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On Completeness and Minimality of Random Exponential System in a Weighted Banach Space of Functions Continuous on the Real Line***

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Abstract In this paper, the completeness and minimality properties of some random exponential system in a weighted Banach space of complex functions continuous on the real line for convex nonnegative weight are studied. The results may be viewed as a probabilistic version of Malliavin's classical results.

Keywords Complete, Minimal, Closure, Random exponential polynomials
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1 Introduction: Problem and Results

The probabilistic approach to classical question on exponential systems gives a new insight and leads to new results combining the methods of probability theory and function theory. In contrast to the well developed theory of classical Fourier series with random coefficients (see [4, 7]), only a few facts are known in the case of random exponents. Here we mention [1, 2, 9, 10] which are devoted to the completeness, minimality and series expansion of random exponential systems. (On the definition of completeness and minimality of an exponential system, the readers can refer to [12].) Motivated by their works, we will study some random exponential system in a weighted Banach space with the help of probability theory.

The main purpose of the paper is to prove a probabilistic analogy of Malliavin's celebrated theorem on completeness of real exponential system in a weighted Banach space (see [6]). Furthermore, we study the minimality property of the random exponential system in the space.

Before formulating the main results of this paper, we first introduce some notations for convenience of the readers.

Let *a weight* α be a nonnegative convex function on $\mathbb{R}$ such that

$$\lim_{t\to+\infty} \alpha(t)/t = +\infty. \tag{1.1}$$

Consider the weighted Banach space

$$C_\alpha = \{f \in C(\mathbb{R}) : \lim_{|t|\to\infty} |f(t)e^{-\alpha(t)}| = 0 \ \text{ and } \ \|f\|_\alpha < \infty\},$$

where $C(\mathbb{R})$ is the set of complex functions continuous on $\mathbb{R}$ and

$$\|f\|_\alpha = \sup\{|f(t)e^{-\alpha(t)}| : t \in \mathbb{R}\}.$$

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Suppose that $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is a complex sequence such that

$$\delta(\Lambda) = \inf\{\mathrm{Re}(\lambda_{n+1} - \lambda_n) : n = 0, 1, 2, \cdots ; \lambda_0 = 0\} > 0, \tag{1.2}$$

$$\Theta(\Lambda) = \sup\{|\arg \lambda_n| : n = 1, 2, \cdots\} < \frac{\pi}{2}. \tag{1.3}$$

Let $\{\xi_n(\omega)\}$ be a sequence of independent real random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\xi_n = 0, \quad n = 1, 2, \cdots, \tag{1.4}$$

$$c_1 = \sup\{\mathbb{E}(|\xi_n|^{1+\sigma}) : n = 1, 2, \cdots\} < +\infty, \tag{1.5}$$

where $\sigma > 0$, and $\mathbb{E}(\xi)$ denotes the mathematical expectation of ξ.

Under the above assumption, we define

$$\lambda_n(\omega) = \lambda_n + i\xi_n(\omega), \quad \Lambda_\omega = \{\lambda_n(\omega)\}_{n=1}^{\infty} \quad \text{and} \quad \mathcal{E}(\Lambda_\omega) = \{e^{\lambda_n(\omega)t}\}_{n=1}^{\infty}. \tag{1.6}$$

The condition (1.1) guarantees that $\mathcal{E}(\Lambda_\omega) \subset C_\alpha$. Then we ask whether $\mathcal{E}(\Lambda_\omega)$ is complete or minimal in C_α. Our results are as follows.

Theorem 1.1 *Let $\alpha(t)$ be a nonnegative convex function satisfying* (1.1)*. Let $\mathcal{E}(\Lambda_\omega)$ be defined by* (1.6)*, where $\Lambda = \{\lambda_n\}$ is a complex sequence satisfying* (1.2)–(1.3) *and $\{\xi_n(\omega)\}$ is a sequence of independent real random variables satisfying* (1.4)–(1.5)*. Define $\lambda(r)$ as follows:*

$$\lambda(r) = \sum_{\mathrm{Re}\lambda_n \le r} \mathrm{Re}\frac{1}{\lambda_n}, \quad \text{if } r \ge \mathrm{Re}\lambda_1; \quad \lambda(r) = 0, \quad \text{otherwise}. \tag{1.7}$$

Then

(i) *If*

$$\int_0^{+\infty} \frac{\alpha(\lambda(t) - a)}{1 + t^2} dt = +\infty, \quad \forall\, a \in \mathbb{R}, \tag{1.8}$$

then $\mathcal{E}(\Lambda_\omega)$ is complete with probability 1.

(ii) *If there is some $b \in \mathbb{R}$ such that*

$$\int_0^{+\infty} \frac{\alpha(\lambda(t) - b)}{1 + t^2} dt < +\infty, \tag{1.9}$$

then $\mathcal{E}(\Lambda_\omega)$ is incomplete with probability 1.

(iii) *If* (1.8) *holds, then $\mathcal{E}(\Lambda_\omega)$ is not minimal with probability* 1*. If* (1.9) *holds, then $\mathcal{E}(\Lambda_\omega)$ is minimal with probability* 1.

Remark 1.2 The theorem when $\{\xi_n(\omega)\} = \{0\}$ includes Malliavin's result in [6]. So the conclusions (i) and (ii) can be rewritten into:

If $\{e^{\lambda_n t}\}$ is complete/incomplete in C_α, then so is $\{e^{\lambda_n(\omega)t}\}$ with probability 1.

Theorem 1.1 may be viewed as a probabilistic generalization of [6].

2 Preliminary Results

We need some auxiliary facts to prove our theorem.

Lemma 2.1 (See [6]) *Let $\beta(t)$ be a nonnegative convex function on $\mathbb{R}$ satisfying* (1.1)*, and assume that*

$$\beta^*(t) = \sup\{xt - \beta(x) : x \in \mathbb{R}\}, \quad t \in \mathbb{R} \tag{2.1}$$

is the Young transform (see [8]) *of the function* $\beta(x)$. *Suppose that* $\lambda(r)$ *is an increasing function on* $[0,\infty)$ *satisfying*

$$\lambda(R)-\lambda(r)\le A(\log R-\log r+1),\quad R>r>1. \tag{2.2}$$

Then there exists an analytic function $f(z)\not\equiv 0$ *in* $\mathbb{C}_+$ *satisfying*

$$|f(z)|\le A\exp\{Ax+\beta(x)-x\lambda(|z|)\},\quad z=x+iy\in\mathbb{C}_+, \tag{2.3}$$

if and only if there exists $a\in\mathbb{R}$ *such that*

$$\int_0^{+\infty}\frac{\beta^*(\lambda(t)-a)}{1+t^2}dt<\infty. \tag{2.4}$$

We note that in the whole paper A denotes the constant and A_ω denotes the positive number only depending on ω, whose values may be different in different cases.

Lemma 2.2 *Assume that the hypothesis of Theorem* 1.1 *holds. Then there is* $\Omega'\subset\Omega$ *such that* $\mathbb{P}(\Omega')=1$, *and for every* $\omega\in\Omega'$, *the function*

$$G(z)=\prod_{n=1}^{\infty}\Big(\frac{1-z/\lambda_n(\omega)}{1+z/\overline{\lambda}_n(\omega)}\Big)\exp\Big(\frac{z}{\lambda_n(\omega)}+\frac{z}{\overline{\lambda}_n(\omega)}\Big) \tag{2.5}$$

is analytic in the closed right half plane $\overline{\mathbb{C}}_+=\{z=x+iy:x\ge 0\}$, *and satisfies the following inequalities:*

$$|G(z)|\le\exp\{x\lambda(r)+A_\omega x\},\quad z\in\mathbb{C}_+, \tag{2.6}$$

$$|G(z)|\ge\exp\{x\lambda(r)-A_\omega x\},\quad z\in\Sigma(\Lambda_\omega), \tag{2.7}$$

where $r=|z|$, $\Sigma(\Lambda_\omega)=\{z\in\mathbb{C}_+:|z-\lambda_n(\omega)|\ge\frac{\delta(\Lambda)}{4},n=1,2,\cdots\}$ *and* $\lambda(r)$ *is defined by* (1.7).

We recall some theorems from probability theory that we need before we proceed to the proof of Lemma 2.2. We refer to [11] for complete proofs and comments.

Theorem A (Chebyshev Inequality) *Let* ξ *be a real-valued random variable and* $f(x)$ *be a non-decreasing positive continuous function. Then, for each* $a>0$,

$$\mathbb{P}\{|\xi|>a\}\le\frac{\mathbb{E}(f(|\xi|))}{f(a)}.$$

Theorem B (Borel-Cantelli Lemma) *Let* $E_1,E_2,\cdots,E_n,\cdots$ *be a sequence of events from a probability space and* $E=\bigcap\limits_{k=1}^{\infty}\bigcup\limits_{n=k}^{\infty}E_n$.

(1) *If* $\sum\limits_{n=1}^{+\infty}\mathbb{P}(E_n)<\infty$, *then* $\mathbb{P}(E)=0$;

(2) *If the events* E_n *are independent and* $\sum\limits_{n=1}^{+\infty}\mathbb{P}(E_n)=\infty$, *then* $\mathbb{P}(E)=1$.

Theorem C (Two-Series Theorem) *A sufficient condition for the convergence of the series* $\sum\limits_{n=1}^{+\infty}\xi_n$ *of independent random variables with probability* 1 *is that both series* $\sum\limits_{n=1}^{+\infty}D(\xi_n)$ *and*

$\sum_{n=1}^{+\infty}\mathbb{E}(\xi_n)$ *converge, where* $\mathbb{E}(\xi_n)$ *and* $D(\xi_n)$ *denote the mathematical expectation and variance of* ξ_n*, respectively.*

Now we begin to prove Lemma 2.2.

Proof of Lemma 2.2 For a fixed positive number τ, consider the truncated random variables

$$\xi_n^* = \begin{cases} \xi_n, & \text{if } |\xi_n| \le \tau|\lambda_n|, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Let $\lambda_n^*(\omega) = \lambda_n + i\xi_n^*(\omega)$. By the Chebyshev inequality,

$$\mathbb{P}\{\xi_n^* \ne \xi_n\} = \mathbb{P}\{|\xi_n| > \tau|\lambda_n|\} \le \frac{\mathbb{E}(|\xi_n|^{1+\sigma})}{\tau^{1+\sigma}|\lambda_n|^{1+\sigma}} \le \frac{c_1\tau^{-(1+\sigma)}}{|\lambda_n|^{1+\sigma}}. \tag{2.9}$$

The separation condition (1.2) yields $\sum_{n=1}^{\infty}|\lambda_n|^{-(1+\sigma)} < +\infty$. Hence

$$\sum_{n=1}^{+\infty}\mathbb{P}\{\xi_n^* \ne \xi_n\} < +\infty. \tag{2.10}$$

By the Borel-Cantelli Lemma we have

$$\mathbb{P}\Big\{\bigcap_{k=1}^{+\infty}\bigcup_{n=1}^{+\infty}\{\omega : \xi_n(\omega) \ne \xi_n^*(\omega)\}\Big\} = 0. \tag{2.11}$$

Set

$$\Omega_1 = \Omega\Big\backslash\bigcap_{k=1}^{+\infty}\bigcup_{n=1}^{+\infty}\{\omega : \xi_n^*(\omega) \ne \xi_n(\omega)\}.$$

Then $\mathbb{P}(\Omega_1) = 1$ and for each $\omega \in \Omega_1$,

$$\#\{n : \lambda_n^*(\omega) \ne \lambda_n(\omega)\} = \#\{n : \xi_n^*(\omega) \ne \xi_n(\omega)\} < +\infty, \tag{2.12}$$

where $\#E$ denotes the number of elements in the set E. The condition (1.3) yields that

$$\Big|\frac{\operatorname{Im}\lambda_n}{\operatorname{Re}\lambda_n}\Big| \le \tan\Theta(\Lambda), \qquad \frac{1}{\operatorname{Re}|\lambda_n|} \le \frac{1}{|\lambda_n|\cos\Theta(\Lambda)}.$$

Then

$$\begin{aligned} |\arg\lambda_n^*(\omega)| &= \Big|\arctan\frac{\operatorname{Im}\lambda_n + \xi_n^*}{\operatorname{Re}\lambda_n}\Big| \le \arctan\Big(\Big|\frac{\operatorname{Im}\lambda_n}{\operatorname{Re}\lambda_n}\Big| + \Big|\frac{\xi_n^*}{\operatorname{Re}\lambda_n}\Big|\Big) \\ &\le \arctan\Big(\frac{\sin\Theta(\Lambda)+\tau}{\cos\Theta(\Lambda)}\Big) = A_1 < \frac{\pi}{2}, \end{aligned}$$

where A_1 is a positive constant independent of n. Now by (2.12), for every $\omega \in \Omega_1$, there exists a positive number $\Theta(\Lambda_\omega)$ such that

$$|\arg\lambda_n(\omega)| \le \max\{\{|\arg\lambda_n(\omega)| : \lambda_n(\omega) \ne \lambda_n^*(\omega)\} \cup \{A_1\}\} = \Theta(\Lambda_\omega) < \frac{\pi}{2}.$$

Besides,

$$\inf_n\{\operatorname{Re}(\lambda_{n+1}(\omega) - \lambda_n(\omega))\} = \inf_n\{\operatorname{Re}(\lambda_{n+1} - \lambda_n)\} = \delta(\Lambda).$$

Then using a method similar to [3], we have the following estimates:

For every $\omega \in \Omega_1$, the function $G_\omega(z)$ defined by (2.5) is analytic in the closed right half plane $\overline{\mathbb{C}}_+ = \{z = x + iy : x \geq 0\}$ and satisfies:

$$|G_\omega(z)| \leq \exp\{x\lambda_\omega(|z|) + A_\omega x\}, \quad z \in \mathbb{C}_+,$$
$$|G_\omega(z)| \geq \exp\{x\lambda_\omega(|z|) - A_\omega x\}, \quad z \in \Sigma(\Lambda_\omega),$$

where A_ω is a positive number only depending on ω and

$$\lambda_\omega(r) = \sum_{\mathrm{Re}\lambda_n \leq r} \mathrm{Re}\Big(\frac{1}{\lambda_n(\omega)}\Big), \quad \text{if } r \geq \mathrm{Re}\,\lambda_1; \quad \lambda(r) = 0, \quad \text{otherwise.}$$

To complete the proof, we only need to prove that $|\lambda_\omega(r) - \lambda(r)| \leq A_\omega$ holds for almost every $\omega \in \Omega_1$. For every $\omega \in \Omega_1$, since (2.12) holds, we have

$$\begin{aligned}
&|\lambda_\omega(r) - \lambda(r)| \\
&\leq 2\Big|\sum_{\mathrm{Re}\,\lambda_n \leq r} \mathrm{Re}\,\frac{1}{\lambda_n + i\xi_n^*} - \sum_{\mathrm{Re}\,\lambda_n \leq r} \mathrm{Re}\,\frac{1}{\lambda_n}\Big| + A_\omega = 2\Big|\sum_{\mathrm{Re}\,\lambda_n \leq r}\Big[\frac{\mathrm{Re}\,\lambda_n}{|\lambda_n + i\xi_n^*|^2} - \frac{\mathrm{Re}\,\lambda_n}{|\lambda_n|^2}\Big]\Big| + A_\omega \\
&= 2\Big|\sum_{\mathrm{Re}\,\lambda_n \leq r} \frac{\mathrm{Re}\,\lambda_n[(\xi_n^*)^2 + 2\mathrm{Im}\,\lambda_n\xi_n^*]}{|\lambda_n|^2[(\mathrm{Re}\,\lambda_n)^2 + (\xi_n^* + \mathrm{Im}\,\lambda_n)^2]}\Big| + A_\omega \leq 2\sum_{\mathrm{Re}\,\lambda_n \leq r} \frac{|\xi_n^*|^2 + 2|\xi_n^*||\mathrm{Im}\,\lambda_n|}{|\lambda_n|^2\mathrm{Re}\,\lambda_n} + A_\omega \\
&\leq \frac{2}{\cos(\Theta(\Lambda))}\sum_{n=1}^{\infty} \frac{\tau|\lambda_n||\xi_n^*| + 2|\lambda_n||\xi_n^*|}{|\lambda_n|^3} + A_\omega \leq \frac{2(\tau+2)}{\cos(\Theta(\Lambda))}\sum_{n=1}^{\infty}\frac{|\xi_n^*|}{|\lambda_n|^2} + A_\omega.
\end{aligned}$$

By (1.4) and Hölder inequality,

$$\mathbb{E}|\xi_n^*| \leq \mathbb{E}(|\xi_n|) \leq [\mathbb{E}(|\xi_n|^{1+\sigma})]^{\frac{1}{1+\sigma}} \leq c_1^{\frac{1}{1+\sigma}} < +\infty.$$

Combining it with (1.2), we have

$$\sum_{n=1}^{+\infty} \frac{\mathbb{E}|\xi_n^*|}{|\lambda_n|^2} \leq c_1^{\frac{1}{1+\sigma}} \sum_{n=1}^{+\infty} \frac{1}{|\mathrm{Re}\,\lambda_n|^2} < +\infty,$$
$$\sum_{n=1}^{+\infty} \frac{\mathbb{E}|\xi_n^*|^2}{|\lambda_n|^4} \leq \sum_{n=1}^{+\infty} \frac{\tau\mathbb{E}|\xi_n^*|}{|\lambda_n|^3} \leq \tau c_1^{\frac{1}{1+\sigma}} \sum_{n=1}^{+\infty} \frac{1}{|\mathrm{Re}\,\lambda_n|^3} < +\infty.$$

Then according to Two-series Theorem, there exists $\Omega_2 \subset \Omega$ such that $\mathbb{P}(\Omega_2) = 1$ and $A_\omega = \sum_{n=1}^{+\infty} \frac{|\xi_n^*|}{|\lambda_n|^2} < +\infty$ for every $\omega \in \Omega_2$. Let $\Omega' = \Omega_1 \bigcap \Omega_2$. Then it follows that $\mathbb{P}(\Omega') = 1$ and $|\lambda_\omega(r) - \lambda(r)| \leq A_\omega$ for $\omega \in \Omega'$. Taking account of the above estimates, we have now proved the lemma.

3 Proof of the Main Theorem

Proof of Theorem 1.1 Below we will prove the main theorem in order.

(i) Suppose its contrary holds, i.e., there exists $E \subset \Omega$ such that $\mathbb{P}(E) > 0$ and for every $\omega \in E$, $\mathcal{E}(\Lambda_\omega)$ is incomplete in C_α. By Hahn-Banach Theorem, it is equivalent to say there exists a nonzero bounded linear functional T_ω on C_α vanishing on $\mathcal{E}(\Lambda_\omega)$. So by the Riesz representation Theorem, there exists a complex measure μ_ω satisfying

$$\|\mu_\omega\| = \int_{-\infty}^{+\infty} e^{\alpha(t)}|d\mu_\omega| = \|T_\omega\|,$$
$$T_\omega(h) = \int_{-\infty}^{+\infty} h(t)d\mu_\omega, \quad \forall\, h \in C_\alpha,$$

where $\omega \in E$. Because $\mathbb{P}(E\bigcap\Omega') = \mathbb{P}(E) > 0$, we can take one $\varpi \in E\bigcap\Omega'$, where Ω' is as mentioned in Lemma 2.2. Then the function

$$f_\varpi(z) = \frac{1}{G_\varpi(z)}\int_{-\infty}^{+\infty} e^{tz}d\mu_\varpi$$

is analytic in the open right half plane $\mathbb{C}_+$ and continuous in the closed right half plane $\overline{\mathbb{C}}_+$ satisfying

$$|f_\varpi(z)| \le \|\mu_\varpi\|\exp\{\alpha^*(x) - x\lambda(|z|) + A_\varpi x\}.$$

Lemma 2.1 shows that there exists a constant b such that (1.9) holds, which contradicts (1.8). Hence the assertion of (i) follows from the contradiction.

(ii) Suppose that (1.9) holds for some real number b. Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\lambda(t) - b)$ for $t \ge 0$ and let $u(z)$ be the Poisson integral of $\varphi(t)$, i.e.,

$$u(x+iy) = \frac{x}{\pi}\int_{-\infty}^{+\infty}\frac{\varphi(t)}{x^2+(y-t)^2}dt.$$

Then $u(x+iy)$ is harmonic in the half plane $\mathbb{C}_+$ and there exists an analytic function $T(z)$ on $\mathbb{C}_+$ satisfying

$$Ax \ge \operatorname{Re} T(z) = u(z) \ge (x-1)(\lambda(r) - b) - \alpha^*(x-1), \quad x \ge 1. \tag{3.1}$$

Therefore for every $\omega \in \Omega'$, taking Lemma 2.2 and the above inequality into account and properly choosing the number N_ω, we can establish that the function

$$g_\omega(z) = \frac{G_\omega(z)}{(1+z)^{N_\omega}}\exp\{-T(z) - N_\omega(z) - N_\omega\} \tag{3.2}$$

satisfies the following inequality

$$|g_\omega(z)| \le \frac{1}{1+|z|^2}\exp\{\alpha^*(x-1) - x\}. \tag{3.3}$$

Set

$$h_\omega(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_\omega(1+iy)e^{-(1+iy)t}dy. \tag{3.4}$$

Then $h_\omega(t)$ is continuous on $\mathbb{R}$. Moreover, by (3.3) and Cauchy contour theorem,

$$h_\omega(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g_\omega(x+iy)e^{-(x+iy)t}dy, \quad x > 0 \tag{3.5}$$

is independent of x and hence the Young transform formula $\alpha = (\alpha^*)^*$ yields

$$|h_\omega(t)e^{\alpha(t)}| \le \exp\{-|t|\}. \tag{3.6}$$

Since $h_\omega(t)e^{xt}$ can be viewed as the Fourier transform of $g_\omega(x+iy)$, the inverse transform shows

$$g_\omega(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_\omega(t)e^{zt}dt, \quad \mathrm{Re}\, z > 0. \tag{3.7}$$

Thus the functional T_ω defined by

$$T_\omega(f) = \int_{-\infty}^{+\infty} f(t)h_\omega(t)dt \tag{3.8}$$

satisfies

$$T_\omega(e^{\lambda_n(\omega)t}) = \sqrt{2\pi} g_\omega(\lambda_n(\omega)) = 0, \tag{3.9}$$

and according to (3.6),

$$\|T_\omega\| = \int_{-\infty}^{+\infty} h_\omega(t)e^{\alpha(t)}dt < +\infty.$$

So T_ω is a nonzero bounded linear functional on C_α and hence by Hahn-Banach Theorem, $\mathcal{E}(\Lambda_\omega)$ is incomplete in C_α for every $\omega \in \Omega'$. This completes the proof.

(iii) If (1.8) holds, then by (i) $\mathcal{E}(\Lambda_\omega)\backslash\{e^{\lambda_k(\omega)t}\}$ ($\forall\, k \in \mathbb{N}$) is complete with probability 1. So $\mathcal{E}(\Lambda_\omega)$ is not minimal with probability 1.

If (1.9) holds, then we can construct $g_\omega(z)$ as in (ii) for every $\omega \in \Omega'$ that is defined in Lemma 2.2.

Letting $A_{n,\omega}$ be the coefficient of the singular part of the Laurent series of $1/g_\omega(z)$ in $U(\lambda_n(\omega)) = \{z : |z - \lambda_n(\omega)| \le \frac{\delta(\Lambda)}{2}\}$, we have

$$\frac{1}{g_\omega(z)} = \frac{A_{n,\omega}}{z - \lambda_n(\omega)} + g_{n,\omega}(z), \tag{3.10}$$

where $g_{n,\omega}(z)$ is analytic in $U(\lambda_n(\omega))$. Then

$$A_{n,\omega} = \frac{1}{2\pi i} \int_{|z-\lambda_n(\omega)|=\frac{\delta(\Lambda)}{4}} \frac{1}{g_\omega(z)}dz. \tag{3.11}$$

But by (2.7) and (3.2), we get

$$|A_{n,\omega}| \le \frac{1}{1+|z|^2} \exp\{-\mathrm{Re}\,\lambda_n \lambda(\mathrm{Re}\,\lambda_n) + A_\omega \mathrm{Re}\,\lambda_n + A_\omega\}. \tag{3.12}$$

Consider the analytic functions on $\mathbb{C}_+$,

$$H_{n,\omega}(z) = \frac{A_{n,\omega} g_\omega(z)}{z - \lambda_n(\omega)}, \quad n \in \mathbb{N}. \tag{3.13}$$

Combining (3.3) with (3.12), we obtain that for $n \in \mathbb{N}$,

$$|H_{n,\omega}(z)| \le \frac{1}{1+|z|^2} \exp\{\alpha^*(x-1) - x - \mathrm{Re}\,\lambda_n \lambda(|\lambda_n(\omega)|) + A_\omega \mathrm{Re}\,\lambda_n + A_\omega\}. \tag{3.14}$$

Let

$$h_{n,\omega}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_{n,\omega}(1+iy)e^{-(1+iy)t}dy, \quad n \in \mathbb{N}. \tag{3.15}$$

Then $h_{n,\omega}(t)$ is continuous on $\mathbb{R}$. By Cauchy contour theorem, the above estimates (3.14) yield that

$$h_{n,\omega}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H_{n,\omega}(x+iy)e^{-(x+iy)t}dy, \quad n \in \mathbb{N},\ \forall\, x > 0, \tag{3.16}$$

$$|h_{n,\omega}(t)e^{\alpha(t)}| \le \exp\{-\operatorname{Re}\lambda_n\lambda(\operatorname{Re}\lambda_n) + A_\omega \operatorname{Re}\lambda_n + A_\omega - |t|\}. \tag{3.17}$$

Then by Fourier transform theory,

$$H_{n,\omega}(z) = \int_{-\infty}^{+\infty} h_{n,\omega}(t)e^{zt}dt, \quad \operatorname{Re} z > 0. \tag{3.18}$$

It is easy to see that

$$H_{n,\omega}(\lambda_j(\omega)) = \delta_{nj} \quad \text{(the Kronecker notion)} \tag{3.19}$$

and hence

$$\int_{-\infty}^{+\infty} h_{n,\omega}(t)e^{\lambda_j(\omega)t}dt = \delta_{nj}. \tag{3.20}$$

Define functionals $T_{n,\omega}$ on C_α by

$$T_{n,\omega}(f) = \int_{-\infty}^{+\infty} f(t)h_{n,\omega}(t)dt, \quad \forall f \in C_\alpha. \tag{3.21}$$

Then by (3.17) we see that $T_{n,\omega} \in (C_\alpha)^*$ $(n \in \mathbb{N})$ and

$$\|T_{n,\omega}\| \le 2\exp\{-\operatorname{Re}\lambda_n\lambda(\operatorname{Re}\lambda_n) + A_\omega \operatorname{Re}\lambda_n + A_\omega\}. \tag{3.22}$$

According to [12, Problem 2, p.24] or [5, Lecture 18], the following proposition holds:

A system $\{x_k\}$ of elements of the Banach space X is minimal if and only if there is a biorthogonal system of functionals $\{f_k\} \subset X^*$, which means $f_k(x_m) = \delta_{km}$.

By (3.20) we see that the system $\{T_{n,\omega}\}$ is biorthogonal to $\mathcal{E}(\Lambda_\omega)$ and hence $\mathcal{E}(\Lambda_\omega)$ is minimal. The above argument works for each $\omega \in \Omega'$. So $\mathcal{E}(\Lambda_\omega)$ is minimal with probability 1.

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