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η-Invariant and Flat Vector Bundles***

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(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract We present an alternate definition of the mod **Z** component of the Atiyah-Patodi-Singer η invariant associated to (not necessary unitary) flat vector bundles, which identifies explicitly its real and imaginary parts. This is done by combining a deformation of flat connections introduced in a previous paper with the analytic continuation procedure appearing in the original article of Atiyah, Patodi and Singer.

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1 Introduction

Let M be an odd dimensional oriented closed spin manifold carrying a Riemannian metric g^{TM}. Let $S(TM)$ be the associated Hermitian bundle of spinors. Let E be a Hermitian vector bundle over M carrying a unitary connection ∇^E. Moreover, let F be a Hermitian vector bundle over M carrying a unitary flat connection ∇^F. Let

$$D^{E\otimes F} : \Gamma(S(TM) \otimes E \otimes F) \longrightarrow \Gamma(S(TM) \otimes E \otimes F) \tag{1.1}$$

denote the corresponding (twisted) Dirac operator, which is formally self-adjoint (cf. [**?**]).

For any $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) \gg 0$, following [**?**], set

$$\eta(D^{E\otimes F}, s) = \sum_{\lambda\in \mathrm{Spec}(D^{E\otimes F})\setminus\{0\}} \frac{\mathrm{Sgn}(\lambda)}{|\lambda|^s}. \tag{1.2}$$

Then by [**?**], one knows that $\eta(D^{E\otimes F}, s)$ is a holomorphic function in s when $\mathrm{Re}\,(s) > \frac{\dim M}{2}$. Moreover, it extends to a meromorphic function over $\mathbf{C}$, which is holomorphic at $s = 0$. The η invariant of $D^{E\otimes F}$, in the sense of Atiyah-Patodi-Singer [**?**], is defined by

$$\eta(D^{E\otimes F}) = \eta(D^{E\otimes F}, 0), \tag{1.3}$$

while the corresponding *reduced* η invariant is defined and denoted by

$$\bar\eta(D^{E\otimes F}) = \frac{\dim(\ker D^{E\otimes F}) + \eta(D^{E\otimes F})}{2}. \tag{1.4}$$

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The η and reduced η invariants play an important role in the Atiyah-Patodi-Singer index theorem for Dirac operators on manifolds with boundary (cf. [**?**]).

In [**?**] and [**?**], it is shown that the following quantity

$$\rho(D^{E\otimes F}) := \bar{\eta}(D^{E\otimes F}) - \mathrm{rk}(F)\,\bar{\eta}(D^{E}) \mod \mathbf{Z} \tag{1.5}$$

does not depend on the choice of g^{TM} as well as the metrics and (Hermitian) connections on E. Also, a Riemann-Roch theorem is proved in [**?**, (5.3)], which gives a K-theoretic interpretation of the analytically defined invariant $\rho(D^{E\otimes F}) \in \mathbf{R}/\mathbf{Z}$. Moreover, it is pointed out in [**?**, Remark (1), p. 89] that the above mentioned K-theoretic interpretation applies also to the case where F is a non-unitary flat vector bundle, while on [**?**, p. 93] it shows how one can define the reduced η-invariant in case F is non-unitary, by working on non-self-adjoint elliptic operators, and then extend the Riemann-Roch result [**?**, (5.3)] to an identity in $\mathbf{C}/\mathbf{Z}$ (instead of $\mathbf{R}/\mathbf{Z}$). The idea of analytic continuation plays a key role in obtaining this Riemann-Roch result, as well as its non-unitary extension.

In this paper, we show that by using the idea of analytic continuation, one can construct the $\mathbf{C}/\mathbf{Z}$ component of $\bar{\eta}(D^{E\otimes F})$ directly, without passing to analysis of non-self-adjoint operators, in the case where F is a non-unitary flat vector bundle. Consequently, this leads to a direct construction of $\rho(D^{E\otimes F})$ in this case. We will use a deformation introduced in [**?**] for flat connections in our construction.

In the next section, we will first recall the above mentioned deformation from [**?**] and then give our construction of $\bar{\eta}(D^{E\otimes F}) \mod \mathbf{Z}$ and $\rho(D^{E\otimes F}) \in \mathbf{C}/\mathbf{Z}$ in the case where F is a non-unitary flat vector bundle.

2 The η and ρ Invariants Associated to Non-unitary Flat Vector Bundles

This section is organized as follows. In Subsection **??**, we construct certain secondary characteristic forms and classes associated to non-unitary flat vector bundles. In Subsection **??**, we present our construction of the mod $\mathbf{Z}$ component of the reduced η-invariant, as well as the ρ-invariant, associated to non-unitary flat vector bundles. Finally, we include some further remarks in Subsection **??**.

2.1 Chern-Simons classes and flat vector bundles

We fix a square root of $\sqrt{-1}$ and let $\varphi : \Lambda(T^*M) \to \Lambda(T^*M)$ be the homomorphism defined by $\varphi : \omega \in \Lambda^i(T^*M) \to (2\pi\sqrt{-1})^{-i/2}\omega$. The formulas in what follows will not depend on the choice of the square root of $\sqrt{-1}$.

If W is a complex vector bundles over M and ∇_0^W, ∇_1^W are two connections on W. Let W_t, $0 \le t \le 1$, be a smooth path of connections on W connecting ∇_0^W and ∇_1^W. We define Chern-Simons form $\mathrm{CS}(\nabla_0^W, \nabla_1^W)$ to be the differential form given by

$$\mathrm{CS}(\nabla_0^W, \nabla_1^W) = -\Big(\frac{1}{2\pi\sqrt{-1}}\Big)^{\frac{1}{2}}\varphi\int_0^1 \mathrm{Tr}\Big[\frac{\partial\nabla_t^W}{\partial t}\exp(-(\nabla_t^W)^2)\Big]dt. \tag{2.1}$$

Then (cf. [**?**, Chapter 1])

$$d\,\mathrm{CS}(\nabla_0^W,\nabla_1^W)=\mathrm{ch}(W,\nabla_1^W)-\mathrm{ch}(W,\nabla_0^W). \tag{2.2}$$

Moreover, it is well known that up to exact forms, $\mathrm{CS}(\nabla_0^W,\nabla_1^W)$ does not depend on the path of connections on W connecting ∇_0^W and ∇_1^W.

Let (F,∇^F) be a flat vector bundle carrying the flat connection ∇^F. Let g^F be a Hermitian metric on F. We do not assume that ∇^F preserves g^F. Let $(\nabla^F)^*$ be the adjoint connection of ∇^F with respect to g^F.

From [**?**, (4.1), (4.2)] and [**?**, §1, (g)], one has

$$(\nabla^F)^*=\nabla^F+\omega(F,g^F) \tag{2.3}$$

with

$$\omega(F,g^F)=(g^F)^{-1}(\nabla^F g^F). \tag{2.4}$$

Then

$$\nabla^{F,e}=\nabla^F+\frac{1}{2}\omega(F,g^F) \tag{2.5}$$

is a Hermitian connection on (F,g^F) (cf. [**?**, (1.33)] and [**?**, (4.3)]).

Following [**?**, (2.47)], for any $r\in\mathbf{C}$, set

$$\nabla^{F,e,(r)}=\nabla^{F,e}+\frac{\sqrt{-1}\,r}{2}\omega(F,g^F). \tag{2.6}$$

Then for any $r\in\mathbf{R}$, $\nabla^{F,e,(r)}$ is a Hermitian connection on (F,g^F).

On the other hand, following [**?**, (0.2)], for any integer $j\geq 0$, let $c_{2j+1}(F,g^F)$ be the Chern form defined by

$$c_{2j+1}(F,g^F)=(2\pi\sqrt{-1}\,)^{-j}2^{-(2j+1)}\mathrm{Tr}[\omega^{2j+1}(F,g^F)]. \tag{2.7}$$

Then $c_{2j+1}(F,g^F)$ is a closed form on M. Let $c_{2j+1}(F)$ be the associated cohomology class in $H^{2j+1}(M,\mathbf{R})$, which does not depend on the choice of g^F.

For any $j\geq 0$ and $r\in\mathbf{R}$, let $a_j(r)\in\mathbf{R}$ be defined as

$$a_j(r)=\int_0^1(1+u^2r^2)^j\,du. \tag{2.8}$$

With these notation we can now state the following result first proved in [**?**, Lemma 2.12].

Proposition 2.1 *The following identity in $H^{\mathrm{odd}}(M,\mathbf{R})$ holds for any $r\in\mathbf{R}$,*

$$\mathrm{CS}(\nabla^{F,e},\nabla^{F,e,(r)})=-\frac{r}{2\pi}\sum_{j=0}^{+\infty}\frac{a_j(r)}{j!}c_{2j+1}(F). \tag{2.9}$$

2.2 η and ρ invariants associated to flat vector bundles

We now make the same assumptions as in the beginning of Section 1, except that we no longer assume ∇^F there is unitary.

For any $r \in \mathbf{C}$, let

$$D^{E\otimes F}(r) : \Gamma(S(TM)\otimes E\otimes F) \longrightarrow \Gamma(S(TM)\otimes E\otimes F) \tag{2.10}$$

denote the Dirac operator associated to the connection $\nabla^{F,e,(r)}$ on F. Since when $r \in \mathbf{R}$, $\nabla^{F,e,(r)}$ is Hermitian on (F, g^F), $D^{E\otimes F}(r)$ is formally self-adjoint and one can define the associated reduced η-invariant as in (**??**).

By the variation formula for the reduced η-invariant (cf. [**?**, **?**]), one gets that for any $r \in \mathbf{R}$,

$$\bar\eta(D^{E\otimes F}(r)) - \bar\eta(D^{E\otimes F}(0)) \equiv \int_M \widehat{A}(TM)\,\mathrm{ch}(E)\,\mathrm{CS}(\nabla^{F,e}, \nabla^{F,e,(r)}) \mod \mathbf{Z}, \tag{2.11}$$

where $\widehat{A}$ and ch are standard notations for the Hirzebruch $\widehat{A}$-class and Chern character respectively (cf. [**?**, Chapter 1]).

Let $D^{E\otimes F,e}$ denote the Dirac operator $D^{E\otimes F}(0)$.

From (**??**) and (**??**), one gets that for any $r \in \mathbf{R}$,

$$\bar\eta(D^{E\otimes F}(r)) \equiv \bar\eta(D^{E\otimes F,e}) - \frac{r}{2\pi}\int_M \widehat{A}(TM)\,\mathrm{ch}(E)\sum_{j=0}^{+\infty}\frac{a_j(r)}{j!}c_{2j+1}(F) \mod \mathbf{Z}. \tag{2.12}$$

Recall that even though when $\mathrm{Im}\,(r) \neq 0$, $D^{E\otimes F}(r)$ might not be formally self-adjoint, the η-invariant can still be defined, as outlined in [**?**, p. 93]. On the other hand, from (**??**) and (**??**), one sees that

$$\nabla^F = \nabla^{F,e,(\sqrt{-1}\,)}. \tag{2.13}$$

We denote the associated Dirac operator $D^{E\otimes F}(\sqrt{-1}\,)$ by $D^{E\otimes F}$.

We also recall that

$$\int_0^1 (1-u^2)^j du = \frac{2^{2j}(j!)^2}{(2j+1)!}. \tag{2.14}$$

We can now state the main result of this paper as follows.

Theorem 2.2 *Formula* (**??**) *holds indeed for any* $r \in \mathbf{C}$. *In particular, one has*

$$\bar\eta(D^{E\otimes F}) \equiv \bar\eta(D^{E\otimes F,e}) - \frac{\sqrt{-1}}{2\pi}\int_M \widehat{A}(TM)\,\mathrm{ch}(E)\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F) \mod \mathbf{Z}. \tag{2.15}$$

Equivalently,

$$\begin{aligned}&\mathrm{Re}\,(\bar\eta(D^{E\otimes F})) \equiv \bar\eta(D^{E\otimes F,e}) \mod \mathbf{Z},\\ &\mathrm{Im}\,(\bar\eta(D^{E\otimes F})) = -\frac{1}{2\pi}\int_M \widehat{A}(TM)\,\mathrm{ch}(E)\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F).\end{aligned} \tag{2.16}$$

Proof Clearly, the right-hand side of (**??**) is a holomorphic function in $r \in \mathbf{C}$. On the other hand, by [**?**, p. 93], $\bar{\eta}(D^{E\otimes F}(r)) \bmod \mathbf{Z}$ is also holomorphic in $r \in \mathbf{C}$. By (**??**) and the uniqueness of the analytic continuation, one sees that (**??**) holds indeed for any $r \in \mathbf{C}$. In particular, by putting together (**??**) and (**??**), one gets (**??**).

Recall that when ∇^F preserves g^F, the ρ-invariant has been defined in (**??**). Now if we no longer assume that ∇^F preserves g^F, then by Theorem 2.2, one sees that one gets the following formula of the associated (extended) ρ-invariant.

Corollary 2.3 *The following identity holds:*

$$\rho(D^{E\otimes F}) \equiv \bar{\eta}(D^{E\otimes F,e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E) - \frac{\sqrt{-1}}{2\pi}\int_M \widehat{A}(TM)\,\mathrm{ch}(E)\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F) \mod \mathbf{Z}. \tag{2.17}$$

Equivalently,

$$\begin{aligned} \mathrm{Re}\,(\rho(D^{E\otimes F})) &\equiv \bar{\eta}(D^{E\otimes F,e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E) \mod \mathbf{Z}, \\ \mathrm{Im}\,(\rho(D^{E\otimes F})) &= -\frac{1}{2\pi}\int_M \widehat{A}(TM)\,\mathrm{ch}(E)\sum_{j=0}^{+\infty}\frac{2^{2j}j!}{(2j+1)!}c_{2j+1}(F). \end{aligned} \tag{2.18}$$

It is pointed out in [**?**] that the Riemann-Roch formula proved in [**?**, (5.3)] still holds for $\rho(D^{E\otimes F})$ in the case where ∇^F does not preserve g^F. One way to understand this is that the argument in the proof of [**?**, (5.3)] given in [**?**] works line by line to give a K-theoretic interpretation of $\bar{\eta}(D^{E\otimes F,e}) - \mathrm{rk}(F)\,\bar{\eta}(D^E)$. By (**??**) it then gives such an interpretation for $\rho(D^{E\otimes F})$.

2.3 Further remarks

Remark 2.4 The argument in proving Theorem 2.2 works indeed for any twisted vector bundles F, not necessary a flat vector bundle. This gives a direct formula for the mod **Z** part of the η-invariant for non-self-adjoint Dirac operators.

Remark 2.5 In [**?**, Theorem 2.2], a K-theoretic formula for $D^{E\otimes F}(r) \bmod \mathbf{Z}$ has been given in the $r \in \mathbf{R}$ case. As a consequence, one gets an alternate K-theoretic formula for $\rho(D^{E\otimes F})$ in [**?**, (4.6)] which holds in the case where ∇^F preserves g^F. By combining the arguments in [**?**] with Theorem 2.2 proved above, one can indeed extend [**?**, Theorem 2.2] and [**?**, (4.6)] to the case where ∇^F might not preserve g^F. We leave this to the interested reader. Here we only mention that this will provide an alternate K-theoretic interpretation of ρ-invariants in the case where ∇^F does not preserve g^F.

Remark 2.6 We refer to [**?**] where we have employed the deformation (**??**) to study and generalize certain Riemann-Roch-Grothendieck formulas due to Bismut-Lott [**?**] and Bismut [**?**], for flat vector bundles over fibred spaces.

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