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Curvature Estimates for Irreducible Symmetric Spaces

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Abstract By making use of the classification of real simple Lie algebra, we get the maximum of the squared length of restricted roots case by case, and thus get the upper bounds of sectional curvature for irreducible Riemannian symmetric spaces of compact type. As an application, this paper verifies Sampson's conjecture in most cases for irreducible Riemannian symmetric spaces of noncompact type.

Keywords Symmetric space, Semi-simple Lie algebra, Harmonic map
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1 Introduction and Statement of Results

There are many Liouville type theorems for harmonic maps. It was conjectured by J. H. Sampson [9] that any harmonic map with finite energy from a complete simply connected Riemannian manifold with non positive sectional curvature whose dimension exceeds 2 must be constant. This is valid for space forms, but unsolved in general case. For Cartan-Hadamard manifolds, Y. L. Xin [11] proved a general vanishing theorem as follows.

Theorem 1.1 *Let M be an m-dimensional Cartan-Hadamard manifold with the sectional curvature $-a^2 \le K \le 0$ and the Ricci curvature bounded from above by $-b^2$. Let f be a harmonic map from M into any Riemannian manifold with the moderate divergent energy. If $b \ge 2a$, then f has to be constant.*

Moreover, by using L^2-cohomology techniques, from Theorem 2.2 of [5], the above condition can be relaxed to $b \ge \sqrt{2}a$.

For the irreducible symmetric space, the Ricci curvature is constant. In this article we get the bounds of sectional curvature by using the method of Lie algebra. As an application, we verify Sampson's conjecture in most cases for Riemannian symmetric spaces of noncompact type by using the above Theorem 1.1.

Theorem 1.2 *Let M be one of the irreducible symmetric spaces of noncompact type in the following cases* (*see* [4, *Table* V, *p.*518]):

$$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n),\quad n\ge 4;\quad \mathrm{SU}^*(2n)/\mathrm{Sp}(n);\quad \mathrm{SU}(p,q)/S(U_p\times U_q),\quad p+q\ge 4;$$
$$\mathrm{SO}_o(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q),\quad \textit{for } r=1,\ p+q\ge 4,$$
$$\textit{for } r>1,\ p+q\ge 6,\ \textit{where } r=\min(p,q);$$
$$\mathrm{SO}^*(2n)/\mathrm{U}(n),\quad n\ge 3;\quad \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n),\quad n\ge 3;\quad \mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q);$$
$$\mathrm{EI},\ \mathrm{EII},\ \mathrm{EIII},\ \mathrm{EIV},\ \mathrm{EV},\ \mathrm{EVI},\ \mathrm{EVII},\ \mathrm{EVIII},\ \mathrm{EIX},\ \mathrm{FI},\ \mathrm{FII}\ \textit{and}\ \mathrm{G}.$$

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Let f be a harmonic map from M into any Riemannian manifold with the moderate divergent energy. Then f has to be constant.

In §2, we collect basic facts about real simple Lie algebras and symmetric spaces. In §3, we give the upper bounds of sectional curvature for Riemannian symmetric spaces of compact type by choosing the Cartan subalgebra with maximal vector part. In §4, we give the method of obtaining Cartan subalgebra with maximal vector part from a Cartan subalgebra with maximal torus part. In §5, we carry out the detailed calculation on estimates of sectional curvature by using the root space decompositions of complex simple Lie algebras. The final results will be shown in Table 5.1. This is our main result. It is interesting in its own right, although some of the classical cases were given by Y. C. Wong [10].

From Table 5.1 and Theorem 1.1, Theorem 1.2 follows immediately.

2 Some Basic Facts for Real Semisimple Lie Algebra and Irreducible Symmetric Space

Let $M = U/K$ be a compact irreducible Riemannian symmetric space of type I, its non-compact dual is G/K, which is of type III. There are decompositions for corresponding Lie algebras

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}. \tag{2.1}$$

Let $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}} = \mathfrak{g}_0^{\mathbb{C}}$ be its complexified simple Lie algebra, the rank of $\mathfrak{g}$ (resp. M) is the dimension of the maximal abelian complex (resp. real) subspace. The Killing form of $\mathfrak{g}$ is

$$(X, Y) = B(X, Y) = \mathrm{tr}\,\mathrm{ad}X\mathrm{ad}Y. \tag{2.2}$$

Let

$$\langle X, Y\rangle = \epsilon(X, Y), \tag{2.3}$$

where $\epsilon = 1$ for noncompact type; -1 for compact type. The restriction on $i\mathfrak{p}$ (resp. $\mathfrak{p}$) of $\langle\,,\,\rangle$ gives the Riemannian inner product at $o = \pi(e)$ for the corresponding symmetric space, where e is the identity of U. The curvature tensor is $R(X, Y) = -\mathrm{ad}[X, Y]$, from the invariance of Killing form we get

$$R(X, Y, Z, W) = \epsilon(-[[X, Y], W], Z) = \epsilon([X, Y], [Z, W]). \tag{2.4}$$

Since the Killing form is positive definite on $\mathfrak{p}$ and negative definite on $i\mathfrak{p}$, the curvature is non-negative for symmetric space of compact type, non positive for symmetric space of noncompact type. Moreover, by choosing an adapted base we have ([8, p.180])

$$\mathrm{Ric}\,(X, Y) = -\frac{1}{2}(X, Y). \tag{2.5}$$

So the Ricci curvature is $\frac{1}{2}$ for compact type and $-\frac{1}{2}$ for noncompact type. Owing to the dual relation, we only need to calculate the upper bounds of sectional curvature for the irreducible Riemannian symmetric spaces of compact type, and the lower bounds are 0 except for the case of rank 1.

The irreducible Riemannian symmetric spaces of type II and IV are dual to each other. Let M be of type II, it can be identified as a connected compact simple Lie group U with the bi-invariant metric, see [4, p.516]. We give the Riemannian inner product at e

$$\langle X, Y\rangle = -(X, Y). \tag{2.6}$$

We have the Cartan decomposition of $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ and the decomposition of root spaces. For $X, Y \in \mathfrak{u}$, we have (see [8, p.185])

$$\nabla_X Y = \frac{1}{2}[X, Y]. \tag{2.7}$$

So

$$R(X, Y) = -\frac{1}{4}\mathrm{ad}[X, Y]. \tag{2.8}$$

In next section, we get that the Ricci curvature is $\frac{1}{4}$.

Recall that the pair $(\mathfrak{l}, \theta)$ is called effective orthogonal symmetric Lie algebra if (i) $\mathfrak{l}$ is a real Lie algebra; (ii) θ is an involutive automorphism of $\mathfrak{l}$, i.e., $\theta \neq \mathrm{id}$ but $\theta^2 = \mathrm{id}$; (iii) the set of fixed points of $\theta, \mathfrak{k}$, is a compactly imbedded subalgebra of $\mathfrak{l}$; (iv) $\mathfrak{k} \cap \mathfrak{z}(\mathfrak{l}) = 0$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. The pair (G, K) is said to be associated with $(\mathfrak{l}, \theta)$ if G is a connected Lie group with Lie algebra $\mathfrak{l}$, and K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$. If K is closed in G, G/K endowed with the G-invariant metric becomes a Riemannian symmetric space. The orthogonal symmetric Lie algebra $(\mathfrak{l}, \theta)$ is called irreducible if, let $\mathfrak{l} = \mathfrak{k} + \mathfrak{p}$ be the ± 1 characteristic subspace decomposition of θ, $\mathfrak{l}$ be real semisimple Lie algebra, $\mathfrak{k}$ contains no nonzero idea of $\mathfrak{l}$, and the adjoint algebra $\mathrm{ad}_{\mathfrak{l}}\mathfrak{k}$ acts irreducibly on $\mathfrak{p}$. If $\mathfrak{l}$ is a compact simple real Lie algebra, $(\mathfrak{l}, \theta)$ is of type I, the associated Riemannian symmetric space is of type I; if $\mathfrak{l}$ is a noncompact real simple Lie algebra, its complexification $\mathfrak{l}^{\mathbb{C}}$ is a complex simple Lie algebra, then $(\mathfrak{l}, \theta)$ is of type III, the associated Riemannian symmetric space is of type III. Furthermore, every irreducible symmetric Lie algebra of type I corresponds 1-1 to one of type III, they are dual to each other.

A real simple Lie algebra is of first kind if its complexification is a complex simple Lie algebra; otherwise of second kind. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. There exists a compact real form $\mathfrak{u}$, and for any two compact real forms, there exists an inner automorphism of $\mathfrak{g}$ which transforms one to the other, which means there is only one compact real form up to equivalence.

A maximal abelian subalgebra of a complex semisimple Lie algebra is called a Cartan subalgebra of $\mathfrak{g}$. Any two Cartan subalgebras of $\mathfrak{g}$ are conjugate under the group of inner automorphism. A maximal real abelian subalgebra of a real semisimple Lie algebra $\mathfrak{g}_0$ is called a Cartan subalgebra of $\mathfrak{g}_0$. The Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is a real form of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$, namely $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$.

The real simple Lie algebra of first kind can be classified as follows. Given a complex simple Lie algebra $\mathfrak{g}$, let $\mathfrak{u}$ be its compact real form, and let θ be an involution which leaves $\mathfrak{u}$ invariant. The Cartan decomposition is $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$, where $\mathfrak{k}, i\mathfrak{p}$ are ± 1 characteristic subspace of θ, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, and $\mathfrak{k}$ is also a subalgebra of $\mathfrak{u}$, called characteristic subalgebra with respect to θ. Let $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$. If $\theta \neq 1$, then $\mathfrak{g}_0$ is a noncompact real simple Lie algebra. Two involutions determine the same real simple Lie algebra (up to an automorphism) if and only if they are conjugate under an inner automorphism. So every noncompact real simple Lie algebra corresponds 1-1 to a conjugacy class of involution of $\mathfrak{g}$, and every conjugacy class corresponds 1-1 to a Riemannian symmetric space of compact type (or of noncompact type). Here θ is the involution of complex simple Lie algebra $\mathfrak{g}$, meanwhile it is the involution of real simple Lie algebra $\mathfrak{g}_0$ and $\mathfrak{u}$. For details see [3, 7].

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. With respect to $\mathfrak{h}$ there is a root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$. Denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$ (resp. $\mathfrak{g}_0$) by τ (resp. σ). Define

$$\langle X, Y\rangle = -(X, \tau Y), \quad |X| = \sqrt{\langle X, X\rangle}. \tag{2.9}$$

This gives a Hermitian inner product on $\mathfrak{g}$. Any root α can be imbedded into $\mathfrak{h}$ by

$$\alpha(h) = (\alpha, h), \quad h \in \mathfrak{h}. \tag{2.10}$$

We call the real subspace $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \mathbb{R}\alpha$ generated by all roots the real part of $\mathfrak{h}$. Since the root is purely imaginary with respect to $\mathfrak{u}, \alpha \in i\mathfrak{u}$, there exists the decomposition

$$\mathfrak{h}_0 = i\mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p}, \quad \mathfrak{h}_\mathfrak{k} = i\mathfrak{h}_0 \cap \mathfrak{k}, \quad \mathfrak{h}_\mathfrak{p} = \mathfrak{h}_0 \cap \mathfrak{p}. \tag{2.11}$$

Choose the Weyl basis

$$e_{-\alpha} = \tau(e_\alpha), \quad (e_\alpha, e_{-\alpha}) = (e_\alpha, \tau(e_\alpha)) = -1, \tag{2.12}$$

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha\beta}, \quad N_{\alpha\beta} = -N_{\beta\alpha}, \quad N_{-\alpha-\beta} = N_{\alpha\beta}, \tag{2.13}$$

$$[e_\alpha, e_\alpha] = -\alpha, \tag{2.14}$$

where $N_{\alpha\beta}^2 = \frac{1}{2} q_{\beta\alpha}(1 + p_{\beta\alpha})(\alpha, \alpha) \in \mathbb{R}$, $\beta - p_{\beta\alpha}\alpha, \cdots, \beta + q_{\beta\alpha}\alpha$ is the α-chain of β.

$\mathfrak{u}$ is generated over $\mathbb{R}$ by

$$\{i\alpha, u_\alpha, v_\alpha, \alpha \in \Delta^+\}, \quad u_\alpha = \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha}), \quad v_\alpha = \frac{i}{\sqrt{2}}(e_\alpha - e_{-\alpha}). \tag{2.15}$$

Any element of $\mathfrak{u}$ can be expressed as

$$X = ih + \sum_{\alpha\in\Delta^+} a_\alpha e_\alpha + \bar{a}_\alpha e_{-\alpha}, \quad h \in \mathfrak{h}_0, \ a_\alpha \in \mathbb{C}.$$

Moreover

$$[ih, u_\alpha] = (\alpha, h) v_\alpha, \quad [ih, v_\alpha] = -(\alpha, h) u_\alpha, \quad h \in \mathfrak{h}_0. \tag{2.16}$$

The Cartan subalgebra of $\mathfrak{k}$ is called a torus Cartan subalgebra of $\mathfrak{g}_0$. A Cartan subalgebra of $\mathfrak{g}_0$ is called a Cartan subalgebra with maximal torus part if it contains a torus Cartan subalgebra. A maximal abelian subspace of $\mathfrak{p}$ is called a reduced Cartan subalgebra of $(\mathfrak{u}, \theta)$, and a Cartan subalgebra of $\mathfrak{g}_0$ is called a Cartan subalgebra with maximal vector part if it contains a reduced Cartan subalgebra.

In general, we can choose Cartan subalgebra in different ways. The real simple Lie algebra of first kind can be given as follows. Let $\mathfrak{h}_T$ be a Cartan subalgebra of $\mathfrak{k}$, it can be extended to a Cartan subalgebra of $\mathfrak{g}$, say $\mathfrak{h}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with maximal torus part. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We can choose the compatible order. Let

$$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\} \tag{2.17}$$

be a simple root system. Then θ has Gantmacher standard form (see [2, 6]):

$$\theta = \theta_0 \exp(2\pi i\, \mathrm{ad} h_0), \quad h_0 \in i\mathfrak{h}_T, \tag{2.18}$$

where θ_0 is a special rotation with respect to Π, θ_0 preserves Π and is an involution which called a regular involution, $\theta_0(h_0) = h_0$.

If $\theta_0 = 1$, denoting the highest root of Δ by $\delta = \sum_{i=1}^{l} m_i(\delta)\alpha_i$, we can choose $i, m_i(\delta) = 1$ or 2 such that

$$(h_0, \alpha_i) = \frac{1}{2}, \quad (h_0, \alpha_j) = 0, \quad j \neq i. \tag{2.19}$$

If $\theta_0 \neq 1$, let the characteristic subalgebra of θ_0 be $\mathfrak{k}_0 = \{X \in \mathfrak{u} \mid \theta_0(X) = X\}$. The Cartan subalgebra $\mathfrak{h}_0'$ of $\mathfrak{k}_0$ is the intersection of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ with $\mathfrak{k}_0$, $\mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{k}_0$. Let $\{\alpha_i'\}$ be the simple root system of $\mathfrak{h}_0'$, and let $\lambda' = \sum_i m_i(\lambda')\alpha_i'$ be the highest root of $\mathfrak{k}_0'$. Then we can choose i, $m_i(\lambda') = 1$ or 2 so that

$$(h_0, \alpha_i') = \frac{1}{2}, \quad (h_0, \alpha_j') = 0, \quad j \neq i. \tag{2.20}$$

For these two cases we denote h_0 by h_i. So for a given complex simple Lie algebra, by finding all θ_0, h_i, we can exhaust all real simple Lie algebras of first kind. $\theta_0 \neq 1$ only when $\mathfrak{g} = \mathfrak{a}_l, \mathfrak{d}_l, \mathfrak{e}_6$. For details see [7].

3 The Cartan Subalgebra with Maximal Vector Part and the Upper Bounds of Sectional Curvature

Let $M = U/K$ be an irreducible Riemannian symmetric space of compact type, and let $\mathfrak{h}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$, which can be extended to a Cartan algebra $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$ by choosing $\mathfrak{h}_\mathfrak{k} \subset \mathfrak{k}$,

$$\tilde{\mathfrak{h}}_0 = i\mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p}, \quad \tilde{\mathfrak{h}} = \mathfrak{h}_0^C. \tag{3.1}$$

The root system of $\mathfrak{g}$ with respect to $\tilde{\mathfrak{h}}$ is Δ, the simple root system is

$$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_r, \cdots, \alpha_l\}, \tag{3.2}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathfrak{h}_\mathfrak{p}$.

The tangent space of M is generated by

$$\{i\alpha_1, \cdots, i\alpha_r, u_\alpha - \theta(u_\alpha), v_\alpha - \theta(v_\alpha), \alpha \in \Delta^+\}, \tag{3.3}$$

where u_α, v_α is defined by (2.15).

The projection on $\mathfrak{h}_\mathfrak{p}$ of a root α is $\alpha' = \frac{1}{2}(\alpha - \theta(\alpha))$. If $\alpha' \neq 0$, α' is called a restricted root. In the same way by $(h, \alpha') = \alpha'(h), h \in \mathfrak{h}_\mathfrak{p}$, any restricted root can be identified with an element in $\mathfrak{h}_\mathfrak{p}$. It is well known that every restricted root is the restriction of a root. All restricted roots span $\mathfrak{h}_\mathfrak{p}$ over $\mathbb{R}$. We also have

$$[h, e_\alpha] = (\alpha', h)e_\alpha, \quad h \in \mathfrak{h}_\mathfrak{p}. \tag{3.4}$$

Denote the set of restricted roots by $\Sigma' = \{\alpha',\ \alpha' \neq 0\}$, and the set of noncompact roots by $\Sigma = \{\alpha \in \Delta, e_\alpha \in \mathfrak{p}^{\mathbb{C}}\}$.

We remark that there are similar projections by choosing different Cartan subalgebra. In the sequel we choose the Cartan subalgebra with maximal torus part. By abuse of notation, we denote the projection on torus part or vector part by the same symbol α', for torus part $\alpha' = \frac{\alpha+\theta(\alpha)}{2}$, but for vector part , $\alpha' = \frac{\alpha-\theta(\alpha)}{2}$.

From the general theory of symmetric space we know that any tangent vector is conjugate to an element ih of $i\mathfrak{h}_{\mathfrak{p}}$ by the action of isotropy group (see [4, Theorem 6.2, p.246]). So it is only to estimate the sectional curvature of the form

$$K(ih, X),\ h \in \mathfrak{h}_{\mathfrak{p}}, \quad X \perp \mathfrak{h}_{\mathfrak{p}},\ X \in \mathfrak{p}.$$

Then

$$X = \sum_{\alpha\in\Sigma^+} a_\alpha e_\alpha + \bar{a}_\alpha e_{-\alpha}. \tag{3.5}$$

$$[ih, X] = \sum i(\alpha, h)(a_\alpha e_\alpha - \bar{a}_\alpha e_{-\alpha}), \tag{3.6}$$

$$R(ih, X, ih, X) = |[ih, X]|^2 = \sum 2(h, \alpha)^2 |a_\alpha|^2 \tag{3.7}$$

$$= \sum 2(h, \alpha')^2 |a_\alpha|^2 \leq |h|^2 \sum 2(\alpha', \alpha')^2 |a_\alpha|^2. \tag{3.8}$$

If $|h| = 1, |X| = 1$ then $\sum |a_\alpha|^2 = \frac{1}{2}$. Let d be the maximum of squared length of restricted roots. Then

$$K(ih, X) \leq d.$$

On the other hand, if $|\alpha_0'|^2 = d$, and set $h = \alpha_0', X = e_{\alpha_0} + e_{-\alpha_0}$, it can be verified that

$$K(ih, X) = (\alpha_o', \alpha_0) = d. \tag{3.9}$$

So we get the following proposition (see [4, Theorem 11.1, p.334]):

Proposition 3.1 *The maximum of sectional curvature for an irreducible Riemannian symmetric space of compact type is the squared length of the highest restricted root.*

Let δ be highest noncompact root. Then δ' is highest restricted root. So in order to get the maximum of sectional curvature it is sufficient to get the maximum of the squared length of the projection of noncompact roots onto the reduced Cartan subalgebra.

Now we proceed to the irreducible Riemannian symmetric space of type II. Let $M = U$ be a compact simple Lie group with Lie algebra $\mathfrak{u}$, the Riemannian metric is given in (2.6), and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$. The root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is Δ. For $h \in \mathfrak{h}_0$,

$$\begin{aligned}
(h, h) =&\ \operatorname{tr} \operatorname{ad}h \cdot \operatorname{ad}h = \sum_{\alpha\in\Delta} (\alpha, h)^2 = 2\sum_{\alpha>0} (\alpha, h)^2, \\
\operatorname{Ric}(ih, ih) =& \sum_{\alpha>0} R(ih, u_\alpha, ih, u_\alpha) + R(ih, v_\alpha, ih, v_\alpha) \\
=& -\frac{1}{4}\sum_{\alpha>0} ([ih, u_\alpha], [ih, u_\alpha]) + ([ih, v_\alpha], [ih, v_\alpha]) \\
=& -\frac{1}{4}\sum_{\alpha>0} (\alpha, h)^2 (v_\alpha, v_\alpha) + (\alpha, h)^2 (-u_\alpha, -u_\alpha)
\end{aligned}$$

$$= \frac{1}{2}\sum_{\alpha>0}(\alpha,h)^2 = -\frac{1}{4}(ih,ih) = \frac{1}{4}\langle ih,ih\rangle.$$

We see that the Ricci curvature is $\frac{1}{4}$. By the same method, we get that the upper bound of curvature for U is exactly the maximum of the squared length of roots, i.e., the squared length of the highest root.

In fact, let $U^* = \{(u,u)|u \in U\}$, $\mu(u,v) = (v,u)$, and let $U \times U/U^*$ be the Riemannian symmetric pair of compact type associated with the orthogonal symmetric Lie algebra $(\mathfrak{u}\times\mathfrak{u}, d\mu)$. This metric coincides with the previous one in (2.6) up to a constant.

4 The Transformation from the Cartan Subalgebra with Maximal Torus Part to the Cartan Subalgebra with Maximal Vector Part

Let $\mathfrak{g}$ be a complex simple Lie algebra with the Cartan decomposition

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}. \tag{4.1}$$

Let $\mathfrak{h}_0 = i\mathfrak{h}_T + \mathfrak{h}_V, \mathfrak{h}_T \subset \mathfrak{k}, \mathfrak{h}_V \subset \mathfrak{p}, \mathfrak{h} = \mathfrak{h}_0^C$ be a Cartan subalgebra with maximal torus part. The root system is Δ, $\theta = \theta_0\exp(2\pi i\,\mathrm{ad}h_0)$ as in (2.18)–(2.20).
The decomposition of root system is

$$\Delta = \Delta_0 \cup \Delta_n \cup \Delta_c, \tag{4.2}$$

$$\Delta_0 = \{\alpha \in \Delta \mid \theta(\alpha) \neq \alpha\}, \tag{4.3}$$

$$\Delta_n = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha,\ \theta(e_\alpha) = -e_\alpha\}, \tag{4.4}$$

$$\Delta_c = \{\alpha \in \Delta \mid \theta(\alpha) = \alpha,\ \theta(e_\alpha) = e_\alpha\}. \tag{4.5}$$

The centralizer of $\mathfrak{h}_V^C$ in $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{h}_V) = \{X \in g \mid [X,h] = 0, h \in \mathfrak{h}_V\}$. We have

$$\mathfrak{z}(\mathfrak{h}_V) = \mathfrak{h}_V^C + \sum_{\alpha\in\Delta_n} \mathfrak{g}_\alpha. \tag{4.6}$$

We call a root α compact if $\mathfrak{g}_\alpha \subset \mathfrak{k}^{\mathbb{C}}$, otherwise noncompact.

We now give the method of Harish-Chandra for constructing Cartan subalgebra with maximal vector part from a Cartan subalgebra with maximal torus part. It is so-called Cayley's transformation. For details see [12] and [4, pp.385–387, pp.530–534].

Two roots α, β are called strongly orthogonal if $\alpha \pm \beta \notin \Delta$.

Let γ_1 be the highest root in Δ_n^+. Set

$$\Delta_n^+(\gamma_1) = \{\alpha \in \Delta_n^+ \mid \alpha \neq \gamma_1, \alpha \pm \gamma_1 \notin \Delta\}. \tag{4.7}$$

Let γ_2 be the highest root in $\Delta_n^+(\gamma_1)$. Set

$$\Delta_n^+(\gamma_1,\gamma_2) = \{\alpha \in \Delta_n^+(\gamma_1) \mid \alpha \neq \gamma_2, \alpha \pm \gamma_2 \notin \Delta\}. \tag{4.8}$$

In this way we can get the maximal system of positive roots which are strongly orthogonal to each other, say $\{\gamma_1, \gamma_2, \cdots, \gamma_s\}$.

Let $\mathfrak{a}_1$ be the real subspace generated by $\{e_{\gamma_i} - e_{-\gamma_i}, i = 1, 2, \cdots, s\}$. Then

$$\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{h}_V \tag{4.9}$$

is a reduced Cartan subalgebra of $\mathfrak{p}$.

Let

$$\tilde{\mathfrak{h}}_T = \{X \in \mathfrak{h}_T \mid (X, \gamma_i) = 0,\ i = 1, 2, \cdots, s\}. \tag{4.10}$$

Then

$$\tilde{\mathfrak{h}}_0 = \tilde{\mathfrak{h}}_T + \mathfrak{a} \tag{4.11}$$

is a Cartan subalgebra with maximal vector part of $\mathfrak{g}_0$.

Let

$$X_i = \frac{\pi}{2\sqrt{2}|\gamma_i|}(e_{\gamma_i} + e_{-\gamma_i}), \tag{4.12}$$

$$\rho_i = \exp(\mathrm{ad}X_i) \in \mathrm{Int}(\mathfrak{u}), \tag{4.13}$$

$$\rho = \rho_1\rho_2 \cdots \rho_s. \tag{4.14}$$

Then

$$\rho(\mathfrak{h}_0) = \tilde{\mathfrak{h}}_0, \tag{4.15}$$

$$\rho(\gamma_i) = -\frac{|\gamma_i|}{\sqrt{2}}(e_{\gamma_i} - e_{-\gamma_i}), \tag{4.16}$$

$$\rho|\tilde{\mathfrak{h}}_T + \mathfrak{h}_V = \mathrm{id}. \tag{4.17}$$

Being an automorphism, ρ exchanges two Cartan subalgebra, taking roots to roots, eigenvectors to eigenvectors. Let α_0 be a noncompact root which takes maximal root length. Since ρ is an isometry of $\mathfrak{u}$, the projections on two Cartan subalgebra $\mathfrak{h}_0, \tilde{\mathfrak{h}}_0$ of $\alpha_0, \rho(\alpha_0)$ have the same length. We need only to calculate the maximal length of the projection on $\{\gamma_1, \gamma_2, \cdots, \gamma_s\} \cup \mathfrak{h}_V$ of non compact roots with respect to $\mathfrak{h}$. This is the maximal length of restricted roots with respect to $\tilde{\mathfrak{h}}_0$, the Cartan subalgebra with maximal vector part.

We can choose compatible order so that ρ takes the highest noncompact root in $\mathfrak{h}_0$ to the highest noncompact root in $\tilde{\mathfrak{h}}_0$. Let α_0 be the highest noncompact root with respect to $\mathfrak{h}_0$. The projection on $\{\gamma_1, \gamma_2, \cdots, \gamma_s\} \cup \mathfrak{h}_V$ is α_0'. Then $|\alpha_0'|$ attains the maximal length of restricted roots.

Since real simple Lie algebras of the first kind are completely classified according to Cartan subalgebra with maximal torus part, the irreducible symmetric spaces of type I and type III can be given by using the Gantmacher standard form of complex simple Lie algebras. Moreover by estimating the maximum of squared length of restricted roots we get the curvature bounds for symmetric spaces.

5 Estimates of Bounds of Sectional Curvature

Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ be the Cartan decomposition of U/K which is an irreducible symmetric space of compact type, as in (2.9). The Riemannian inner product at $\pi(e)$ is

$$\langle X, Y\rangle = -(X, Y). \tag{5.1}$$

Every root is embedded in $\mathfrak{g}$ through Killing form as in (2.10). Then $\forall\alpha \in \Delta, \alpha \in i\mathfrak{u}, (\alpha, \alpha) > 0$. For every simple Lie algebra, there are at most two different lengths of roots. Let $\mathfrak{h}$ be a Cartan subalgebra. For $H_1, H_2 \in \mathfrak{h}$,

$$(H_1, H_2) = \mathrm{tr}(\mathrm{ad}H_1 \mathrm{ad}H_2) = \sum_{\beta\in\Delta} \beta(H_1)\beta(H_2) = 2 \sum_{\beta\in\Delta^+} \beta(H_1)\beta(H_2). \tag{5.2}$$

We have

$$(\alpha, \alpha) = 2 \sum_{\beta\in\Delta^+} (\beta, \alpha)(\beta, \alpha) = \frac{1}{2} \sum_{\beta\in\Delta^+} \Big(\frac{2(\beta, \alpha)}{(\alpha, \alpha)}\Big)^2 (\alpha, \alpha)^2 \tag{5.3}$$

$$= \frac{1}{2} \sum_{\beta\in\Delta^+} a_{\beta\alpha}^2 (\alpha, \alpha)^2, \tag{5.4}$$

where $a_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = p_{\beta\alpha} - q_{\beta\alpha}$ is Cartan integer, so

$$(\alpha, \alpha) = \frac{2}{\sum\limits_{\beta\in\Delta^+} a_{\beta\alpha}^2}. \tag{5.5}$$

By this normalization we can calculate the maximum of squared length of roots, as listed in [1].

We adopt the Dynkin diagrams of complex simple Lie algebras as in [4, p.476].

For the Gantamacher standard form as in (2.18), $\theta_0 \neq 1$ only when $\mathfrak{g} = \mathfrak{a}_l, \mathfrak{d}_l, \mathfrak{e}_6$. We list these special rotations as follows (for Dynkin diagrams see the following paragraphs).

For $\mathfrak{a}_l$, $\theta_0(\alpha_1) = \alpha_l$, $\theta_0(\alpha_2) = \alpha_{l-1}, \cdots, \theta_0(\alpha_l) = \alpha_1$.

For $\mathfrak{d}_l$, $\theta_0(\alpha_{l-1}) = \alpha_l$, $\theta_0(\alpha_l) = \alpha_{l-1}$, $\theta_0(\alpha_j) = \alpha_j$, $j \le l - 2$.

For $\mathfrak{e}_6$, $\theta_0(\alpha_1) = \alpha_6$, $\theta_0(\alpha_2) = \alpha_2$, $\theta_0(\alpha_3) = \alpha_5$, $\theta_0(\alpha_4) = \alpha_4$, $\theta_0(\alpha_5) = \alpha_3$, $\theta_0(\alpha_6) = \alpha_1$.

We can reduce the calculation to the following cases:

(1) If $\theta = \exp(2\pi i\, \mathrm{ad}h_i)$ is an inner automorphism, from the classification of real simple Lie algebra we have $\mathfrak{g}_0 = \mathfrak{k}+\mathfrak{p}, \mathfrak{h}_0 = i\mathfrak{h}_T$ which means that $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}_0$, its Cartan subalgebra is also a Cartan subalgebra of $\mathfrak{g}_0$, and $\mathfrak{a}$ consists of r strongly orthogonal roots in Δ_n^+. Since γ_1 is the highest noncompact root, $|\gamma_1|^2$ is the upper bound of curvature. Let $\alpha = \sum m_j(\alpha)\alpha_j$. From $(h_i, \alpha_j) = \frac{1}{2}\delta_{ij}$ we have

$$\theta(e_\alpha) = \exp(2\pi i(\alpha, h_i))e_\alpha = (-1)^{m_i(\alpha)} e_\alpha. \tag{5.6}$$

Then

$$\Delta_n = \{\alpha \in \Delta,\ m_i(\alpha) \equiv 1 \pmod 2\} \tag{5.7}$$

and γ_1 is the highest root whose coefficient satisfies $m_i \equiv 1 \pmod 2$. This includes the cases of AIII, BI, DIII, CI, CII, EI, EII, EIII, EV, EVI, EVII, EVIII, EIX, FI, FII, G and some cases of DI.

(2) If $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(M)$, then we can choose a Cartan subalgebra $\mathfrak{h}$ whose real part satisfies $\mathfrak{h}_0 = \mathfrak{h}_\mathfrak{p} \subset \mathfrak{p}$, so the projection on $\mathfrak{h}_\mathfrak{p}$ of any root is itself. We choose the highest root α_0, (α_0, α_0) is the curvature bound. This is in the case of AI.

(3) If $\theta = \theta_0$ is a special rotation, $\theta_0(e_\alpha) = e_{\theta_0(\alpha)}$, then $\Delta_n = 0$, $\mathfrak{h}_\mathfrak{p}$ is generated by

$$\{\alpha - \theta_0(\alpha),\ \alpha \in \Delta_0^+\}.$$

We can get the highest root which satisfies $\alpha \in \Delta_0^+, e_\alpha \in \mathfrak{p}^{\mathbb{C}}$. The projection is $\alpha_0' = \frac{1}{2}(\alpha - \theta_0(\alpha))$, and (α_0, α_0) is the upper bound of curvature. This includes the cases of AII, EIV and some cases of DI.

(4) For the other cases, from the Cayley's transformation, we can get $\{\gamma_1, \gamma_2, \cdots, \gamma_s\} \cup \mathfrak{h}_V$, by calculating the projection on this subspace we can give the upper bounds of sectional curvature. This includes some cases of DI.

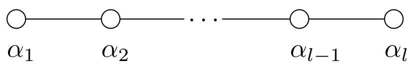
Now we calculate, case by case, the upper bounds of sectional curvature of irreducible symmetric spaces of compact type according to the table in [4, p.518].

Let $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ be a simple root system, every root can be given as the integral linear combination

$$\alpha = m_1(\alpha)\alpha_1 + m_2(\alpha)\alpha_2 + \cdots + m_l(\alpha)\alpha_l. \tag{5.8}$$

In the following calculation, for four classical complex simple Lie algebras we imbed the root system into Euclidean spaces.

(i) The Dynkin diagram of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$ is



Let $\varepsilon_j,\ 1 \le j \le l+1$ be an orthogonal base of $\mathbb{R}^{l+1}$ with $|\varepsilon_j|^2 = \frac{1}{2(l+1)}$. The simple root system of $\mathfrak{a}_l$ is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l\}, \quad |\alpha_j|^2 = \frac{1}{l+1}.$$

The positive roots are

$$\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}, \quad i < j.$$

The highest root is

$$\delta = \varepsilon_1 - \varepsilon_{l+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_l.$$

For AI, M=SU(n)/SO(n), $\mathfrak{g} = \mathfrak{a}_l = \mathfrak{sl}(n, \mathbb{C})$, $l = n-1$, $\mathrm{rank}(M) = \mathrm{rank}(\mathfrak{g}) = l$, $\theta(Z) = -Z^T$, $Z \in \mathfrak{g}$. If n is even, then $n = 2m, \theta = \theta_0 \exp(2\pi i h_m)$. If n is odd, then $n = 2m+1, \theta = \theta_0$, where θ_0 is the special rotation. In this case, let $\mathfrak{h}_0$ consist of all real diagonal matrixes of trace 0. Then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, as well as a reduced Cartan subalgebra of $(\mathfrak{g}_0, \theta)$, $\mathfrak{h}_0 \subset \mathfrak{p}$, so the length of restricted root α' is the same as the length of α. Since for $\mathfrak{a}_l$, all simple roots (as well as all roots) have the same length, $(\alpha, \alpha) = \frac{1}{l+1}$, we see that the upper bound of curvature is $\frac{1}{n}$.

For AII, $M = \mathrm{SU}(2n)/\mathrm{Sp}(n)$, $\mathfrak{g} = \mathfrak{a}_l$, $l = 2n-1$, $\theta = \theta_0$, $\theta_0(Z) = -J_n Z^T J_n^{-1}$ is special rotation,

$$\theta(\alpha_j) = \alpha_{l+1-j} = \alpha_{2n-j}, \quad j = 1, 2, \cdots, 2n-1.$$

If $\alpha = m_1\alpha_1 + \cdots + m_n\alpha_n + \cdots + m_{2n-1}\alpha_{2n-1}$, then

$$\theta(\alpha) = m_{2n-1}\alpha_1 + \cdots + m_n\alpha_n + \cdots + m_1\alpha_{2n-1}, \quad \theta(e_\alpha) = e_{\theta(\alpha)}.$$

In this case Δ_n is empty set, the reduced Cartan subalgebra $\mathfrak{h}_\mathfrak{p}$ is generated by $\{\alpha - \theta(\alpha), \alpha \in \Delta\}$, and the base is

$$\{\sigma_i = \alpha_i - \theta(\alpha_i) = \varepsilon_i - \varepsilon_{i+1} - \varepsilon_{2n-i} + \varepsilon_{2n-i+1},\ 1 \le i \le n-1\}.$$

$\theta_0(\alpha) = \alpha$ if and only if $m_i(\alpha) = m_{2n-i}(\alpha),\ i = 1, 2, \cdots, n-1$. The positive root is in the form of $\alpha = \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$. The highest root which satisfies $\theta_0(\alpha) \neq \alpha$ is

$$\alpha_0 = \varepsilon_1 - \varepsilon_{2n-1} = \alpha_1 + \cdots + \alpha_{2n-2}.$$

We get the orthogonal base $\{\tilde{\sigma}_i\}$ from $\{\sigma_i\}$ such that

$$\tilde{\sigma}_1 = \sigma_1, \quad \tilde{\sigma}_2 = 2\sigma_2 + \sigma_1 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3 - 2\varepsilon_{2n-2} + \varepsilon_{2n-1} + \varepsilon_{2n}, \ \cdots, \quad (\tilde{\sigma}_i, \alpha_0) = 0, \quad i > 1.$$

The projection on $\{\tilde{\sigma}_i, 1 \leq i \leq n-1\}$ of α_0 is

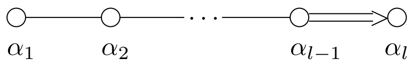
$$\frac{(\alpha_0, \tilde{\sigma}_1)}{(\tilde{\sigma}_1, \tilde{\sigma}_1)}\tilde{\sigma}_1 \quad \text{with the squared length} \quad \frac{(\alpha_0, \tilde{\sigma}_1)^2}{(\tilde{\sigma}_1, \tilde{\sigma}_1)} = \frac{1}{2}(\alpha_0, \alpha_0) = \frac{1}{4n}.$$

We see that the upper bound of curvature is $\frac{1}{4n}$.

For AIII, $M = \mathrm{SU}(p+q)/S(U_p \times U_q),\ \mathfrak{g} = \mathfrak{a}_l,\ l = p+q-1,\ \mathrm{rank}(M) = \min(p,q),\ \theta(Z) = -I_{p,q}X^T I_{p,q},\ \theta = \exp(2\pi i\, \mathrm{ad} h_i),\ \mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$, and $\alpha_0 = \alpha_1 + \cdots + \alpha_l \in \Delta_n$ is the highest root. In the Cayley's transformation we have $\gamma_1 = \alpha_0$, but $|\alpha_0|^2 = \frac{1}{l+1}$ attains the maximal value, so the upper bound of curvature is $|\alpha_0|^2 = \frac{1}{p+q}$.

For BDI, $M = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q), \theta(Z) = I_{p,q}Z^T I_{p,q}$, and these are cases of BI or DI according as $p+q$ is odd or even.

(ii) The Dynkin diagram of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$ is



Let $\varepsilon_j, 1 \leq j \leq l$ be an orthogonal base of $\mathbb{R}^l$, $|\varepsilon_j|^2 = \frac{1}{2(2l-1)}$. The simple root system of $\mathfrak{b}_l$ is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l-1,\ \alpha_l = \varepsilon_l\}.$$

The positive roots are $\varepsilon_i,\ \varepsilon_i \pm \varepsilon_j,\ i < j$, where $\varepsilon_i = \alpha_i + \cdots + \alpha_l,\ \varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_j,\ \varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_l)$. The highest root is

$$\delta = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l.$$

For BI, $M = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q),\ p+q = 2l+1,\ \mathfrak{g} = \mathfrak{b}_l,\ \mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$, and $\theta = \exp(2\pi i\, \mathrm{ad} h_i)$ is an inner automorphism

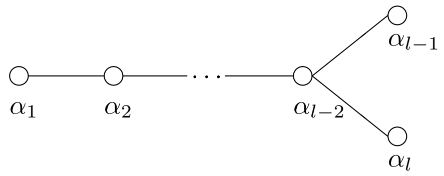
$$\Delta_n = \{\alpha \in \Delta,\ m_i(\alpha) \equiv 1 \pmod 2\}.$$

The highest noncompact root α_0 is the highest root whose coefficient m_i is odd. We get it in these two cases

$$\begin{aligned} \varepsilon_1 + \varepsilon_{i+1} &= \alpha_1 + \cdots + \alpha_i + 2(\alpha_{i+1} + \cdots + \alpha_l), \quad i < l, \\ \varepsilon_1 &= \alpha_1 + \alpha_2 + \cdots + \alpha_l. \end{aligned}$$

So the upper bound of curvature is $\frac{1}{2l-1} = \frac{1}{p+q-2}$ $(i < l)$, $\frac{1}{2(2l-1)} = \frac{1}{2(p+q-2)}$ $(i = l)$. The later case corresponds to $\mathrm{SO}(2n+1)/\mathrm{SO}(2n)$ which is of rank 1.

(iii) The Dynkin diagram of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ is



Let $\varepsilon_j, 1 \leq j \leq l$, be an orthogonal base of $\mathbb{R}^l$, $|\varepsilon_j|^2 = \frac{1}{4(l-1)}$. The simple root system is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l-1,\ \alpha_l = \varepsilon_{l-1} + \varepsilon_l\}.$$

The positive roots are $\varepsilon_i \pm \varepsilon_j,\ i < j$, where $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1},\ \varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{l-2} + \alpha_j + \cdots + \alpha_l = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$. The highest root is

$$\delta = \varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l.$$

For DI, $M = \mathrm{SO}(p+q)/\mathrm{SO}(p) \times \mathrm{SO}(q),\ p + q = 2l$, $\mathfrak{g} = \mathfrak{d}_l$.

θ can be an inner automorphism or an outer automorphism.

If $\theta = \exp(2\pi i\, \mathrm{ad} h_i), 1 \leq i \leq [\frac{l}{2}]$, we can get the highest root which satisfies $m_i(\alpha) = 1$,

$$\alpha_0 = \varepsilon_1 + \varepsilon_{i+1} = \alpha_1 + \cdots + \alpha_i + 2(\alpha_{i+1} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l.$$

So the maximum of curvature is $2 \cdot \frac{1}{4(l-1)} = \frac{1}{p+q-2}$.

If $\theta = \theta_0$ is an outer involution,

$$\theta_0(\alpha_i) = \alpha_i, \quad 1 \leq i \leq l-2, \quad \theta_0(\alpha_{l-1}) = \alpha_l, \quad \theta_0(\alpha_l) = \alpha_{l-1},$$

then Δ_n is empty, the reduced Cartan subalgebra $\mathfrak{h}_\mathfrak{p}$ is generated by

$$\{\sigma_1 = \alpha_{l-1} - \theta(\alpha_{l-1}) = \alpha_{l-1} - \alpha_l\}.$$

$\mathrm{rank}(M) = \dim \mathfrak{h}_\mathfrak{p} = 1, \theta_0(\alpha) = \alpha$ if and only if $m_{l-1}(\alpha) = m_l(\alpha)$. The highest root which satisfies $m_{l-1}(\alpha) \neq m_l(\alpha)$ is

$$\alpha_0 = \varepsilon_1 - \varepsilon_l = \alpha_1 + \cdots + \alpha_{l-1}.$$

The projection of α_0 is

$$\frac{(\alpha_0, \sigma_1)}{(\sigma_1, \sigma_1)}\sigma_1 \quad \text{with the squared length} \quad \frac{(\alpha_0, \sigma_1)^2}{(\sigma_1, \sigma_1)} = \frac{1}{2}(\alpha_0, \alpha_0) = \frac{1}{2(p+q-2)}.$$

We see that the maximum of curvature is $\frac{1}{2(p+q-2)}$.

If $\theta = \theta_0 \exp(2\pi i\, \mathrm{ad} h_i), 1 \leq i \leq [\frac{l}{2}]$, from Cayley's transformation, we know that there exist $\gamma_1, \gamma_2, \cdots, \gamma_{r-1}$ such that the reduced Cartan subalgebra $\mathfrak{h}_\mathfrak{p}$ is generated by

$$\{\gamma_i,\ i = 1, 2, \cdots, r-1,\ \alpha_{l-1} - \alpha_l\}.$$

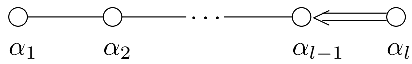
The projection of root γ_1 on $\mathfrak{h}_\mathfrak{p}$ is itself, but $|\gamma_1|$ attains the maximum of length of roots, we see that the maximum of curvature is $\frac{1}{p+q-2}$.

For DIII, $M = \mathrm{SO}(2n)/\mathrm{U}(n)$, $\mathfrak{g} = \mathfrak{g}_l$, $l = n$, $\theta = \exp(2\pi i\, \mathrm{ad} h_i)$, $i = l$ is inner involution, and the highest noncompact root is

$$\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{l-1}) + \alpha_{l-1} + \alpha_l.$$

We see that the maximum of curvature is $\frac{1}{2(l-1)} = \frac{1}{2n-2}$.

(iv) The Dynkin diagram of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ is



Let $\varepsilon_j, 1 \le j \le l$ be an orthogonal base of $\mathbb{R}^l$, $|\varepsilon_j|^2 = \frac{1}{4(l+1)}$. The simple root system is

$$\Pi = \{\alpha_j = \varepsilon_j - \varepsilon_{j+1},\ j = 1, 2, \cdots, l-1,\ \alpha_l = 2\varepsilon_l\}.$$

The positive roots are $2\varepsilon_i$, $\varepsilon_i \pm \varepsilon_j$, $i < j$, where $2\varepsilon_i = 2(\alpha_i + \cdots + \alpha_{l-1}) + \alpha_l$, $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$, $\varepsilon_i + \varepsilon_j = \alpha_i + \cdots + \alpha_{j-2} + 2(\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l$. The highest root is

$$\delta = 2\varepsilon_1 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l.$$

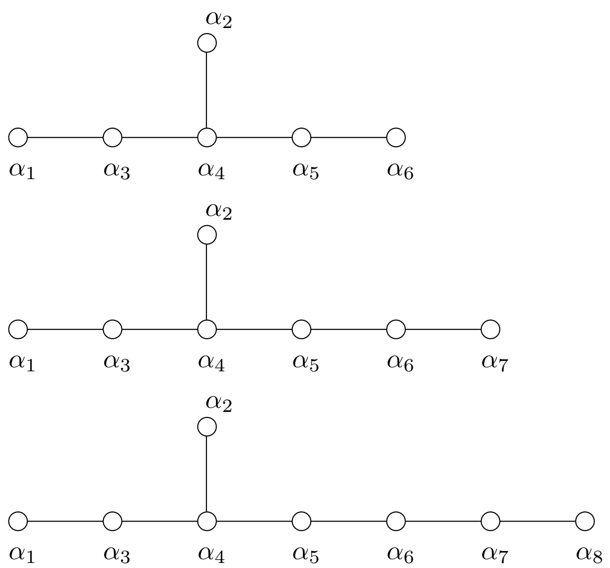
For CI or CII, $\mathfrak{g} = \mathfrak{c}_l$, $\theta = \exp(2\pi i\, \mathrm{ad} h_i)$, $1 \le i \le [\frac{l-1}{2}] + 1$ or $i = l$.

For CI, $i = l$, the noncompact root satisfies that $m_l(\alpha)$ is odd. We see that the highest noncompact root is $\alpha_0 = 2\varepsilon_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}) + \alpha_l$. The maximum of curvature is $(\alpha_0, \alpha_0) = \frac{1}{n+1}$.

For CII, $l = p + q, i < l$, the noncompact root satisfies that $m_i(\alpha)$ is odd. We see that the highest noncompact root is $\alpha_0 = \varepsilon_i + \varepsilon_{i+1} = \alpha_i + 2(\alpha_{i+1} + \cdots + \alpha_{l-1}) + \alpha_l$. The maximum of curvature is $(\alpha_0, \alpha_0) = \frac{1}{2(l+1)} = \frac{1}{2(p+q+1)}$.

Now we consider the cases of exception type.

(v) For type E, $\mathfrak{g} = \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, we draw the corresponding Dynkin diagram as follows.



$\mathfrak{e}_6$

$\mathfrak{e}_7$

$\mathfrak{e}_8$

In these cases all the roots have the same length. All involutions are inner automorphism except for $\mathfrak{e}_6$. From formula (5.5), we get the squared lengths of roots are $\frac{1}{12}$, $\frac{1}{18}$, $\frac{1}{30}$, respectively.

We list the positive roots of $\mathfrak{e}_6$ as follows:

$$\begin{aligned}
&\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6,\\
&\alpha_1+\alpha_3, \quad \alpha_3+\alpha_4, \quad \alpha_2+\alpha_4, \quad \alpha_4+\alpha_5, \quad \alpha_5+\alpha_6,\\
&\alpha_1+\alpha_3+\alpha_4, \quad \alpha_3+\alpha_4+\alpha_2, \quad \alpha_3+\alpha_4+\alpha_5, \quad \alpha_4+\alpha_5+\alpha_2, \quad \alpha_4+\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_3+\alpha_4+\alpha_2, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5, \quad \alpha_2+\alpha_3+\alpha_4+\alpha_5,\\
&\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_2+\alpha_4+\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \quad \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\\
&\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_2+\alpha_3+2\alpha_4+\alpha_5,\\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5,\\
&\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \quad \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5,\\
&\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\\
&\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\\
&\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6.
\end{aligned}$$

For EIV, $\mathfrak{g}=\mathfrak{e}_6,\ \theta=\theta_0$ is outer automorphism

$$\theta_0(\alpha_1)=\alpha_6, \quad \theta_0(\alpha_2)=\alpha_2, \quad \theta_0(\alpha_3)=\alpha_5, \tag{5.9}$$

$$\theta_0(\alpha_4)=\alpha_4, \quad \theta_0(\alpha_5)=\alpha_3, \quad \theta_0(\alpha_6)=\alpha_1. \tag{5.10}$$

The reduced Cartan subalgebra $\mathfrak{h}_\mathfrak{p}$ is generated by

$$\{\alpha_1-\alpha_6, \alpha_3-\alpha_5\}.$$

$\theta_0(\alpha)=\alpha$ if and only if $m_1(\alpha)=m_6(\alpha),\ m_3(\alpha)=m_5(\alpha)$. The highest root which satisfies $\theta_0(\alpha)\neq\alpha$ is

$$\alpha_0=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6.$$

An orthogonal base of $\mathfrak{h}_\mathfrak{p}$ is

$$\Big\{\sigma_1=\alpha_1-\alpha_6,\ \sigma_2=\alpha_3-\alpha_5+\frac{1}{2}(\alpha_1-\alpha_6)\Big\}.$$

The projection on $\mathfrak{h}_\mathfrak{p}$ of α_0 is

$$\frac{(\alpha_0,\sigma_1)}{(\sigma_1,\sigma_1)}\sigma_1+\frac{(\alpha_0,\sigma_2)}{(\sigma_2,\sigma_2)}\sigma_2.$$

We see that the maximum of curvature is

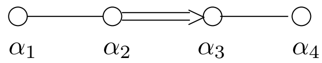
$$\frac{(\alpha_0,\sigma_1)^2}{(\sigma_1,\sigma_1)}+\frac{(\alpha_0,\sigma_2)^2}{(\sigma_2,\sigma_2)}=\frac{1}{2}(\alpha_0,\alpha_0)=\frac{1}{24}.$$

For EI, EII, EIII, we need the Cayley's transformation. There exists at least one γ_1, but all roots have the same length. $|\gamma_1|^2=\frac{1}{12}$ attains the maximum. We see that the maximum of curvature is $|\gamma_1|^2=\frac{1}{12}$.

For EV, EVI, EVII, $\mathfrak{g}=\mathfrak{e}_7$, all involutions are inner automorphisms, and with the same reason as previous case we get that the maximum of curvature is $\frac{1}{18}$.

For EVIII, EIX, $\mathfrak{g} = \mathfrak{e}_8$, all involutions are inner automorphisms, and with the same reason as previous case we get that the maximum of curvature is $\frac{1}{30}$.

(vi) The Dynkin diagram of $\mathfrak{f}_4$ is



For FI, FII, $\mathfrak{g} = \mathfrak{f}_4$, $|\alpha_1|^2 = |\alpha_2|^2 = \frac{1}{9}$, $|\alpha_3|^2 = |\alpha_4|^2 = \frac{1}{18}$. We list all positive roots

$$\begin{aligned}
&\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4,\\
&\alpha_1+\alpha_2, \quad \alpha_2+\alpha_3, \quad \alpha_3+\alpha_4,\\
&2\alpha_2+\alpha_3, \quad \alpha_1+\alpha_2+\alpha_3, \quad \alpha_2+\alpha_3+\alpha_4,\\
&\alpha_1+\alpha_2+2\alpha_3, \quad \alpha_1+\alpha_2+\alpha_3+\alpha_4, \quad \alpha_2+2\alpha_3+\alpha_4,\\
&\alpha_1+2\alpha_2+2\alpha_3, \quad \alpha_2+2\alpha_3+2\alpha_4, \quad \alpha_1+\alpha_2+2\alpha_3+\alpha_4,\\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \quad \alpha_1+2\alpha_2+2\alpha_3+\alpha_4,\\
&\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4, \quad \alpha_1+2\alpha_2+3\alpha_3+\alpha_4,\\
&\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4,\\
&\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4,\\
&\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4,\\
&2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4.
\end{aligned}$$

All involutions are inner automorphisms, $\theta = \exp(2\pi i\,\mathrm{ad}h_i)$, $i = 1, 4$.

For $i = 1$, the highest noncompact root which satisfies $m_1 = 1 \pmod 2$ is

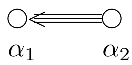
$$\alpha_0 = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \quad |\alpha_0|^2 = \frac{1}{9}.$$

For $i = 4$, the highest noncompact root which satisfies $m_4 = 1 \pmod 2$ is

$$\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \quad |\alpha_0|^2 = \frac{1}{18}.$$

We get that the maximum of curvature is $\frac{1}{9}$ for FI, $\frac{1}{18}$ for FII.

(vii) The Dynkin diagram of $\mathfrak{g}_2$ is



For G, $\mathfrak{g} = \mathfrak{g}_2$, $|\alpha_1|^2 = \frac{1}{12}, |\alpha_2|^2 = \frac{1}{4}$, $\theta = \exp(2\pi i\,\mathrm{ad}h_i)$, $i = 2$. We list all positive roots

$$\begin{aligned}
&\alpha_1, \quad \alpha_2,\\
&\alpha_1+\alpha_2, \quad 2\alpha_1+\alpha_2,\\
&3\alpha_1+\alpha_2, \quad 3\alpha_1+2\alpha_2.
\end{aligned}$$

The highest noncompact root which satisfies $m_2 = 1 \pmod 2$ is

$$\alpha_0 = 3\alpha_1 + \alpha_2, \quad |\alpha_0|^2 = \frac{1}{4}.$$

We get that the maximum of curvature is $\frac{1}{4}$.

Now we get the upper bounds of sectional curvature for all cases of irreducible Riemannian symmetric spaces of compact type. In summary, we have the following table.

Table 5.1

Type	compact type	rank	dimension	bound
AI	$\mathrm{SU}(n)/\mathrm{SO}(n)$	$n-1$	$\frac{1}{2}(n-1)(n+2)$	$\frac{1}{n}$
AII	$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$n-1$	$(n-1)(2n+1)$	$\frac{1}{4n}$
AIII	$\mathrm{SU}(p+q))/S(U_p\times U_q))$	$\min(p,q)$	$2pq$	$\frac{1}{p+q}$
BDI	$\mathrm{SO}(p+q))/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\min(p,q)>1$ $=1$	pq	$\frac{1}{p+q-2}$ $\frac{1}{2(p+q-2)}$
DIII	$\mathrm{SO(2n)}/\mathrm{U}(n)$	$[\frac{1}{2}n]$	$n(n-1)$	$\frac{1}{2n-2}$
CI	$\mathrm{Sp}(n)/\mathrm{U}(n)$	n	$n(n+1)$	$\frac{1}{n+1}$
CII	$\mathrm{Sp}(p+q))/\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$\min(p,q)$	$4pq$	$\frac{1}{2(p+q+1)}$
EI – III		6/4/2	42/40/32	$\frac{1}{12}$
EIV		2	26	$\frac{1}{24}$
EV – VII		7/4/3	70/64/54	$\frac{1}{18}$
EVIII – IX		8/4	128/112	$\frac{1}{30}$
FI		4	28	$\frac{1}{9}$
FII		1	16	$\frac{1}{18}$
G		2	8	$\frac{1}{4}$

Remark 5.1 Finally, we would like to emphasize the normality for a compact irreducible symmetric space $M=U/K$ of compact type. The invariant Riemannian inner product at $o=\pi(e)$ is

$$\langle X,Y\rangle=-c\cdot B_{\mathfrak{u}}(X,Y)=-c\cdot B_{\mathfrak{g}}(X,Y),\quad X,Y\in\mathfrak{u},\ \mathfrak{g}=\mathfrak{u}^{\mathbb{C}}, \tag{5.11}$$

where $c>0$ is a positive constant, $B_{\mathfrak{u}}(X,Y)$ is the Killing form of $\mathfrak{u}$, and it is the restriction of the Killing form of $\mathfrak{g}$. In the above calculations, we choose $c=1$ (see (2.3)). Under this normalization, the Ricci curvature is $\frac{1}{2}$ for any case.

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