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Asymptotic Behavior of Global Classical Solutions of Quasilinear Non-strictly Hyperbolic Systems with Weakly Linear Degeneracy**

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Abstract In this paper, we study the asymptotic behavior of global classical solutions of the Cauchy problem for general quasilinear hyperbolic systems with constant multiple and weakly linearly degenerate characteristic fields. Based on the existence of global classical solution proved by Zhou Yi et al., we show that, when t tends to infinity, the solution approaches a combination of C^1 travelling wave solutions, provided that the total variation and the L^1 norm of initial data are sufficiently small.

Keywords Asymptotic behavior, Characteristic fields with constant multiplicity, Weakly linear degeneracy, Global classical solution, Normalized coordinates, Travelling wave
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1 Introduction and Main Result

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector-valued function of (t, x) standing for the density of physical quantities, $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$ and $B(u) = (B_1(u), B_2(u), \cdots, B_n(u))^T$ is a given suitably smooth vector-valued function.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors $l_1(u), \cdots, l_n(u)$ (resp. $r_1(u), \cdots, r_n(u)$). In this paper, we assume that

(H_1) (1.1) *is a hyperbolic system with constant multiple characteristic fields.*

Without loss of generality, we may suppose that

$$\lambda(u) \stackrel{\triangle}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u). \tag{1.2}$$

When $p = 1$, the system (1.1) is strictly hyperbolic; while, when $p > 1$, (1.1) is a non-strictly hyperbolic system.

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For $i = 1, \cdots, n$, let

$$l_i(u) = (l_{i1}(u), \cdots, l_{in}(u)) \quad (\text{resp. } r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T)$$

be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$, i.e.,

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

By [4], the components of the eigenvalues and eigenvectors, i.e. $\lambda_i(u)$, $r_{ij}(u)$ and $l_{ij}(u)$ $(i, j = 1, \cdots, n)$, all have the same singularity as $A(u)$ on the domain under consideration. Furthermore, the eigenvectors can be chosen such that the following normalized conditions hold on the domain under consideration:

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i, j = 1, \cdots, n, \tag{1.5}$$

$$r_i^T(u)r_i(u) \equiv 1, \quad i = 1, \cdots, n, \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

In order to introduce some basic known results and state our main result precisely, we first recall the concepts of *weakly linear degeneracy*, *normalized coordinates* and *matching condition* (see [10, 11]).

Definition 1.1 *The p multiple characteristic $\lambda(u)$ is weakly linearly degenerate, if, along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by*

$$\frac{du}{ds} = r_i(u), \quad u(0) = 0, \quad i = 1, \cdots, p, \tag{1.7}$$

we have

$$\nabla\lambda(u)r_i(u) \equiv 0, \quad \forall\, |u| \text{ small}, \quad \forall\, i = 1, \cdots, p, \tag{1.8}$$

namely,

$$\lambda(u^{(i)}(s)) \equiv \lambda(0), \quad \forall\, |s| \text{ small}, \quad \forall\, i = 1, \cdots, p. \tag{1.9}$$

The j-th characteristic $\lambda_j(u)$ $(j = p+1, \cdots, n)$ is weakly linearly degenerate, if, along the j-th characteristic trajectory $u = u^{(j)}(s)$ passing through $u = 0$, we have

$$\nabla\lambda_j(u)r_j(u) \equiv 0, \quad \forall\, |u| \text{ small}, \tag{1.10}$$

namely,

$$\lambda_j(u^{(j)}(s)) \equiv \lambda_j(0), \quad \forall\, |s| \text{ small}. \tag{1.11}$$

If all characteristics are weakly linearly degenerate, then system (1.1) *is called to be weakly linearly degenerate.*

In this thesis, we suppose that

(H_2) (1.1) *is a hyperbolic system with weakly linearly degenerate characteristics.*

Definition 1.2 *If there exists an invertible smooth transformation $u = u(\tilde{u})$ $(u(0) = 0)$ such that in the $\tilde{u}$-space,*

$$\tilde{r}_i\Big(\sum_{h=1}^{p} \tilde{u}_h e_h\Big) \equiv e_i, \quad i = 1, \cdots, p, \quad \forall\, |\tilde{u}_h| \ \textit{small}, \quad h = 1, \cdots, p, \tag{1.12}$$

$$\tilde{r}_j(\tilde{u}_j e_j) \equiv e_j, \quad \forall\, |\tilde{u}_j| \ \textit{small}, \quad j = p+1, \cdots, n, \tag{1.13}$$

where

$$e_i = (0, \cdots, 0, \overset{(i)}{1}, 0, \cdots, 0)^T. \tag{1.14}$$

Such a transformation $u = u\,(\tilde{u})$ is called the normalized transformation and the corresponding unknown variables $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n)^T$ are called the normalized variables or normalized coordinates.

In this paper, we suppose that

(H_3) *There exist the normalized coordinates for the system* (1.1).

Remark 1.1 For the strictly hyperbolic systems, the normalized coordinates always exist (see [9, Lemma 2.5]). However for the non-strictly hyperbolic systems, the normalized coordinates maybe do not exist.

Obviously, in the normalized coordinates (if any), (1.9) and (1.11) simply reduces to

$$\lambda\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv \lambda_i\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv \lambda_i(0) \equiv \lambda(0), \quad \forall\, |u_h| \text{ small}, \quad \forall\, i = 1, \cdots, p \tag{1.9a}$$

and

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall\, |u_i| \text{ small}, \quad \forall\, i = p+1, \cdots, n. \tag{1.11a}$$

Definition 1.3 *The inhomogeneous term $B(u)$ is said to satisfy the matching condition, if, along the i-th characteristic trajectory, we have*

$$\nabla B(u) r_i(u) \equiv 0, \quad \forall\, i = 1, 2, \cdots, n, \quad \forall\, |u| \ \textit{small},$$

i.e., in the normalized coordinates (*if any*)

$$B\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\, |u_h| \ \textit{small}, \quad h = 1, \cdots, p, \tag{1.15}$$

$$B(u_j e_j) \equiv 0, \quad \forall\, |u_j| \ \textit{small}, \quad j = p+1, \cdots, n. \tag{1.16}$$

In this paper, we further suppose that

(H_4) *For the system* (1.1), *$B(u)$ satisfies the matching condition.*

Consider the Cauchy problem for the system (1.1) with the following initial data

$$t = 0: \quad u(0, x) = f(x), \tag{1.17}$$

where $f(x)$ is a C^1 vector-valued function of x. For the case that the initial data $f(x)$ satisfies the following decay property: there exists a constant $\mu > 0$ such that

$$\varrho \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|f(x)|+|f'(x)|)\} < +\infty \tag{1.18}$$

is sufficiently small, by means of the normalized coordinates Li et al proved that the Cauchy problem (1.1) and (1.17) admits a unique global classical solution, provided that the system (1.1) is weakly linearly degenerate (see [5, 8–11]). In their works, the condition $\mu > 0$ is essential. If $\mu = 0$, a counterexample was constructed in [6] showing that the classical solution may blow up in a finite time, even when the system (1.1) is weakly linearly degenerate.

Recently, Zhou [13] proved the global existence of classical solution when the system (1.1) is strictly hyperbolic and homogeneous provided that the total variation and L^1 norm of the initial data are sufficient small. Before long, this result was generalized by Du [3] and Wu [12] respectively.

Theorem A (See [13, 3, 12]) *Suppose that* (H_1)–(H_4) *hold. Suppose furthermore that* $A(u),\ B(u) \in C^2$ *in a neighborhood of* $u = 0$ *and* $f \in C^1$ *with bounded* C^1 *norm. Let*

$$M \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}} |f'(x)| < +\infty. \tag{1.19}$$

Then there exists a small constant $\epsilon > 0$ *such that the Cauchy problem* (1.1) *and* (1.17) *admits a unique global* C^1 *solution* $u = u(t, x)$ *for all* $t \in \mathbb{R}^+$*, provided that*

$$\int_{-\infty}^{+\infty} |f'(x)|dx \le \epsilon, \tag{1.20}$$

$$\int_{-\infty}^{+\infty} |f(x)|dx \le \frac{\epsilon}{M+1}. \tag{1.21}$$

Our goal in this paper is to describe the exact time asymptotic behavior of the classical solution of the Cauchy problem (1.1) and (1.17). Based on Theorem A, we prove the following theorem.

Theorem 1.1 (Asymptotic Behavior) *Suppose that* $A(u),\ B(u)$ *are* $C^{2,\rho}$ $(0 < \rho \le 1)$ *continuous. Then, under the assumptions of Theorem A, there exists a unique* C^1 *vector-valued function* $\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))^T$ *such that in the normalized coordinates*

$$u(t, x) \longrightarrow \sum_{i=1}^{n} \varphi_i(x - \lambda_i(0)t)e_i, \quad \text{as} \quad t \to +\infty; \tag{1.22}$$

moreover, $\varphi_i(x)$ $(i = 1, \cdots, n)$ *are globally Lipschitz continuous, more precisely, there exists a positive constant* κ_1 *independent of* $\epsilon,\ M,\ x_1$ *and* x_2 *such that for every* $i \in \{1, \cdots, n\}$*, it holds that*

$$|\varphi_i(x_1) - \varphi_i(x_2)| \le \kappa_1 M|x_1 - x_2|, \quad \forall\, x_1, x_2 \in \mathbb{R}. \tag{1.23}$$

Furthermore, if $f'(x)$, the derivative of the initial data, is globally ρ-Hölder continuous, that is, there exists a positive constant ς such that

$$|f'(x_1) - f'(x_2)| \leq \varsigma |x_1 - x_2|^\rho, \quad \forall\, x_1, x_2 \in \mathbb{R}, \tag{1.24}$$

then $\varphi'(x)$ satisfies that, for all $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned}|\varphi'(x_1) - \varphi'(x_2)| \leq \kappa_2 \Big[\varsigma + \frac{M}{M+1}(M+\epsilon)\max(M,\epsilon)\Big] |\alpha - \beta|^\rho \\ + \kappa_2 (M^2 + M\epsilon + \epsilon^3)|\alpha - \beta|,\end{aligned} \tag{1.25}$$

where κ_2 is a positive constant independent of ϵ, M, ς, x_1 and x_2.

Remark 1.2 Theorem 1.1 gives the exact time asymptotic behavior of the global classical solutions presented in Theorem A. For the initial data satisfying the decay property (1.18) and $B(u) \equiv 0$, Kong and Yang [7] proved that, when t tends to infinity, the global classical solution approaches a combination of C^1 travelling wave solutions at algebraic rate $(1+t)^{-\mu}$. The goal of the present paper is, in fact, to generalize the result in [7] to the case of the initial data with small BV norm. By [1] and [6], we observe that the BV norm is suitable and almost sharp in [13]. Comparing with [7], because of the lack of the decay rate of the initial data, in the present situation there is no any estimate on the convergence rate.

The paper is organized as follows. For the sake of completeness, in Section 2 we recall John's formula on the decomposition of waves with some supplements. Section 3 is devoted to establishing some new estimates, these estimates will play an important role in the proof of main result. The main result, Theorem 1.1, is proved in Section 4. By analyzing carefully the global propagation properties of the classical waves, we use the estimates given in Section 3 to describe the large time behavior of the global classical solutions, and then construct the desired travelling waves.

2 Preliminaries

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which play an important role in our proof.

Let

$$v_i(u) = l_i(u)u, \quad i = 1, \cdots, n, \tag{2.1}$$

$$w_i(u) = l_i(u)u_x, \quad i = 1, \cdots, n, \tag{2.2}$$

$$b_i(u) = l_i(u)B(u), \quad i = 1, 2, \cdots, n. \tag{2.3}$$

By (1.5), we have

$$u = \sum_{k=1}^n v_k r_k(u), \tag{2.4}$$

$$u_x = \sum_{k=1}^n w_k r_k(u), \tag{2.5}$$

$$B(u)=\sum_{k=1}^{n} b_k(u)r_k(u). \tag{2.6}$$

Let

$$\frac{d}{d_i t}=\frac{\partial}{\partial t}+\lambda_i(u)\frac{\partial}{\partial x} \tag{2.7}$$

be the directional derivative along the i-th characteristic. We have (see [9–11] or [5])

$$\frac{dv_i}{d_i t}=\sum_{j,k=1}^{n}\beta_{ijk}(u)v_j w_k+\sum_{j,k=1}^{n}\nu_{ijk}(u)v_j b_k(u)+b_i(u)\triangleq F_i(t,x), \tag{2.8}$$

$$\frac{dw_i}{d_i t}=\sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j w_k+\sum_{j,k=1}^{n}\sigma_{ijk}(u)w_j b_k(u)+(b_i(u))_x\triangleq G_i(t,x), \tag{2.9}$$

where

$$\beta_{ijk}(u)=(\lambda_k(u)-\lambda_i(u))l_i(u)\nabla r_j(u)r_k(u), \tag{2.10}$$

$$\nu_{ijk}(u)=-l_i(u)\nabla r_j(u)r_k(u), \tag{2.11}$$

$$\gamma_{ijk}(u)=(\lambda_k(u)-\lambda_j(u))l_i(u)\nabla r_j(u)r_k(u)-\nabla\lambda_j(u)r_k(u)\delta_{ij}, \tag{2.12}$$

$$\sigma_{ijk}(u)=l_i(u)(\nabla r_k(u)r_j(u)-\nabla r_j(u)r_k(u)). \tag{2.13}$$

Equivalently we also get

$$\begin{aligned} d[v_i(dx-\lambda_i(u)dt)]&=\Big[\sum_{j,k=1}^{n}\tilde\beta_{ijk}(u)v_j w_k+\sum_{j,k=1}^{n}\nu_{ijk}(u)v_j b_k(u)+b_i(u)\Big]dt\wedge dx\\ &\triangleq \widetilde F_i(t,x)dt\wedge dx, \end{aligned} \tag{2.14}$$

$$\begin{aligned} d[w_i(dx-\lambda_i(u)dt)]&=\Big[\sum_{j,k=1}^{n}\tilde\gamma_{ijk}(u)w_j w_k+\sum_{j,k=1}^{n}\sigma_{ijk}(u)w_j b_k(u)+(b_i(u))_x\Big]dt\wedge dx\\ &\triangleq \widetilde G_i(t,x)dt\wedge dx, \end{aligned} \tag{2.15}$$

where

$$\tilde\beta_{ijk}(u)=\beta_{ijk}(u)+\nabla\lambda_i(u)r_k(u)\delta_{ij}, \tag{2.16}$$

$$\tilde\gamma_{ijk}(u)=\gamma_{ijk}(u)+\frac{1}{2}[\nabla\lambda_j(u)r_k(u)\delta_{ij}+\nabla\lambda_k(u)r_j(u)\delta_{ik}]. \tag{2.17}$$

From above, we see that

$$\beta_{iji}(u)\equiv 0,\quad \forall\, j\in\{1,2,\cdots,n\}, \tag{2.18}$$

$$\gamma_{ijj}(u)\equiv 0,\quad j\neq i, \tag{2.19}$$

$$\tilde\beta_{iji}(u)\equiv 0,\quad \forall\, j\neq i, \tag{2.20}$$

$$\tilde\gamma_{ijj}(u)\equiv 0,\quad \forall\, i,j\in\{1,2,\cdots,n\}. \tag{2.21}$$

In the normalized coordinates (if any), making use of (1.12)–(1.13), we see that the following relations hold (see [5]):

$$\beta_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \tag{2.22}$$

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall\,|u_j| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j \in \{p+1,\cdots,n\}, \tag{2.23}$$

$$\nu_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \tag{2.24}$$

$$\nu_{ijj}(u_j e_j) \equiv 0, \quad \forall\,|u_j| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j \in \{p+1,\cdots,n\}, \tag{2.25}$$

$$\gamma_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, i \in \{p+1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \tag{2.26}$$

$$\sigma_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \tag{2.27}$$

$$\sigma_{ijj}(u_j e_j) \equiv 0, \quad \forall\,|u_j| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \quad \forall\, j \in \{p+1,\cdots,n\}, \tag{2.28}$$

$$\tilde{\beta}_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, i \in \{p+1,\cdots,n\}, \quad \forall\, j,k \in \{1,\cdots,p\}, \tag{2.29}$$

$$\tilde{\beta}_{ijj}(u_j e_j) \equiv 0, \quad \forall\,|u_j| \text{ small}, \quad \forall\, j \in \{p+1,\cdots,n\}, \quad j \neq i, \tag{2.30}$$

$$\tilde{\gamma}_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall j,k \in \{1,\cdots,p\}. \tag{2.31}$$

If the system (1.1) is weakly linearly degenerate, we further have (see [5])

$$\tilde{\beta}_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \text{when } i \in \{1,\cdots,p\}, \tag{2.32}$$

$$\tilde{\beta}_{iii}(u_i e_i) \equiv 0, \quad \forall\,|u_i| \text{ small}, \quad \text{when } i \in \{p+1,\cdots,n\}, \tag{2.33}$$

$$\gamma_{ijk}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, j,k \in \{1,\cdots,p\}, \quad \text{when } i \in \{1,\cdots,p\}, \tag{2.34}$$

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall\,|u_i| \text{ small}, \quad \forall\, i \in \{p+1,\cdots,n\}. \tag{2.35}$$

When the inhomogeneous term $B(u)$ satisfies the matching condition, in the normalized coordinates (if any) we have (see [5])

$$b_i(u) = \sum_{\lambda_j(u)\neq\lambda_k(u)} b_{ijk}(u) u_j u_k, \quad \forall\,|u| \text{ small}, \quad \forall\, i \in \{1,2,\cdots,n\}, \tag{2.36}$$

where $b_{ijk}(u)$ is a C^1 function which can be obtained by Taylor formula.

$$(b_i(u))_x = \sum_{k=1}^{n} \tilde{b}_{ik}(u) w_k, \tag{2.37}$$

where $\tilde{b}_{ik}(u) = \sum\limits_{l=1}^{n} \frac{\partial b_i(u)}{\partial u_l} r_{kl}(u)$ and $\tilde{b}_{ik}(u)$ satisfies that

$$\tilde{b}_{ik}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad \forall\,|u_h| \text{ small}, \quad \forall\, k \in \{1,\cdots,p\}, \tag{2.38}$$

$$\tilde{b}_{ik}(u_k e_k) \equiv 0, \quad \forall\, |u_k| \text{ small}, \quad \forall\, k \in \{p+1, \cdots, n\}. \tag{2.39}$$

3 Uniform Estimates

In this section, we shall establish some new uniform estimates which play a key role in the proof of Theorem 1.1.

For the system (1.1), we suppose that there exists the normalized coordinates. Without loss of generality, we assume that $u = (u_1, \cdots, u_n)^T$ are already the normalized coordinates.

For the sake of ease, we denote

$$\Gamma \overset{\triangle}{=} \{(j,k) \mid j,k \in \{1,2,\cdots,n\},\ j \text{ or } k \notin \{1,2,\cdots,p\}, j \neq k\}. \tag{3.1}$$

It is easy to see that

$$\text{if } (j,k) \in \Gamma, \quad \text{then } (i,j) \in \Gamma \text{ or } (i,k) \in \Gamma, \quad \forall\, i = 1,2,\cdots,n. \tag{3.2}$$

By (1.2), there exist positive constants δ and δ_0 so small that

$$|\lambda_i(u) - \lambda_j(v)| \geq \delta_0, \quad \forall\, |u|,\, |v| \leq \delta, \quad (i,j) \in \Gamma, \tag{3.3}$$

$$|\lambda_i(u) - \lambda_i(v)| \leq \frac{\delta_0}{2}, \quad \forall\, |u|,\, |v| \leq \delta, \quad i = 1, \cdots, n. \tag{3.4}$$

On the other hand, noting (3.1.3) in [3] or Lemma 4.1 in [12], we observe that the global classical solution $u = u(t,x)$ of the Cauchy problem (1.1) and (1.17) satisfies

$$|u| \leq K_0 \epsilon, \tag{3.5}$$

where $K_0 > 0$ is a constant independent of ϵ and M. Therefore, taking ϵ small enough, we always have

$$|u| \leq \delta. \tag{3.6}$$

For any fixed $T \geq 0$, we introduce

$$\begin{aligned}
U_\infty(T) &= \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |u(t,x)|, \\
V_\infty(T) &= \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |v(t,x)|, \\
W_\infty(T) &= \sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |w(t,x)|, \\
U_1(T) &= \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |u(t,x)| dx, \\
V_1(T) &= \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |v(t,x)| dx, \\
W_1(T) &= \sup_{0 \leq t \leq T} \int_{-\infty}^{+\infty} |w(t,x)| dx, \\
\widetilde{U}_1(T) &= \max_{(i,j) \in \Gamma} \sup_{\widetilde{C}_j} \int_{\widetilde{C}_j} |u_i(t,x)| dt, \\
\widetilde{V}_1(T) &= \max_{(i,j) \in \Gamma} \sup_{\widetilde{C}_j} \int_{\widetilde{C}_j} |v_i(t,x)| dt,
\end{aligned}$$

$$\widetilde{W}_1(T) = \max_{(i,j)\in\Gamma} \sup_{\widetilde{C}_j} \int_{\widetilde{C}_j} |w_i(t,x)|dt,$$

$$\overline{U}_1(T) = \max_{(i,j)\in\Gamma} \sup_{L_j} \int_{L_j} |u_i(t,x)|dt,$$

$$\overline{V}_1(T) = \max_{(i,j)\in\Gamma} \sup_{L_j} \int_{L_j} |v_i(t,x)|dt,$$

$$\overline{W}_1(T) = \max_{(i,j)\in\Gamma} \sup_{L_j} \int_{L_j} |w_i(t,x)|dt,$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$, $v = (v_1,\cdots,v_n)^T$ and $w = (w_1,\cdots,w_n)^T$ in which v_i and w_i are defined by (2.1) and (2.2) respectively, $\widetilde{C}_j$ stands for any given j-th characteristic on the domain $[0,T]\times\mathbb{R}$, while L_j stands for any given ray with the slope $\lambda_j(0)$ on the region $[0,T]\times\mathbb{R}$.

Combining Lemma 3.1 and (3.1.3) in [3] or using Lemma 4.1 in [12] gives the following lemma.

Lemma 3.1 *Under the assumptions of Theorem* 1.1, *there exists a positive constant* K_1 *independent of* ϵ, M *and* T *such that*

$$U_1(T),\ V_1(T),\ \widetilde{V}_1(T) \le K_1\frac{\epsilon}{M+1}, \tag{3.7}$$

$$W_1(T),\ \widetilde{W}_1(T) \le K_1\epsilon, \tag{3.8}$$

$$U_\infty(T),\ V_\infty(T) \le K_1\epsilon, \tag{3.9}$$

$$W_\infty(T) \le K_1 M. \tag{3.10}$$

On the other hand, we have

Lemma 3.2 *Under the assumptions of Theorem* 1.1, *there exists a positive constant* K_2 *independent of* ϵ, M *and* T *such that*

$$\widetilde{U}_1(T) \le K_2\frac{\epsilon}{M+1}, \tag{3.11}$$

$$\overline{U}_1(T),\ \overline{V}_1(T) \le K_2\frac{\epsilon}{M+1}, \tag{3.12}$$

$$\overline{W}_1(T) \le K_2\epsilon. \tag{3.13}$$

Proof We first prove (3.11).

Noting (1.12)–(1.13), by Hadamard's formula we have

$$\begin{aligned} u_i &= \sum_{k=1}^n v_k r_k(u) e_i = v_i + \sum_{k=1}^p v_k\Big(r_k(u) - r_k\Big(\sum_{h=1}^p u_h e_h\Big)\Big)e_i + \sum_{k=p+1}^n v_k(r_k(u) - r_k(u_k e_k))e_i \\ &= v_i + \sum_{(j,k)\in\Gamma} \Xi_{ijk}(u) u_j v_k, \end{aligned} \tag{3.14}$$

where

$$\Xi_{ijk}(u)=\int_0^1 \frac{\partial r_k(su_1,\cdots,su_{k-1},u_k,su_{k+1},\cdots,su_n)}{\partial u_j}e_i ds,\quad k=1,\cdots,p,\ j\ge p+1,$$
$$\Xi_{ijk}(u)=\int_0^1 \frac{\partial r_k(su_1,\cdots,su_{k-1},u_k,su_{k+1},\cdots,su_n)}{\partial u_j}e_i ds,\quad k=p+1,\cdots,n,\ j\ne k \tag{3.15}$$

are $C^{1,\rho}$ functions of u. Integrating (3.14) along the j-th characteristic $\widetilde{C}_j$: $x=x_j(s,\alpha)$ ($(i,j)\in \Gamma$) and noting (3.6) gives

$$\int_{\widetilde{C}_j}|u_i(t,x)|dt\le \widetilde{V}_1(T)+C_1\{U_\infty(T)\widetilde{V}_1(T)+V_\infty(T)\widetilde{U}_1(T)\}.$$

Here and hereafter C_j $(j=1,2,\cdots)$ stand for some positive constants independent of ϵ, M and T. Noting (3.7) and (3.9), we get (3.11) immediately.

We next prove (3.12)–(3.13).

Similarly to [3, 12], we introduce

$$Q_V(T)=\sum_{(i,j)\in\Gamma}\int_0^T\int_{\mathbb{R}}|v_i(t,x)||v_j(t,x)|dtdx,$$
$$Q_{VW}(T)=\sum_{(i,j)\in\Gamma}\int_0^T\int_{\mathbb{R}}|v_i(t,x)||w_j(t,x)|dtdx,$$
$$Q_W(T)=\sum_{(i,j)\in\Gamma}\int_0^T\int_{\mathbb{R}}|w_i(t,x)||w_j(t,x)|dtdx.$$

Noting [3, Lemma 3.1] or [12, Lemma 4.1], we have

$$Q_V(T)\le C_2\frac{\epsilon^2}{(M+1)^2},\quad Q_{VW}(T)\le C_2\min\Big(\frac{\epsilon^2}{M+1},\epsilon^2\Big),\quad Q_W(T)\le C_2\epsilon^2. \tag{3.16}$$

For any fixed $\alpha\in\mathbb{R}$, let L_j be the ray with the slope $\lambda_j(0)$ passing through $(0,\alpha)$, and let P be the intersection point of L_j with the line $t=T$. Passing through the point P, we draw the i-th characteristic and denote the intersection point of this characteristic with the x-axis by $(0,\beta)$, where $(i,j)\in\Gamma$. For fixing the idea we may suppose that $\alpha<\beta$. Let Ω be the domain bounded by L_j, the x-axis and the i-th characteristic passing through the point P. Using Green formula on the region Ω, we obtain from (2.14) that

$$\int_{L_j}v_i(t,x)(\lambda_j(0)-\lambda_i(u))dt=\int_\alpha^\beta v_i(0,x)dx-\iint_\Omega \widetilde{F}_i(t,x)dtdx. \tag{3.17}$$

Since $(i,j)\in\Gamma$, it follows from (3.3) that

$$|\lambda_j(0)-\lambda_i(u)|\ge\delta_0. \tag{3.18}$$

On the other hand, noting (2.23)–(2.24), (2.28)–(2.29), (2.31)–(2.32), (2.35) and using Hadamard's formula we have

$$\begin{aligned}\widetilde{F}_i(t,x) &= \sum_{k,l=1}^{n} \tilde{\beta}_{ikl}(u)v_k w_l + \sum_{k,l=1}^{n} \nu_{ikl}(u)v_k b_l(u) + b_i(u)\\ &= \sum_{(k,l)\in\Gamma} \tilde{\beta}_{ikl}(u)v_k w_l + \sum_{k,l\in\{1,\cdots,p\}} \Big(\tilde{\beta}_{ikl}(u) - \tilde{\beta}_{ikl}\Big(\sum_{h=1}^{p} u_h e_h\Big)\Big)v_k w_l\\ &\quad + \sum_{k\in\{p+1,\cdots,n\}} (\tilde{\beta}_{ikk}(u) - \tilde{\beta}_{ikk}(u_k e_k))v_k w_k + \sum_{(k,l)\in\Gamma} b_{ikl}(u)u_k u_l\\ &\quad + \sum_{k,l=1}^{n} \nu_{ikl}(u)v_k \sum_{(m,q)\in\Gamma} b_{lmq}(u)u_m u_q\\ &\equiv \sum_{(k,l)\in\Gamma} \tilde{\beta}_{ikl}(u)v_k w_l + \sum_{k,l\in\{1,\cdots,p\}} \sum_{m\in\{p+1,\cdots,n\}} \beta^{(1)}_{iklm}(u)v_k w_l u_m\\ &\quad + \sum_{k\in\{p+1,\cdots,n\}} \sum_{l\neq k} \beta^{(2)}_{ikl}(u)v_k w_k u_l + \sum_{(k,l)\in\Gamma} \beta^{(3)}_{ikl}(u)u_k u_l, \end{aligned} \tag{3.19}$$

where we have made use of Hadamard's formula similar to (3.14) and $\beta^{(m)}_{\cdots}(u)$ $(m = 1, 2, 3)$ are all ρ-Hölder continuous functions with respect to u.

By (3.1.20)–(3.1.21) in [3], i.e.,

$$\sum_{j=p+1}^{n} |u_j| \le C_3 \sum_{j=p+1}^{n} |v_j|, \quad \sum_{j\neq k} |u_j| \le C_3 \sum_{j\neq k} |v_j|, \quad \text{when } k \in \{p+1,\cdots,n\}, \tag{3.20}$$

we have

$$|\widetilde{F}_i(t,x)| \le C_4 \sum_{(k,l)\in\Gamma} [|v_k v_l| + |v_k w_l|]. \tag{3.21}$$

Thus, noting (3.21) and using (1.21), (3.16) and (3.18), we obtain from (3.17) that

$$\begin{aligned}\int_{L_j} |v_i(t,x)|dt &\le \frac{1}{\delta_0}[V_1(0) + C_4(Q_V(T) + Q_{VW}(T))]\\ &\le C_5\Big\{\frac{\epsilon}{M+1} + \frac{\epsilon^2}{(M+1)^2} + \epsilon^2\Big\} \le C_6 \frac{\epsilon}{M+1},\end{aligned}$$

provided that ϵ is small. Therefore, we have

$$\overline{V}_1(T) \le C_6 \frac{\epsilon}{M+1}. \tag{3.22}$$

This is nothing but the desired second inequality in (3.12).

On the other hand, integrating (3.14) along the ray L_j gives

$$\int_{L_j} |u_i(t,x)|dt \le \overline{V}_1(T) + C_7\{U_\infty(T)\overline{V}_1(T) + V_\infty(T)\overline{U}_1(T)\}. \tag{3.23}$$

Noting (3.9) and (3.22), we obtain from (3.23) that

$$\overline{U}_1(T) \le C_8 \frac{\epsilon}{M+1}. \tag{3.24}$$

Combining (3.22) and (3.24) proves (3.12).

We next prove (3.13)

Similarly to (3.17), we obtain from (2.15) that

$$\int_{L_j} w_i(t,x)(\lambda_j(0)-\lambda_i(u))dt = \int_\alpha^\beta w_i(0,x)dx - \iint_\Omega \widetilde{G}_i(t,x)dtdx. \tag{3.25}$$

Noting (2.22), (2.26)–(2.27), (2.30) and (2.35)–(2.38), we have

$$\begin{aligned}
\widetilde{G}_i(t,x) &= \sum_{k,l=1}^n \tilde{\gamma}_{ikl}(u)w_k w_l + \sum_{k,l=1}^n \sigma_{ikl}(u)w_k b_l(u) + (b_i(u))_x \\
&= \sum_{(k,l)\in\Gamma} \tilde{\gamma}_{ikl}(u)w_k w_l + \sum_{k,l\in\{1,\cdots,p\}} \Big[\tilde{\gamma}_{ikl}(u) - \tilde{\gamma}_{ikl}\Big(\sum_{h=1}^p u_h e_h\Big)\Big] w_k w_l \\
&\quad + \sum_{k,l=1}^n \sigma_{ikl}(u)w_k \sum_{(m,q)\in\Gamma} b_{lmq}u_m u_q + \sum_{k\in\{1,\cdots,p\}} \Big[\tilde{b}_{ik}(u) - \tilde{b}_{ik}\Big(\sum_{h=1}^p u_h e_h\Big)\Big] w_k \\
&\quad + \sum_{k\in\{p+1,\cdots,n\}} [\tilde{b}_{ik}(u) - \tilde{b}_{ik}(u_k e_k)]w_k \\
&\equiv \sum_{(k,l)\in\Gamma} \tilde{\gamma}_{ikl}(u)w_k w_l + \sum_{\substack{k,l\in\{1,\cdots,p\}\\ m\in\{p+1,\cdots,n\}}} \gamma^{(1)}_{iklm}(u)w_k w_l u_m + \sum_{(k,l)\in\Gamma} \gamma^{(2)}_{ikl}(u)u_k w_l,
\end{aligned} \tag{3.26}$$

where we have made use of Hadamard's formula similar to (3.14) and $\gamma^{(m)}_{\cdots}(u)$ $(m=1,2)$ are all ρ-Hölder continuous functions with respect to u.

By (3.6) and (3.26), we get

$$|\widetilde{G}_i(t,x)| \le C_9 \sum_{(j,k)\in\Gamma} [|w_j w_k| + |v_j w_k|]. \tag{3.27}$$

Thus, noting (3.7) and (3.27), it follows from (3.25) that

$$\int_{L_j} |w_i(t,x)|dx \le \frac{1}{\delta_0}(W_1(0) + C_9(Q_{VW}(T) + Q_W(T))) \le C_{10}\epsilon.$$

Therefore,

$$\overline{W}_1(T) \le C_{10}\epsilon.$$

This proves (3.13). Thus, the proof of Lemma 3.2 is completed.

Combining Lemmas 3.1 and 3.2 gives

Lemma 3.3 *Under the assumptions of Theorem* 1.1*, there exists a positive constant* K_3 *independent of* ϵ *and* M *such that*

$$U_1(\infty),\, V_1(\infty),\, \overline{U}_1(\infty),\, \overline{V}_1(\infty),\, \widetilde{U}_1(\infty),\, \widetilde{V}_1(\infty) \le K_3 \frac{\epsilon}{M+1}, \tag{3.28}$$

$$W_1(\infty),\, \overline{W}_1(\infty),\, \widetilde{W}_1(\infty) \le K_3\epsilon, \tag{3.29}$$

$$U_\infty(\infty),\, V_\infty(\infty) \le K_3\epsilon, \tag{3.30}$$

$$W_\infty(\infty) \le K_3 M, \tag{3.31}$$

where

$$V_1(\infty) = \sup_{0 \le t \le \infty} \int_{-\infty}^{+\infty} |v(t,x)| dx, \quad etc.$$

Lemma 3.4 *Under the assumptions of Theorem* 1.1*, for any* $t \in \mathbb{R}^+$ *and arbitrary* $\alpha, \beta \in \mathbb{R}$*, it holds that*

$$|u(t, \alpha + \lambda_i(0)t) - u(t, \beta + \lambda_i(0)t)| \le c_1 M |\alpha - \beta|, \tag{3.32}$$

$$|u(t, x_i(t,\alpha)) - u(t, x_i(t,\beta))| \le c_2 M |\alpha - \beta|; \tag{3.33}$$

moreover, for any given ρ*-Hölder continuous function* $g(u)$*,*

$$|g(u(t, \alpha + \lambda_i(0)t)) - g(u(t, \beta + \lambda_i(0)t))| \le c_3 M |\alpha - \beta|^\rho, \tag{3.34}$$

$$|g(u(t, x_i(t,\alpha))) - g(u(t, x_i(t,\beta)))| \le c_4 M |\alpha - \beta|^\rho, \tag{3.35}$$

where $x = x_i(t, \cdot)$ *stands for the* i*-th characteristic passing through the point* $(0, \cdot)$*. Here and hereafter* c_i $(i = 1, 2, \cdots)$ *stand for some positive constants independent of* ϵ*,* M*,* t*,* α *and* β*.*

Proof For fixing the idea we suppose that $\alpha \le \beta$. Since the solution $u = u(t,x)$ is classical, i.e., $u \in C^1([0,+\infty) \times \mathbb{R})$, noting (2.5), (3.6) and (3.31), we can easily get (3.32) and (3.34) by using Taylor's formula.

We next prove (3.33) and (3.35).

Using (2.5) and noting (3.6) and (3.31) again, we have

$$\begin{aligned} |u(t, x_i(t,\alpha)) - u(t, x_i(t,\beta))| &\le \sup_{x \in \mathbb{R}} \{|u_x(t,x)|\} \sup_{\xi \in \mathbb{R}} \Big\{ \Big| \frac{\partial x_i(t,\xi)}{\partial \xi} \Big| \Big\} |\alpha - \beta| \\ &\le c_5 M |\alpha - \beta| \sup_{\xi \in \mathbb{R}} \Big\{ \Big| \frac{\partial x_i(t,\xi)}{\partial \xi} \Big| \Big\}. \end{aligned} \tag{3.36}$$

In what follows, we estimate $\big|\frac{\partial x_i(t,\xi)}{\partial \xi}\big|$.

Noting

$$\frac{\partial x_i(t,\xi)}{\partial t} = \lambda_i(u)(t, x_i(t,\xi)),$$

we have

$$\frac{\partial}{\partial t}\Big(\frac{\partial x_i(t,\xi)}{\partial \xi}\Big) = \nabla \lambda_i(u) u_x \frac{\partial x_i(t,\xi)}{\partial \xi}. \tag{3.37}$$

Noticing $x_i(0,\xi) = \xi$ gives

$$\frac{\partial x_i(0,\xi)}{\partial \xi} = 1. \tag{3.38}$$

Then it follows from (3.37)–(3.38) that

$$\frac{\partial x_i(t,\xi)}{\partial \xi} = \exp\Big\{ \int_0^t (\nabla \lambda_i(u) u_x)(s, x_i(s,\xi)) ds \Big\}. \tag{3.39}$$

Noting (1.9a) and (1.11a), we have

$$\frac{\partial\lambda_i}{\partial u_j}\Big(\sum_{h=1}^{p} u_h e_h\Big) \equiv 0, \quad i,j \in \{1,\cdots,p\}, \quad \frac{\partial\lambda_i}{\partial u_i}(u_i e_i) \equiv 0, \quad i \in \{p+1,\cdots,n\}. \tag{3.40}$$

Therefore, by (1.12)–(1.13), (2.5) and (3.40), we have

$$\nabla\lambda_i(u)u_x = \sum_{(k,l)\in\Gamma} \Theta_{ikl}(u)u_k w_l + \sum_{(j,i)\in\Gamma} \widetilde{\Theta}_{ij}(u)w_j, \tag{3.41}$$

where $\Theta_{ikl}(u)$, $\widetilde{\Theta}_{ij}$ are ρ-Hölder continuous functions with respect to u which can be obtained by Hadamard formula similar to (3.14). Therefore, noting (3.41) and using (3.28)–(3.31), we obtain

$$\int_0^t |(\nabla\lambda_i(u)u_x)(s,x_i(s,\xi))|ds \le c_6\{\widetilde{W}_1(t) + W_\infty(t)\widetilde{U}_1(t) + U_\infty(t)\widetilde{W}_1(t)\} \le c_7\epsilon, \tag{3.42}$$

where the constants c_6, c_7 are independent of not only ϵ, M, t but also α, β and ξ. Combining (3.39) and (3.42) gives

$$\sup_{(t,\xi)\in\mathbb{R}^+\times\mathbb{R}} \Big\{\Big|\frac{\partial x_i(t,\xi)}{\partial\xi}\Big|\Big\} \le e^{c_7\epsilon}. \tag{3.43}$$

Substituting (3.43) into (3.36) yields (3.33) immediately. Finally, noting (3.33) we get (3.35) easily. Thus, the proof of Lemma 3.4 is completed.

For any fixed $T \ge 0$ and for arbitrary α, $\beta \in \mathbb{R}$, we introduce

$$\begin{aligned}
U_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |u_j(s,\alpha+\lambda_i(0)s) - u_j(s,\beta+\lambda_i(0)s)|ds,\\
V_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |v_j(s,\alpha+\lambda_i(0)s) - v_j(s,\beta+\lambda_i(0)s)|ds,\\
W_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |w_j(s,\alpha+\lambda_i(0)s) - w_j(s,\beta+\lambda_i(0)s)|ds,\\
\widetilde{U}_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |u_j(s,x_i(s,\alpha)) - u_j(s,x_i(s,\beta))|ds,\\
\widetilde{V}_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |v_j(s,x_i(s,\alpha)) - v_j(s,x_i(s,\beta))|ds,\\
\widetilde{W}_\alpha^\beta(T) &= \max_{(i,j)\in\Gamma} \int_0^T |w_j(s,x_i(s,\alpha)) - w_j(s,x_i(s,\beta))|ds,
\end{aligned}$$

where $x = x_i(s,\cdot)$, as before, stands for the i-th characteristics passing through the point $(0,\cdot)$.

Lemma 3.5 *Under the assumptions of Theorem* 1.1*, there exists a positive constant* K_4 *independent of* ϵ, M, T, α *and* β *such that*

$$\widetilde{U}_\alpha^\beta(T) \le K_4\epsilon|\alpha-\beta|, \tag{3.44}$$

$$\widetilde{V}_\alpha^\beta(T) \le K_4\epsilon|\alpha-\beta|, \tag{3.45}$$

$$\widetilde{W}_\alpha^\beta(T) \le K_4(M+\epsilon^2)|\alpha-\beta|. \tag{3.46}$$

Proof We first prove (3.45).

Let $\widetilde{C}_i(\alpha)$ and $\widetilde{C}_i(\beta)$ be the i-th characteristics passing through the points $P_1 : (0, \alpha)$ and $P_2 : (0, \beta)$, respectively. For the sake of ease, we assume that $\alpha < \beta$. Denote the intersection point of $\widetilde{C}_i(\alpha)$ (resp. $\widetilde{C}_i(\beta)$) with the straight line $t = T$ by $P_4 : (T, x_i(T, \alpha))$ (resp. $P_3 : (T, x_i(T, \beta))$). Let $\widetilde{\Omega}$ be the region bounded by the curves $\widetilde{C}_i(\alpha)$, $\widetilde{C}_i(\beta)$, $t = 0$ and $t = T$, i.e., the curved-quadrilateral $P_1P_2P_3P_4$. By Green formula, it follows from (2.14) that

$$\begin{aligned}\iint_{\widetilde{\Omega}} \widetilde{F}_j(s,x)dsdx = &\int_\alpha^\beta v_j(0,x)dx + \int_0^T [v_j(\lambda_i(u) - \lambda_j(u))](s, x_i(s,\beta))ds \\ &- \int_\alpha^\beta v_j(T, x_i(T,\gamma))d\gamma - \int_0^T [v_j(\lambda_i(u) - \lambda_j(u))](s, x_i(s,\alpha))ds,\end{aligned}$$

i.e.,

$$\begin{aligned}&\int_0^T [v_j(s, x_i(s,\alpha)) - v_j(s, x_i(s,\beta))][\lambda_i(u)(s, x_i(s,\beta)) - \lambda_j(u)(s, x_i(s,\alpha))]ds \\ = &\int_0^T v_j(s, x_i(s,\beta))[\lambda_j(u)(s, x_i(s,\alpha)) - \lambda_j(u)(s, x_i(s,\beta))]ds \\ &- \int_0^T v_j(s, x_i(s,\alpha))[\lambda_i(u)(s, x_i(s,\alpha)) - \lambda_i(u)(s, x_i(s,\beta))]ds \\ &+ \int_\alpha^\beta [v_j(0,\gamma) - v_j(T, x_i(T,\gamma))]d\gamma - \iint_{\widetilde{\Omega}} \widetilde{F}_j(s,x)dsdx.\end{aligned}$$

When $(i, j) \in \Gamma$, noting (3.3) and using Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned}&\int_0^T |v_j(s, x_i(s,\alpha)) - v_j(s, x_i(s,\beta))|ds \\ &\le \frac{1}{\delta_0}\Big\{[2V_\infty(T) + 2c_4 M\widetilde{V}_1(T)]|\alpha - \beta| + \iint_{\widetilde{\Omega}} |\widetilde{F}_j(s,x)|dsdx\Big\} \\ &\le c_8\Big\{\epsilon|\alpha - \beta| + \iint_{\widetilde{\Omega}} |\widetilde{F}_j(s,x)|dsdx\Big\}.\end{aligned} \tag{3.47}$$

On the other hand, noting (3.20) and using (3.7)–(3.8) and (3.10)–(3.11), we have

$$\begin{aligned}\iint_{\widetilde{\Omega}} |\widetilde{F}_j(s,x)|dsdx &\le c_4 \sum_{(k,l)\in\Gamma} \int_\alpha^\beta d\gamma \int_0^T (|v_k w_l| + |v_k v_l|)(s, x_i(s,\gamma))ds \\ &\le c_9\{V_\infty(T)\widetilde{W}_1(T) + W_\infty(T)\widetilde{V}_1(T) + V_\infty(T)\widetilde{V}_1(T)\}|\alpha - \beta| \\ &\le c_{10}\Big(\frac{M}{M+1} + \epsilon\Big)\epsilon|\alpha - \beta|.\end{aligned} \tag{3.48}$$

Substituting (3.48) into (3.47) gives

$$\int_0^T |v_j(s, x_i(s,\alpha)) - v_j(s, x_i(s,\beta))|ds \le c_{11}\epsilon|\alpha - \beta|.$$

This proves (3.45).

(3.46) can be proved similarly by (2.15) and using Lemmas 3.3 and 3.4 again.

We finally prove (3.44).

Noting (3.14), we have

$$u_j = v_j + \sum_{(k,l)\in\Gamma} \Xi_{jkl}(u)u_k v_l, \tag{3.49}$$

where $\Xi_{jkl}(u)$ is defined by (3.15). Integrating (3.49) from 0 to T along the characteristic: $x = x_i(s,\alpha)$ and $x = x_i(s,\beta)$, respectively, and subtracting the last integral from the first gives

$$\begin{aligned}
&\int_0^T \{[u_j](\alpha) - [u_j](\beta)\}ds \\
&= \int_0^T \{[v_j](\alpha) - [v_j](\beta)\}ds + \sum_{(k,l)\in\Gamma}\int_0^T \{[\Xi_{jkl}(u)u_k v_l](\alpha) - [\Xi_{jkl}(u)u_k v_l](\beta)\}\, ds \\
&= \int_0^T \{[v_j](\alpha) - [v_j](\beta)\}ds + \sum_{(k,l)\in\Gamma}\int_0^T \{[\Xi_{jkl}(u)](\alpha) - [\Xi_{jkl}(u)](\beta)\}\,[u_k](\alpha)[v_l](\alpha)ds \\
&\quad + \sum_{(k,l)\in\Gamma}\int_0^T [\Xi_{jkl}(u)](\beta)\,\{[u_k](\alpha) - [u_k](\beta)\}\,[v_l](\alpha)ds \\
&\quad + \sum_{(k,l)\in\Gamma}\int_0^T [\Xi_{jkl}(u)](\beta)[u_k](\beta)\,\{[v_l](\alpha) - [v_l](\beta)\}\, ds,
\end{aligned} \tag{3.50}$$

where $[\,\cdot\,](\alpha)$ stands for $[\,\cdot\,](s, x_i(s,\alpha))$, while $[\,\cdot\,](\beta)$ stands for $[\,\cdot\,](s, x_i(s,\beta))$. Thus, noting (3.3), (3.6) and using (2.1), (3.33) and (3.35), we obtain from (3.50) that

$$\begin{aligned}
\int_0^T |[u_j](\alpha) - [u_j](\beta)|ds \le \widetilde{V}_\alpha^\beta(T) + c_{12}\{&c_4 M|\alpha-\beta|[U_\infty(T)\widetilde{V}_1(T) + V_\infty(T)\widetilde{U}_1(T)] \\
&+ c_2 M|\alpha-\beta|\widetilde{V}_1(T) + \widetilde{U}_\alpha^\beta(T)V_\infty(T) \\
&+ \widetilde{V}_\alpha^\beta(T)U_\infty(T) + c_2 M|\alpha-\beta|\widetilde{U}_1(T)\}.
\end{aligned}$$

Then, using (3.7), (3.10) and (3.45), we have

$$\int_0^T |[u_j](\alpha) - [u_j](\beta)|ds \le c_{13}\epsilon|\alpha-\beta| + c_{14}K_1\epsilon\widetilde{U}_\alpha^\beta(T). \tag{3.51}$$

It follows from (3.51) that

$$\widetilde{U}_\alpha^\beta(T) \le c_{15}\epsilon|\alpha-\beta| + c_{14}K_1\epsilon\widetilde{U}_\alpha^\beta(T). \tag{3.52}$$

(3.44) comes from (3.52) directly. Thus, the proof of Lemma 3.5 is completed.

Similarly, we can prove the following lemma.

Lemma 3.6 *Under the assumptions of Theorem* 1.1*, there exists a positive constant* K_5 *independent of* ϵ, M, α *and* β *such that*

$$U_\alpha^\beta(\infty),\ \widetilde{U}_\alpha^\beta(\infty) \le K_5\epsilon|\alpha-\beta|, \tag{3.53}$$

$$V_\alpha^\beta(\infty),\ \widetilde{V}_\alpha^\beta(\infty) \le K_5\epsilon|\alpha-\beta|, \tag{3.54}$$

$$W_\alpha^\beta(\infty),\ \widetilde{W}_\alpha^\beta(\infty) \le K_5(M+\epsilon^2)|\alpha-\beta|, \tag{3.55}$$

where

$$U_\alpha^\beta(\infty) = \max_{(i,j)\in\Gamma} \int_0^\infty |u_j(s, \alpha + \lambda_i(0)s) - u_j(s, \beta + \lambda_i(0)s)| ds, \quad etc. \tag{3.56}$$

We finally estimate the difference of w_i on two differential i-th characteristic at the same time.

For arbitrary $\alpha, \beta \in \mathbb{R}$, we introduce

$$W^*_{\alpha,\beta}(\infty) = \sum_{i=1}^{p} \sup_{t\in[0,\infty)} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|,$$
$$W^i_{\alpha,\beta}(\infty) = \sup_{t\in[0,\infty)} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))|, \quad \text{if } \ i = p+1, \cdots, n,$$

where $x = x_i(t, \cdot)$ stands for the i-th characteristic passing through the point $(0, \cdot)$.

Lemma 3.7 *For any given $i \in \{1, \cdots, n\}$ and for any fixed $\alpha \in \mathbb{R}$, the limit*

$$\lim_{t\to+\infty} w_i(t, x_i(t,\alpha))$$

exists, denoted it by $\Psi_i(\alpha)$, that is,

$$\lim_{t\to+\infty} w_i(t, x_i(t,\alpha)) = \Psi_i(\alpha), \quad \forall\, \alpha \in \mathbb{R}, \tag{3.57}$$

where $x = x_i(t,\alpha)$ stands for the i-th characteristic passing through the point $(0,\alpha)$. Moreover, $\Psi_i(\alpha)$ is a continuous function of $\alpha \in \mathbb{R}$ and satisfies that there exists a positive constant K_6 independent of ϵ, M and α such that

$$|\Psi_i(\alpha)| \le (1 + K_6\epsilon)M, \quad \forall\, \alpha \in \mathbb{R}. \tag{3.58}$$

Furthermore, there exists a positive constant K_7 independent of ϵ, M, α and β such that

$$\begin{aligned} W^*_{\alpha,\beta}(\infty) \le{}& (1 + K_7\epsilon) \sum_{i=1}^{p} |w_i(0,\alpha) - w_i(0,\beta)| \\ &+ K_7(M^2 + M\epsilon + \epsilon^3)|\alpha - \beta| + K_7 \frac{M^2}{M+1}(M+\epsilon)|\alpha-\beta|^\rho, \\ W^i_{\alpha,\beta}(\infty) \le{}& (1 + K_7\epsilon)|w_i(0,\alpha) - w_i(0,\beta)| + K_7(M^2 + M\epsilon + \epsilon^3)|\alpha - \beta| \\ &+ K_7 \frac{M\epsilon}{M+1}(M+\epsilon)|\alpha-\beta|^\rho, \quad \textit{if} \ \ i = p+1, \cdots, n. \end{aligned} \tag{3.59}$$

In particular, if (1.24) *is satisfied, then there exists a positive constant K_8 independent of ϵ, M, ς, α and β such that, for all $\alpha, \beta \in \mathbb{R}$,*

$$\begin{aligned} |\Psi_i(\alpha) - \Psi_i(\beta)| \le{}& K_8\Big[\varsigma + \frac{M}{M+1}(M+\epsilon)\max(M,\epsilon)\Big]|\alpha-\beta|^\rho \\ &+ K_8(M^2 + M\epsilon + \epsilon^3)|\alpha - \beta|. \end{aligned} \tag{3.60}$$

Proof It follows from (2.9) that

$$w_i(t,x_i(t,\alpha))=w_i(0,\alpha)+\int_0^t G_i(s,x_i(s,\alpha))ds. \tag{3.61}$$

By (2.19), (2.27)–(2.28) and (2.34)–(2.39), we can rewrite $G_i(t,x)$ similar to (3.26) as follows

$$G_i(t,x)\equiv\sum_{(k,l)\in\Gamma}\gamma_{ikl}(u)w_kw_l+\sum_{\substack{k,l\in\{1,\cdots,p\}\\ m\in\{p+1,\cdots,n\}}}\Gamma^{(1)}_{iklm}(u)w_kw_lu_m+\sum_{(k,l)\in\Gamma}\Gamma^{(2)}_{ikl}(u)u_kw_l, \tag{3.62}$$

where $\Gamma^{(1)}_{jklm}(u)$ and $\Gamma^{(2)}_{jkl}(u)$ are similar to that in (3.26). Substituting (3.62) into (3.61), we get

$$\begin{aligned}w_i(t,x_i(t,\alpha))=w_i(0,\alpha)+\int_0^t\Big[&\sum_{(k,l)\in\Gamma}\gamma_{ikl}(u)w_kw_l+\sum_{\substack{k,l\in\{1,\cdots,p\}\\ m\in\{p+1,\cdots,n\}}}\Gamma^{(1)}_{iklm}(u)w_kw_lu_m\\ &+\sum_{(k,l)\in\Gamma}\Gamma^{(2)}_{ikl}(u)u_kw_l\Big]ds.\end{aligned} \tag{3.63}$$

Then, Lemma 3.3 implies that the integrals in the right-hand side of (3.63) converge absolutely when t tends to $+\infty$. Then, the right-hand side of (3.63) converges when t tends to $+\infty$. We denote the limit by $\Psi_i(\alpha)$. That is,

$$\lim_{t\to\infty}w_i(t,x_i(t,\alpha))=\Psi_i(\alpha).$$

It follows from (3.63) that

$$|w_i(t,x_i(t,\alpha))|\le|w_i(0,\alpha)|+(c_{16}+c_{17}K_2)K_1^2M\epsilon\le(1+K_6\epsilon)M. \tag{3.64}$$

(3.58) follows from (3.64) directly.

Furthermore,

$$\begin{aligned}&w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))\\ &=w_i(0,\alpha)-w_i(0,\beta)+\int_0^t[G_i(s,x_i(s,\alpha))-G_i(s,x_i(s,\beta))]ds.\end{aligned} \tag{3.65}$$

Noting Lemmas 3.3, 3.4, 3.6, and making use of the method of (3.50), we get

$$\begin{aligned}&\int_0^t\Big|\sum_{(k,l)\in\Gamma}\gamma_{ikl}(u)w_kw_l(s,x_i(s,\alpha))-\sum_{(k,l)\in\Gamma}\gamma_{ikl}(u)w_kw_l(s,x_i(s,\beta))\Big|ds\\ &\le c_{18}(M^2+M\epsilon^2)|\alpha-\beta|+c_{19}\epsilon\sum_{(i,k)\notin\Gamma}\sup_{t\in[0,+\infty)}|w_k(t,x_i(t,\alpha))-w_k(t,x_i(t,\beta))|,\end{aligned} \tag{3.66}$$

where $[\,\cdot\,](\alpha)$ stands for $[\,\cdot\,](s,x_i(s,\alpha))$, while $[\,\cdot\,](\beta)$ stands for $[\,\cdot\,](s,x_i(s,\beta))$.

Similarly, we get

$$\begin{aligned}&\int_0^t\Big|\sum_{\substack{k,l\in\{1,\cdots,p\}\\ m\in\{p+1,\cdots,n\}}}[\Gamma^{(1)}_{iklm}(u)w_kw_lu_m(s,x_i(s,\alpha))-\Gamma^{(1)}_{iklm}(u)w_kw_lu_m(s,x_i(s,\beta))]\Big|ds\\ &\le c_{20}\frac{M^3}{M+1}\epsilon|\alpha-\beta|^\rho+c_{20}M^2\epsilon|\alpha-\beta|\end{aligned}$$

$$+ c_{21}\frac{M}{M+1}\epsilon\sum_{k=1}^{p}\sup_{t\in[0,+\infty)}|w_k(t,x_i(t,\alpha))-w_k(t,x_i(t,\beta))|, \quad \text{if } \ i\le p\,,$$

$$\int_0^t\Big|\sum_{\substack{k,l\in\{1,\cdots,p\}\\ m\in\{p+1,\cdots,n\}}}[\Gamma^{(1)}_{iklm}(u)w_kw_lu_m(s,x_i(s,\alpha))-\Gamma^{(1)}_{iklm}(u)w_kw_lu_m(s,x_i(s,\beta))]\Big|ds$$

$$\le c_{22}M^2\epsilon^2|\alpha-\beta|^{\rho}+c_{22}(M^2\epsilon+M\epsilon^3)|\alpha-\beta|, \quad \text{if } \ i\ge p+1, \tag{3.67}$$

$$\int_0^t\Big|\sum_{(k,l)\in\Gamma}\Gamma^{(2)}_{ikl}(u)u_kw_l(s,x_i(s,\alpha))-\sum_{(k,l)\in\Gamma}\Gamma^{(2)}_{ikl}(u)u_kw_l(s,x_i(s,\beta))\Big|ds$$

$$\le c_{23}\Big[\Big(M\epsilon^2+\frac{M^2}{M+1}\epsilon\Big)|\alpha-\beta|^{\rho}+(M\epsilon+\epsilon^3)|\alpha-\beta|\Big]$$

$$+ c_{24}\frac{\epsilon}{M+1}\sum_{(i,k)\notin\Gamma}\sup_{t\in[0,+\infty)}|w_k(t,x_i(t,\alpha))-w_k(t,x_i(t,\beta))|. \tag{3.68}$$

We divide it into two cases to discuss.

Case I $i\in\{1,\cdots,p\}$**.**

When $i\in\{1,2,\cdots,p\}$ and $(i,k)\notin\Gamma$, we must have $k\in\{1,2,\cdots,p\}$. Substituting (3.66)–(3.68) into (3.65), we get

$$|w_i(t,x_i(t,\alpha))-w_i(t,x_i(t,\beta))|$$
$$\le |w_i(0,\alpha)-w_i(0,\beta)|+c_{25}(M^2+M\epsilon+\epsilon^3)|\alpha-\beta|+c_{26}\frac{M^2}{M+1}(M+\epsilon)|\alpha-\beta|^{\rho}$$
$$+ c_{27}\epsilon\sum_{k=1}^{p}\sup_{t\in[0,+\infty)}|w_k(t,x_i(t,\alpha))-w_k(t,x_i(t,\beta))|.$$

By the definition of i-th characteristic $x_i(t,\alpha)$, we have

$$x_1(t,\alpha)=x_2(t,\alpha)=\cdots=x_p(t,\alpha)\overset{\triangle}{=}x(t,\alpha), \quad \forall\, t\in[0,+\infty), \quad \forall\,\alpha\in\mathbb{R}.$$

Then we get

$$|w_i(t,x(t,\alpha))-w_i(t,x(t,\beta))|$$
$$\le |w_i(0,\alpha)-w_i(0,\beta)|+c_{25}(M^2+M\epsilon+\epsilon^3)|\alpha-\beta|+c_{26}\frac{M^2}{M+1}(M+\epsilon)|\alpha-\beta|^{\rho}$$
$$+ c_{27}\epsilon\sum_{k=1}^{p}\sup_{t\in[0,+\infty)}|w_k(t,x(t,\alpha))-w_k(s,x(t,\beta))|. \tag{3.69}$$

Summing (3.69) with respect to $i=1,\cdots,p$, we get

$$\sum_{k=1}^{p}\sup_{t\in[0,+\infty)}|w_k(t,x(t,\alpha))-w_k(s,x(t,\beta))|$$
$$\le (1+K_7\epsilon)\sum_{k=1}^{p}|w_k(0,\alpha)-w_k(0,\beta)|+K_7(M^2+M\epsilon+\epsilon^3)|\alpha-\beta|$$
$$+ K_7\frac{M^2}{M+1}(M+\epsilon)|\alpha-\beta|^{\rho}. \tag{3.70}$$

Case II $i \in \{p+1, \cdots, n\}$**.**

When $i \in \{p+1, \cdots, n\}$ and $(i,k) \notin \Gamma$, we must have $k = i$. Substituting (3.66)–(3.68) into (3.65), we get

$$\begin{aligned}
&|w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))| \\
&\le |w_i(0,\alpha) - w_i(0,\beta)| + c_{28}(M^2 + M\epsilon + \epsilon^3)|\alpha-\beta| + c_{29}\frac{M}{M+1}(M+\epsilon)\epsilon|\alpha-\beta|^\rho \\
&\quad + c_{30}\epsilon \sup_{t\in[0,+\infty)} |w_i(t, x_i(t,\alpha)) - w_k(t, x_k(t,\beta))|. \qquad (3.71)
\end{aligned}$$

It follows from (3.71) that

$$\begin{aligned}
&\sup_{t\in[0,+\infty)} |w_i(t, x_i(t,\alpha)) - w_i(t, x_i(t,\beta))| \\
&\le (1+K_7\epsilon)|w_i(0,\alpha) - w_i(0,\beta)| + K_7(M^2 + M\epsilon + \epsilon^3)|\alpha-\beta| \\
&\quad + K_7\frac{M}{M+1}(M+\epsilon)\epsilon|\alpha-\beta|^\rho. \qquad (3.72)
\end{aligned}$$

Then, (3.59) follows from (3.70) and (3.72) directly. Because $w_i(0,x)$ is continuous, it follows from (3.59) that $\Psi_i(\alpha) \in C^0(\mathbb{R})$.

If (1.24) holds, we see that $w_i(0,x)$ is globally ρ-Hölder continuous. (3.60) follows from (3.59) easily. Thus, the proof of Lemma 3.7 is completed.

4 Asymptotic Behavior of Global Classical Solutions — Proof of Theorem 1.1

This section is devoted to the study of asymptotic behavior of the global classical solution of Cauchy problem (1.1) and (1.17) and gives the proof of Theorem 1.1.

Let

$$\frac{D}{D_i t} = \frac{\partial}{\partial t} + \lambda_i(0)\frac{\partial}{\partial x}. \qquad (4.1)$$

Noting (1.1) and using (2.4)–(2.5), (2.36), we have

$$\begin{aligned}
\frac{Du}{D_i t} &= \frac{\partial u}{\partial t} + \lambda_i(0)\frac{\partial u}{\partial x} = -A(u)\frac{\partial u}{\partial x} + B(u) + \lambda_i(0)\frac{\partial u}{\partial x} \\
&= \sum_{j=1}^n (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) + B(u). \qquad (4.2)
\end{aligned}$$

Therefore,

$$\frac{Du_i}{D_i t} = \frac{Du}{D_i t} e_i = \sum_{j=1}^n (\lambda_i(0) - \lambda_j(u)) w_j r_j(u) e_i + B_i(u).$$

Noting (1.2), (1.9a), (1.11a) and (1.15)–(1.16), we may rewrite $\frac{Du_i}{D_i t}$ as follows

$$\frac{Du_i}{D_i t} = \sum_{(j,k)\in\Gamma} B_{ijk}(u) u_j w_k + \sum_{(i,j)\in\Gamma} B_{ij}(u) u_j, \qquad (4.3)$$

where $B_{ijk}(u)$ and $B_{ij}(u)$ are all $C^{1,\rho}$ continuous function with respect to u which can be obtained by Hadmard's formula similar to (3.14).

For any fixed $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, define

$$\alpha = x - \lambda_i(0)t. \tag{4.4}$$

It follows from (4.3) that

$$\begin{aligned} u_i(t,x) &= u_i(t,\alpha+\lambda_i(0)t) \\ &= u_i(0,\alpha) + \int_0^t \Big[\sum_{(j,k)\in\Gamma} B_{ijk}(u)u_j w_k + \sum_{(i,j)\in\Gamma} B_{ij}(u)u_j\Big](s,\alpha+\lambda_i(0)s)ds. \end{aligned} \tag{4.5}$$

Then we can get the following lemma.

Lemma 4.1 *For every $i \in \{1,\cdots,n\}$ and any given $\alpha \in \mathbb{R}$, the limit*

$$\lim_{t\to+\infty} u_i(t,\alpha+\lambda_i(0)t) = \Phi_i(\alpha) \tag{4.6}$$

exists; moreover, there exists a positive constant K_9 independent of ϵ, M and α such that

$$|\Phi_i(\alpha)| \le K_9\epsilon. \tag{4.7}$$

In what follows, we shall investigate the regularity of the limit function $\Phi_i(\alpha)$.

First, we prove that $\Phi_i(\alpha)$ is a globally Lipschitz continuous function of α.

For any fixed $(t,\alpha+\lambda_i(0)t)$, there exists a unique $\theta_i(t,\alpha) \in \mathbb{R}$ such that

$$\theta_i(t,\alpha) + \int_0^t \lambda_i(u(s,x_i(s,\theta_i(t,\alpha))))ds = \alpha + \lambda_i(0)t, \tag{4.8}$$

namely,

$$\theta_i(t,\alpha) = \alpha + \int_0^t [\lambda_i(0) - \lambda_i(u(s,x_i(s,\theta_i(t,\alpha))))]ds, \tag{4.9}$$

where $x = x_i(s,\theta_i(t,\alpha))$ stands for the i-th characteristic passing through the point $(0,\theta_i(t,\alpha))$, which is defined by

$$\frac{dx_i(s,\theta_i(t,\alpha))}{ds} = \lambda_i(u(s,x_i(s,\theta_i(t,\alpha)))), \quad x_i(0,\theta_i(t,\alpha)) = \theta_i(t,\alpha). \tag{4.10}$$

Lemma 4.2 *Under the assumptions of Theorem* 1.1, *for any given $\alpha \in \mathbb{R}$ there exists a unique $\vartheta_i(\alpha)$ such that $\theta_i(t,\alpha)$ converges to $\vartheta_i(\alpha)$ when t tends to ∞; moreover, $\vartheta_i(\alpha)$ satisfies*

$$|\vartheta_i(\alpha) - \alpha| \le K_{10}\frac{\epsilon}{M+1} \tag{4.11}$$

and is a globally Lip-continuous function of α, more precisely, the following estimate holds

$$|\vartheta_i(\alpha) - \vartheta_i(\beta)| \le (1+K_{11}\epsilon)|\alpha-\beta|, \quad \forall\, \alpha,\beta \in \mathbb{R}, \tag{4.12}$$

where K_{10} is a positive constant independent of ϵ, M and α, while K_{11} is another positive constant independent of ϵ, M, α and β.

Proof By (1.9a) and (1.11a), we have

$$\lambda_i(0)-\lambda_i(u)=\lambda_i\Big(\sum_{h=1}^{p}u_he_h\Big)-\lambda_i(u)=\sum_{j=p+1}^{n}\Lambda_{ij}(u)u_j,\quad \text{if}\ \ i=1,2,\cdots,p,$$
$$\lambda_i(0)-\lambda_i(u)=\lambda_i(u_ie_i)-\lambda_i(u)=\sum_{j\neq i}\Lambda_{ij}(u)u_j,\quad \text{if}\ \ i=p+1,\cdots,n, \tag{4.13}$$

where $\Lambda_{ij}(u)$ is a $C^{1,\rho}$ smooth function which can be obtained by Hadamard's formula. Then, it follows from (4.9) that

$$\begin{aligned}\theta_i(t,\alpha)&=\alpha+\int_0^t(\lambda_i(0)-\lambda_i(u))(s,x_i(s,\theta_i(t,\alpha)))ds\\&=\alpha+\sum_{(i,j)\in\Gamma}\int_0^t(\Lambda_{ij}(u)u_j)(s,x_i(s,\theta_i(t,\alpha)))ds.\end{aligned}\tag{4.14}$$

By Lemma 3.3, we denote the limit by $\vartheta_i(\alpha)$ as t tends to $+\infty$. That is,

$$\vartheta_i(\alpha)=\lim_{t\to\infty}\theta_i(t,\alpha). \tag{4.15}$$

The proof of Lemma 4.2 can be obtained immediately.

Lemma 4.3 *For every $i\in\{1,\cdots,n\}$, there exists a positive constant K_{12} independent of ϵ, M, α and β such that*

$$|\Phi_i(\alpha)-\Phi_i(\beta)|\le K_{12}M|\alpha-\beta|,\quad \forall\,\alpha,\beta\in\mathbb{R}. \tag{4.16}$$

Proof By (4.8) and (4.10), for any $t\in\mathbb{R}^+$ and any $\alpha\in\mathbb{R}$ it holds that

$$u_i(t,\alpha+\lambda_i(0)t)=u_i(t,x_i(t,\theta_i(t,\alpha))), \tag{4.17}$$

where, as before, $x=x_i(s,\theta_i(t,\alpha))$ stands for the i-th characteristic passing through the point $(0,\theta_i(t,\alpha))$. Noting Lemma 4.2 and using (4.6), we have

$$\Phi_i(\alpha)-\Phi_i(\beta)=\lim_{t\to\infty}\{u_i(t,x_i(t,\vartheta_i(\alpha)))-u_i(t,x_i(t,\vartheta_i(\beta)))\}. \tag{4.18}$$

Then, using Taylor's formula and noting (2.5), (3.4), (3.46), (3.59) and (4.12), we obtain

$$|\Phi_i(\alpha)-\Phi_i(\beta)|\le\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}}\Big|\frac{\partial u_i}{\partial x}(t,x)\Big|\sup_{(t,\xi)\in\mathbb{R}^+\times\mathbb{R}}\Big|\frac{\partial x_i}{\partial\xi}(t,\xi)\Big||\vartheta_i(\alpha)-\vartheta_i(\beta)|\le c_{31}M|\alpha-\beta|, \tag{4.19}$$

(4.19) is nothing but the desired estimate (4.16). Thus, the proof of Lemma 4.3 is completed.

Lemma 4.4 *For every $i\in\{1,\cdots,n\}$, the limit $\lim\limits_{t\to+\infty}w_i(t,\alpha+\lambda_i(0)t)$ exists, and*

$$\frac{d\Phi_i(\alpha)}{d\alpha}=\lim_{t\to+\infty}w_i(t,\alpha+\lambda_i(0)t)=\Psi_i(\vartheta_i(\alpha))\in C^0(\mathbb{R}). \tag{4.20}$$

Moreover, if (1.24) is satisfied, then the following estimate holds, for all $\alpha,\beta\in\mathbb{R}$,

$$\begin{aligned}\Big|\frac{d\Phi_i}{d\alpha}(\alpha)-\frac{d\Phi_i}{d\alpha}(\beta)\Big|&\le K_{13}\Big[\varsigma+\frac{M}{M+1}(M+\epsilon)\max(M,\epsilon)\Big]|\alpha-\beta|^\rho\\&\quad+K_{13}(M^2+M\epsilon+\epsilon^3)|\alpha-\beta|,\end{aligned}\tag{4.21}$$

where K_{14} is a positive constant independent of ϵ, M, ς, α and β.

Proof It follows from (4.8) that

$$w_i(t, \alpha + \lambda_i(0)t) = w_i(t, x_i(t, \theta_i(t, \alpha))). \tag{4.22}$$

Then noting Lemma 4.2, we have

$$\lim_{t\to+\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t\to+\infty} w_i(t, x_i(t, \theta_i(t, \alpha))) = \lim_{t\to+\infty} w_i(t, x_i(t, \vartheta_i(\alpha))), \tag{4.23}$$

and then by Lemma 3.7,

$$\lim_{t\to+\infty} w_i(t, \alpha + \lambda_i(0)t) = \lim_{t\to+\infty} w_i(t, x_i(t, \vartheta_i(\alpha))) = \Psi_i(\vartheta_i(\alpha)). \tag{4.24}$$

Since, by Lemma 3.7 and (4.12), $\Psi_i(\,\cdot\,)$ and $\vartheta_i(*)$ are continuous with respect to $\cdot$ and $*$ respectively, $\Psi_i(\vartheta_i(\alpha))$ is a continuous function of $\alpha \in \mathbb{R}$.

On the other hand, by the definition (4.6),

$$\begin{aligned}
\frac{d\Phi_i(\alpha)}{d\alpha} &= \lim_{\Delta\alpha\to 0} \frac{\Phi_i(\alpha + \Delta\alpha) - \Phi_i(\alpha)}{\Delta\alpha} = \lim_{t\to+\infty} u_x(t, \alpha + \lambda_i(0)t)e_i \\
&= \lim_{t\to+\infty} \sum_{j=1}^{n} w_j(t, \alpha + \lambda_i(0)t) r_j(u(t, \alpha + \lambda_i(0)t))e_i \\
&= \lim_{t\to+\infty} \Big\{ \sum_{j=1}^{p} w_j \Big(r_j(u) - r_j\Big(\sum_{h=1}^{p} u_h e_h \Big)\Big) e_i \Big\}(t, \alpha + \lambda_i(0)t) \\
&\quad + \Big\{ \sum_{j=p+1}^{n} w_j(r_j(u) - r_j(u_j e_j))e_i + w_i \Big\}(t, \alpha + \lambda_i(0)t) \\
&= \lim_{t\to+\infty} \Big\{ \sum_{(j,k)\in\Gamma} O_{ijk}(u) w_j u_k + w_i \Big\}(t, \alpha + \lambda_i(0)t),
\end{aligned}$$

where $O_{ijk}(u)$ can be obtained by Hadamard's formula. Then, noting Lemma 3.3, we obtain (4.20) immediately.

Moreover, if (1.24) is satisfied, then using (3.85) and (4.12), we obtain (4.21) immediately. Thus, the proof of Lemma 4.4 is completed.

Proof of Theorem 1.1 Taking

$$\varphi_i(x - \lambda_i(0)t) = \Phi_i(x - \lambda_i(0)t), \quad i = 1, \cdots, n \tag{4.25}$$

and noting Lemmas 4.1, 4.3 and Lemma 4.4, we get the conclusion of Theorem 1.1 immediately. Thus, the proof of Theorem 1.1 is completed.

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