In the meantime, taking $\theta_0 > 0$ suitably small, (3.29) also implies that the hypothesis (3.11) is reasonable.

Let

$$U^c_\infty(T) = \max_{i=1,\cdots,n} \sup_{(t,x)\in D(T)\setminus D^T_i} \{(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(t,x)|\}, \tag{3.30}$$

$$\widetilde{U}_1(T) = \max_{i=1,\cdots,n} \max_{j\ne i} \sup_{c_j} \int_{c_j} |u_i(t,x)| dt, \tag{3.31}$$

$$U_1(T) = \max_{i=1,\cdots,n} \sup_{0\le t\le T} \int_{-\infty}^{\infty} |u_i(t,x)| dx \tag{3.32}$$

and

$$W_\infty(T) = \|w(t,x)\|_{L^\infty(D(T))}, \tag{3.33}$$

where c_j denotes any given j-th characteristic on $D(T)$, $w(t,x) = (w_1(t,x), \cdots, w_n(t,x))^T$.

Lemma 3.3 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^\infty$ and system* (1.1) *is strictly hyperbolic. Suppose furthermore that* (1.1) *is not weakly linearly degenerate and* (1.13) *holds. Suppose finally that the initial data* (1.2) *satisfy* (1.6)*. Then, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, for any given positive integer N, on any given existence domain $D(T)$ of C^1 solution $u = u(t,x)$ to the Cauchy problem* (1.1)-(1.2) *with*

$$T\theta^N \le 1, \tag{3.34}$$

we have the following uniform a priori estimates

$$U^c_\infty(T) \le \kappa_4\theta \tag{3.35}$$

and

$$\widetilde{U}_1(T),\ U_1(T) \le \kappa_5\theta; \tag{3.36}$$

moreover, there exists $\theta_0 = \theta_0(N) > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ of C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1)-(1.2)*, where T still satisfies* (3.34)*, we have*

$$W_\infty(T) \le \kappa_6\theta, \tag{3.37}$$

where κ_i $(i = 4, 5, 6)$ are positive constants independent of θ and T but possibly depending on N.

Proof Without loss of generality, in order to prove Lemma 3.3, we assume that $u = (u_1, \cdots, u_n)$ are normalized coordinates.

We first estimate $U_\infty^c(T)$.

Similarly to (3.19), integrating (2.7) along c_i from 0 to t gives

$$u_i(t,x) = u_i(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \rho_{ijk}(u) u_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{3.38}$$

Then, noting (1.6) and (2.8) and using Lemma 3.1, similarly to (3.20), we have

$$\begin{aligned} &(1 + |x - \lambda_i(0)t|)^{1+\mu} |u_i(t,x)| \\ \le\ & C\{\theta + U_\infty^c(T) W_\infty^c(T) + U_\infty^c(T) \widetilde{W}_1(T) + \widetilde{U}_1(T) W_\infty^c(T)\}. \end{aligned} \tag{3.39}$$

Hence, using Lemma 3.2 we get

$$U_\infty^c(T) \le C\theta\{1 + U_\infty^c(T) + \widetilde{U}_1(T)\}. \tag{3.40}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

Similarly to (3.22), from (2.11) we have

$$\begin{aligned} &\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\ \le\ & \int_{x_C}^{x_D} |u_i(0,x)| dx + \iint_{ACDB} \Big| \sum_{j,k=1}^n F_{ijk}(u) u_j w_k(t,x) \Big| dt dx. \end{aligned} \tag{3.41}$$

Noting (1.13) and (2.15), for any given integer $N \ge 1$, we have

$$|F_{iii}(u_i e_i)| \le C_N |u_i|^N, \tag{3.42}$$

here and in what follows, C_N denotes a positive constant possibly depending on N. Then, noting (1.6) and (2.13) and using Hadamard's formula and Lemmas 3.1 and 3.2, it follows from (3.41) that

$$\begin{aligned} &\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\ \le\ & \int_{x_C}^{x_D} |u_i(0,x)| dx + \iint_{ACDB} \Big[\sum_{\substack{j,k=1 \\ j \ne k}}^n |F_{ijk}(u) u_j w_k(t,x)| \\ & + (|F_{iii}(u) - F_{iii}(u_i e_i)| + |F_{iii}(u_i e_i)|) |u_i w_i(t,x)| \Big] dt dx \\ \le\ & C\{\theta + U_\infty^c(T) W_\infty^c(T) + U_\infty^c(T) W_1(T) + U_1(T) W_\infty^c(T) \\ & + U_\infty^c(T) U_\infty(T) W_1(T)\} + C_N (U_\infty(T))^{N+1} W_1(T) T \\ \le\ & C\theta\{1 + U_\infty^c(T) + U_1(T)\} + C_N \theta^{N+2} T. \end{aligned} \tag{3.43}$$

Thus, noting (3.34), we get

$$\widetilde{U}_1(T) \leq C\theta\{1 + U_\infty^c(T) + U_1(T)\} + C_N\theta^2. \tag{3.44}$$

Similarly, we have

$$U_1(T) \leq C\theta\{1 + U_\infty^c(T) + U_1(T)\} + C_N\theta^2. \tag{3.45}$$

The combination of (3.40), (3.44) and (3.45) gives (3.35) and (3.36).

We finally estimate $W_\infty(T)$.

Similarly to (3.28), according to (2.16) we have

$$w_i(t,x) = w_i(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{3.46}$$

Noting (1.13) and (2.19), for any given integer $N \geq 1$, we have

$$|\gamma_{iii}(u_i e_i)| \leq C_N |u_i|^N. \tag{3.47}$$

Then, noting (2.18) and (3.35) and using Lemmas 3.1 and 3.2, from (3.46) we have

$$\begin{aligned}|w_i(t,x)| \leq{} & |w_i(0,x_{i0})| + \int_0^t \Bigg[\sum_{\substack{j,k=1\\ j\neq k}}^n |\gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau))| \\ & + (|\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| + |\gamma_{iii}(u_i e_i)|) w_i^2(\tau, \xi_i(\tau)) \Bigg] d\tau \\ \leq{} & C_N\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) \\ & + U_\infty^c(T)(W_\infty(T))^2 + (U_\infty(T))^N (W_\infty(T))^2 T\} \\ \leq{} & C_N\{\theta(1 + W_\infty(T) + (W_\infty(T))^2) + \theta^N T (W_\infty(T))^2\}. \end{aligned} \tag{3.48}$$

Hence, noting (3.34), we have

$$W_\infty(T) \leq C_N\{\theta + (W_\infty(T))^2\}. \tag{3.49}$$

Thus, we can obtain (3.36) by means of the method in [12].

Remark 3.1 For any given $i \,\overline{\in}\, J$, $\lambda_i(u)$ is weakly linearly degenerate. By (2.15) and (2.19), we have

$$F_{iii}(u_i e_i) \equiv 0 \quad \text{and} \quad \gamma_{iii}(u_i e_i) \equiv 0, \quad \forall\, |u_i| \text{ small}. \tag{3.50}$$

From the proof of Lemma 3.3, we know that there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ (without the restriction (3.34)) of C^1 solution $u = u(t,x)$ to the Cauchy problem (1.1)-(1.2) satisfying (1.6), we have the following uniform a priori estimate

$$|w_i(t,x)| \leq \kappa_7 \theta, \tag{3.51}$$

where κ_7 is a positive constant independent of θ and T.

4 Proof of the Theorems

Proof of Theorem 1.1 According to the existence and uniqueness of the local C^1 solution to the Cauchy problem (1.1)-(1.2) (see [10]), there exists $\tau_0 > 0$ such that on $[0, \tau_0] \times \mathbb{R}$, the Cauchy problem (1.1)-(1.2) has a unique C^1 solution $u = u(t, x)$. By Lemmas 3.2 and 3.3, we have that for any given integer $N \geq 1$, there exists $\theta_0 = \theta_0(N) > 0$ so small that for any given $\theta \in (0, \theta_0]$, on any given existence domain $[0, T] \times \mathbb{R}$ of C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1)-(1.2), where $0 < T \leq \theta^N$, we have the following uniform a priori estimate on the C^1 norm of $u = u(t, x)$:

$$\|u(t, \cdot)\|_{C^1} \triangleq \|u(t, \cdot)\|_{C^0} + \|u_x(t, \cdot)\|_{C^0} \leq C\theta, \quad \forall t \in [0, T], \tag{4.1}$$

where C is a positive constant independent of θ and T but possibly depending on N. By C^1 extension, we immediately get the existence and uniqueness of C^1 solution $u = u(t, x)$ on $[0, \theta^N] \times \mathbb{R}$. Hence, the life-span $\widetilde{T}(\theta)$ of C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1)-(1.2) satisfies

$$\widetilde{T}(\theta) > \theta^{-N}. \tag{4.2}$$

Moreover, by Lemma 3.2, when C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1)-(1.2) satisfying (1.6) blows up in a finite time, $u = u(t, x)$ itself must be bounded on $[0, \widetilde{T}(\theta))$, hence, the first order partial derivatives should tend to the infinity as $t \nearrow \widetilde{T}(\theta)$.

Proof of Theorem 1.2 Assume that (t^*, x^*) is a starting point of the blowup of C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1)-(1.2). By Theorem 1.1, we have

$$t^* > \theta^{-N}. \tag{4.3}$$

On the other hand, we can find an integer $p > N$ such that

$$t^* < \theta^{-p}. \tag{4.4}$$

For each $i = 1, \cdots, n$, passing through any given point (t, x) with $0 \leq t < t^*$ and $x \in \mathbb{R}$, we draw the i-th characteristic $c_i: \ \xi = x_i(\tau; y_i)$ in which $0 \leq \tau \leq t$ and y_i stands for the x-coordinate of the intersection point of this characteristic with x-axis, i.e., we have

$$\frac{dx_i(\tau; y_i)}{d\tau} = \lambda_i(u(\tau, x_i(\tau; y_i))) \tag{4.5}$$

and

$$x_i(0; y_i) = y_i, \quad x_i(t; y_i) = x. \tag{4.6}$$

In what follows, we prove

$$\Big| w_i(t, x) \frac{\partial x_i(t; y_i)}{\partial y_i} \Big| \leq C_p \theta, \quad \forall (t, x) \in [0, t^*) \times \mathbb{R}, \tag{4.7}$$

henceforth C_p denotes a positive constant possibly depending on p.

Noting (2.4), it follows from (4.5) and (4.6) that

$$\frac{d}{d\tau}\Big(\frac{\partial x_i(\tau; y_i)}{\partial y_i}\Big) = \sum_{k=1}^{n} \nabla\lambda_i(u) w_k r_k(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} \tag{4.8}$$

and

$$\frac{\partial x_i(0;y_i)}{\partial y_i}=1. \tag{4.9}$$

Hence

$$\frac{\partial x_i(t;y_i)}{\partial y_i}=\exp\Big(\int_0^t\sum_{k=1}^n\nabla\lambda_i(u)w_kr_k(\tau,x_i(\tau;y_i))d\tau\Big),\quad \forall\, t\in[0,t^*). \tag{4.10}$$

Noting (4.8), it is easy to get that

$$\frac{d}{d\tau}\Big[w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\Big]=\sum_{j,k=1}^n\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i},$$

where $\Gamma_{ijk}(u)$ is defined by (2.21). Then, noting (4.6), for any given $(t,x)\in[0,t^*)\times\mathbb{R}$, we have

$$w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i}=w_i(0,y_i)+\int_0^t\sum_{j,k=1}^n\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\,d\tau. \tag{4.11}$$

Let

$$I_0=[0,\theta^{-N}], \tag{4.12}$$

$$I_1=[\theta^{-N},t]\cap\{\tau\mid 0\le\tau\le t,|w_i(\tau,x_i(\tau;y_i))|\le\kappa_1\theta\}, \tag{4.13}$$

$$I_2=[\theta^{-N},t]\cap\{\tau\mid 0\le\tau\le t,|w_i(\tau,x_i(\tau;y_i))|>\kappa_1\theta\}, \tag{4.14}$$

where κ_1 is given in Lemma 3.2. Then, noting (2.22), (4.11) can be rewritten as

$$\begin{aligned}w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i}&=w_i(0,y_i)+\int_0^t\sum_{\substack{j=1\\ j\ne i}}^n\Gamma_{iji}(u)w_jw_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\,d\tau\\&\quad+\Big(\int_{I_0}+\int_{I_1}+\int_{I_2}\Big)\sum_{\substack{j,k=1\\ j,k\ne i}}^n\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\,d\tau\\&=w_i(0,y_i)+E+E_0+E_1+E_2.\end{aligned} \tag{4.15}$$

Now we estimate every term on the right-hand side of (4.15).

Obviously

$$|w_i(0,y_i)|\le C\theta,\quad \forall\, y_i\in\mathbb{R}. \tag{4.16}$$

Let

$$Q(t)=\sup_{(\tau,y_i)\in[0,t]\times\mathbb{R}}\Big|w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\Big|,\quad \forall\, t\in[0,t^*). \tag{4.17}$$

By Lemma 3.2, we get

$$|E|\le CQ(t)\widetilde{W}_1(t)\le C\theta Q(t). \tag{4.18}$$

By Lemma 3.3, we have

$$|w_i(\tau,x_i(\tau;y_i))|\le C\theta,\quad \forall\tau\in I_0\cup I_1. \tag{4.19}$$

Then, noting (2.19) and (1.13) and using Lemmas 3.2 and 3.3, it follows from (4.10) that

$$\begin{aligned}\Big|\frac{\partial x_i(t;y_i)}{\partial y_i}\Big| &\leq \exp\Big\{C\Big[\int_0^t\Big(\sum_{\substack{k=1\\k\neq i}}^n|w_k|+|(\gamma_{iii}(u)-\gamma_{iii}(u_ie_i))w_i|\\&\qquad+|\gamma_{iii}(u_ie_i)w_i|\Big)(\tau,x_i(\tau;y_i))d\tau\Big]\Big\}\\&\leq \exp\{C_p[\widetilde{W}_1(t)+\theta(\widetilde{U}_1(t)+(U_\infty(t))^pt)]\}\leq C_p,\quad \forall\, t\in[0,t^*).\end{aligned}\tag{4.20}$$

Hence, using Lemmas 3.1 and 3.2 and noting (2.22), we get

$$\begin{aligned}|E_0|+|E_1| &\leq C_p\Big(\int_{I_0}+\int_{I_1}\Big)\sum_{\substack{j,k=1\\j,k\neq i}}^n|\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))|d\tau\\&\leq C_p\widetilde{W}_1(t)W_\infty^c(t)\leq C_p\theta^2.\end{aligned}\tag{4.21}$$

We next estimate the last term.

According to Lemma 3.2, when $\tau\in I_2$, we have

$$(\tau,x_i(\tau;y_i))\in D_i^t.\tag{4.22}$$

Then, using Lemmas 3.1 and 3.2 and noting the definition of I_2, for any given $k\neq i$, we have

$$|w_k(\tau,x_i(\tau;y_i))|\leq CW_\infty^c(\tau)(1+\tau)^{-(1+\mu)}\leq C\theta^2\leq|w_i(\tau,x_i(\tau;y_i))|,\quad\forall\,\tau\in I_2.\tag{4.23}$$

Hence, noting Lemma 3.2, we have

$$\begin{aligned}|E_2| &\leq C\int_{I_2}\sum_{\substack{j=1\\j\neq i}}^n|w_j(\tau,x_i(\tau;y_i))|\Big|w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\Big|d\tau\\&\leq C\widetilde{W}_1(t)Q(t)\leq C\theta Q(t).\end{aligned}\tag{4.24}$$

Noting (4.16), (4.18), (4.21) and (4.24), it follows from (4.15) that

$$\Big|w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i}\Big|\leq C_p\theta+C\theta Q(t).\tag{4.25}$$

Similarly, we have

$$\Big|w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\Big|\leq C_p\theta+C\theta Q(t),\quad\forall\,\tau\in[0,t].\tag{4.26}$$

Hence, we have

$$Q(t)\leq C_p\theta+C\theta Q(t),\tag{4.27}$$

which implies (4.7).

By (4.7), if

$$w_i(t,x_i(t;y_i))\to\infty,\quad\text{as}\ \ t\to t^*-0,\tag{4.28}$$

then

$$\frac{\partial x_i(t,y_i)}{\partial y_i} \to 0, \quad \text{as} \ \ t \to t^* - 0. \tag{4.29}$$

On the other hand, by (4.10) and noting Lemma 3.2, it is easy to see that (4.29) implies (4.28).

This proves Theorem 1.2.

Proof of Theorem 1.3 For each $i \,\overline{\in}\, J$, by Remark 3.1, we have (3.51). Hence, by the equivalence of (4.28) and (4.29), the family of the i-th characteristics never forms any envelope on the domain $[0, \widetilde{T}(\theta)] \times \mathbb{R}$.

Theorem 1.4 can be easily obtained from the second inequality of (3.16) in Lemma 3.2.

References

[1] Hörmander, L., The Lifespan of classical solutions of nonlinear hyperbolic equations, Lecture Notes in Math., **1256**, Springer, 1987, 214–280.

[2] Hoshiga, A., The lifespan of classical solutions to quasilinear hyperbolic systems in the critical case, *Funkcialaj Ekvacioj*, **41**, 1998, 167–188.

[3] John, F., Formation of singularities in one-dimensional nonlinear wave propagation, *Comm. Pure Appl. Math.*, **27**, 1974, 377–405.

[4] Kong, D. X. and Li, T. T., A note on blow-up phenomenon of classical solutions to quasilinear hyperbolic systems, *Nonlinear Analysis*, **49**, 2002, 535–539.

[5] Li, T. T., Global Classical Solutions for Quasilinear Hyperbolic Systems, Research in Applied Mathematics, **32**, J. Wiley/Masson, 1994.

[6] Li, T. T., Une remarque sur les coordonnées normalisées et ses applications aux systèmes hyperboliques quasi linéaires, *C. R. Acad. Sci. Paris*, t. **331**, Série **I**, 2000, 447–452.

[7] Li, T. T., A remark on the normalized coordinates and its applications to quasilinear hyperbolic systems, Optimal Control and Partial Differential Equations, J. L. Menaldi et al. (eds.), IOS Press, 2001, 181–187.

[8] Li, T. T. and Kong, D. X., Breakdown of classical solutions to quasilinear hyperbolic systems, *Nonlinear Analysis, Theory, Methods & Applications, Series A*, **40**, 2000, 407–437.

[9] Li, T. T. and Wang, L. B., Global existence of weakly discontinuous solutions to the Cauchy problem with a kind of non-smooth initial data for quasilinear hyperbolic systems, *Chin. Ann. Math.*, **25B**(3), 2004, 319–334.

[10] Li, T. T. and Yu, W. C., Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Mathematics, Series V, 1985.

[11] Li, T. T., Zhou, Y. and Kong, D. X., Weak linear degeneracy and global classical solutions for general quasilinear hyperbolic systems, *Commun. in Partial Differential Equations*, **19**, 1994, 1263–1317.

[12] Li, T. T., Zhou, Y. and Kong, D. X., Global classical solutions for general quasilinear hyperbolic systems with decay initial data, *Nonlinear Analysis, Theory, Methods & Applications*, **28**, 1997, 1299–1322.

[13] Wang, W. K. and Yang, X. F., Pointwise estimates of solutions to Cauchy problem for quasilinear hyperbolic systems, *Chin. Ann. Math.*, **24B**(4), 2003, 457–468.

[14] Yan, P., Global classical solutions with small initial total variation for quasilinear hyperbolic systems, *Chin. Ann. Math.*, **23B**(3), 2002, 335–348.

[15] Zhou, Y., Global classical solutions to quasilinear hyperbolic systems with weak linear degeneracy, *Chin. Ann. Math.*, **25B**(1), 2004, 37–56.