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Two New Families in the Stable Homotopy Groups of Sphere and Moore Spectrum**

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Abstract This paper proves the existence of an order p element in the stable homotopy group of sphere spectrum of degree p^nq+p^mq+q-4 and a nontrivial element in the stable homotopy group of Moore spectum of degree p^nq+p^mq+q-3 which are represented by $h_0(h_mb_{n-1}-h_nb_{m-1})$ and $i_*(h_0h_nh_m)$ in the E_2-terms of the Adams spectral sequence respectively, where $p\geq 7$ is a prime, $n\geq m+2\geq 4,\ q=2(p-1)$.

Keywords Stable homotopy groups of spheres, Adams spectral sequence, Toda spectrum
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1 Introduction

Let A be the mod p Steenrod algebra and S the sphere spectrum localized at an odd prime p. To determine the stable homotopy groups of spheres π_*S is one of the central problem in homotopy theory. One of the main tools to reach it is the Adams spectral sequence (ASS) $E_2^{s,t}=\mathrm{Ext}_A^{s,t}(Z_p,Z_p)\Longrightarrow\pi_{t-s}S$, where the $E_2^{s,t}$-term is the cohomology of A. If a family of generators x_i in $E_2^{s,t}$ converges nontrivially in the ASS, then we get a family of nontrivial homotopy elements f_i in π_*S and we call f_i is represented by $x_i\in E_2^{s,t}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. For example, a family $\zeta_{n-1}\in\pi_{p^nq+q-3}S$ for $n\geq 2$ which has filtration 3 and is represented by $h_0b_{n-1}\in\mathrm{Ext}_A^{3,p^nq+q}(Z_p,Z_p)$ has been detected in [2], where $q=2(p-1)$.

From [5], $\mathrm{Ext}_A^{1,*}(Z_p,Z_p)$ has Z_p-base consisting of $a_0\in\mathrm{Ext}_A^{1,1}(Z_p,Z_p)$, $h_i\in\mathrm{Ext}_A^{1,p^iq}(Z_p,Z_p)$ for all $i\geq 0$ and $\mathrm{Ext}_A^{2,*}(Z_p,Z_p)$ has Z_p-base consisting of $\tilde{\alpha}_2, a_0^2, a_0h_i\ (i>0),\ g_i\ (i\geq 0),\ k_i\ (i\geq 0),\ b_i\ (i\geq 0)$ and $h_ih_j\ (j\geq i+2, i\geq 0)$ whose internal degrees are $2q+1,\ 2,\ p^iq+1,\ p^{i+1}q+2p^iq,\ 2p^{i+1}q+p^iq,\ p^{i+1}q$ and p^iq+p^jq respectively.

Let M be the Moore spectrum given by the cofibration

$$S\xrightarrow{p}S\xrightarrow{i}M\xrightarrow{j}\Sigma S \tag{1.1}$$

and K be the cofibre of the Adams map $\alpha:\Sigma^qM\to M$ given by the cofibration

$$\Sigma^qM\xrightarrow{\alpha}M\xrightarrow{i'}K\xrightarrow{j'}\Sigma^{q+1}M. \tag{1.2}$$

The above spectrum K actually is the Toda-Smith spectrum $V(1)$.

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From [8, Theorem 1.2.14, p.11], there is a nontrivial differential in the ASS

$$d_2(h_n) = a_0 b_{n-1} \in E_2^{3,tq+1} = \mathrm{Ext}_A^{3,tq+1}(Z_p, Z_p), \quad n \geq 1. \tag{1.3}$$

The elements $h_n \in \mathrm{Ext}_A^{1,p^n q}(Z_p, Z_p)$ and $b_{n-1} \in \mathrm{Ext}_A^{2,p^n q}(Z_p, Z_p)$ are called a pair of a_0-related elements. Theorem IV in [2] states the following result on the a_0-related elements h_n and b_{n-1} : $h_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^n q+q}(Z_p, Z_p)$ is a permanent cycle in the ASS and it converges to a homotopy element $\zeta_{n-1} \in \pi_{p^n q+q-3} S$ of order p; moreover, $i_*(h_0 h_n) \in \mathrm{Ext}_A^{2,p^n q+q}(H^*M, Z_p)$ also is a permanent cycle in the ASS which converges to a nontrivial element in $\pi_{p^n q+q-2} M$.

As a consequence of (1.3) we have

$$d_2(h_n h_m) = a_0(h_m b_{n-1} - h_n b_{m-1}) \in E_2^{4,tq+1} = \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p) \tag{1.4}$$

with $tq = p^n q + p^m q,\ n \geq m+2 \geq 3$. That is, $h_n h_m$ and $(h_m b_{n-1} - h_n b_{m-1})$ are another pair of a_0-related elements. The main purpose of this paper is to prove the following result on these a_0-related elements which is an analogue of Theorem IV in [2].

Theorem A *Let $p \geq 7,\ n \geq m+2 \geq 4$. Then*

$$h_0(h_m b_{n-1} - h_n b_{m-1}) \in \mathrm{Ext}_A^{4,p^n q+p^m q+q}(Z_p, Z_p)$$

is a permanent cycle in the ASS which converges to an element in $\pi_{p^n q+p^m q+q-4} S$ of order p. Moreover

$$i_*(h_0 h_n h_m) \in \mathrm{Ext}_A^{3,p^n q+p^m q+q}(H^*M, Z_p)$$

also is a permanent cycle which converges to a nontrivial element in $\pi_{p^n q+p^m q+q-3} M$.

Remark The $h_0(h_m b_{n-1} - h_n b_{m-1})$-map obtained in Theorem A is represented by

$$\beta_{p^{m-1}/p^{m-1}-1}\beta_{p^{n-1}/p^{n-1}} - \beta_{p^{n-1}/p^{n-1}-1}\beta_{p^{m-1}/p^{m-1}} + \text{other terms} \in \mathrm{Ext}_{BP_*BP}^{4,p^n q+p^m q+q}(BP_*, BP_*)$$

and $i_*(h_0 h_n h_m)$-map in $\pi_{p^n q+p^m q+q-3} M$ is represented by

$$h_0 h_n h_m + \text{other terms} \in \mathrm{Ext}_{BP_*BP}^{3,p^n q+p^m q+q}(BP_*, BP_*M)$$

in the Adams-Novikov spectral sequence, where

$$\beta_{p^{n-1}/p^{n-1}-1} \in \mathrm{Ext}_A^{2,p^n q+q}(BP_*, BP_*), \quad \beta_{p^{n-1}/p^{n-1}} \in \mathrm{Ext}_{BP_*BP}^{2,p^n q}(BP_*, BP_*)$$

such that the images under the Thom map are

$$h_0 h_n \in \mathrm{Ext}_A^{2,p^n q+q}(Z_p, Z_p), \quad b_{n-1} \in \mathrm{Ext}_A^{2,p^n q}(Z_p, Z_p)$$

respectively and $h_n \in \mathrm{Ext}_{BP_*BP}^{1,p^n q}(BP_*, BP_*M)$ is the generator represented by $[t_1^{p^n}]$ in the cobar complex.

Theorem A will be proved by some arguments processing in the Adams resolution of certain spectra related to S and K. The only geometric input used in the proof is the nontrivial differential (1.4). After giving some preliminaries on low dimensional Ext groups in Section 2, the proof of Theorem A will be given in Section 3.

2 Some Preliminaries on Low Dimensional Ext Groups

In this section, we consider some result on low dimensional Ext groups and some spectra closely related to S which will be used in the proof of Theorem A.

Proposition 2.1 *Let* $p \geq 7,\ n \geq m+2 \geq 4,\ tq = p^n q + p^m q$. *Then*

(1) $\mathrm{Ext}_A^{4,tq+rq+u}(Z_p, Z_p) = 0$ *for* $r = 2,3,4,\ u = -1,0$ *or* $r = 3,4,\ u = 1$,
$\mathrm{Ext}_A^{4,tq}(Z_p, Z_p) \cong Z_p\{b_{n-1}b_{m-1}\}$, $\mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p) \cong Z_p\{a_0 h_n b_{m-1}, a_0 h_m b_{n-1}\}$,
$\mathrm{Ext}_A^{4,tq+q}(Z_p, Z_p) \cong Z_p\{h_0 h_n b_{m-1}, h_0 h_m b_{n-1}\}$.

(2) $\mathrm{Ext}_A^{5,tq+rq+1}(Z_p, Z_p) = 0$ *for* $r = 1,3,4$, $\mathrm{Ext}_A^{5,tq+rq}(Z_p, Z_p) = 0$ *for* $r = 2,3$,
$\mathrm{Ext}_A^{5,tq+2q+1}(Z_p, Z_p) \cong Z_p\{\tilde{\alpha}_2 h_n b_{m-1}, \tilde{\alpha}_2 h_m b_{n-1}\}$,
$\mathrm{Ext}_A^{5,tq+2}(Z_p, Z_p) \cong Z_p\{a_0^2 h_n b_{m-1}, a_0^2 h_m b_{n-1}\}$, $\mathrm{Ext}_A^{5,tq+1}(Z_p, Z_p) \cong Z_p\{a_0 b_{n-1} b_{m-1}\}$,
$a_0^2 b_{n-1} b_{m-1} \neq 0 \in \mathrm{Ext}_A^{6,tq+2}(Z_p, Z_p)$.

Proof From [8, Theorem 3.2.5, p.82] , there is a May spectral sequence (MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ with E_1-term

$$E_1^{*,*,*} = E(h_{i,j} \mid i > 0, j \geq 0) \otimes P(b_{i,j} \mid i > 0, j \geq 0) \otimes P(a_i \mid i \geq 0),$$

where E is the exterior algebra and P the polynomial algebra and

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1}, \quad b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}, \quad a_i \in E_1^{1,2p^i-1,2i+1}.$$

Observe the second degree of the following generators (mod $p^n q$) for $0 \leq i \leq n, n \geq m+2 \geq 4$,

$$|h_{s,i}| = \begin{cases} (p^{s+i-1} + \cdots + p^i)q & (\operatorname{mod} p^n q), \quad 0 \leq i < s+i-1 < n, \\ (p^{n-1} + \cdots + p^i)q & (\operatorname{mod} p^n q), \quad 0 \leq i < s+i-1 = n, \end{cases}$$

$$|b_{s,i-1}| = \begin{cases} (p^{s+i-1} + \cdots + p^i)q & (\operatorname{mod} p^n q), \quad 1 \leq i < s+i-1 < n, \\ (p^{n-1} + \cdots + p^i)q & (\operatorname{mod} p^n q), \quad 1 \leq i < s+i-1 = n, \end{cases}$$

$$|a_{i+1}| = (p^i + \cdots + 1)q + 1 \qquad (\operatorname{mod} p^n q), \quad 1 \leq i < n,$$

$$|a_{i+1}| = (p^{n-1} + \cdots + 1)q + 1 \qquad (\operatorname{mod} p^n q), \quad i = n.$$

At degree $k = tq + rq + u$ with $0 \leq r \leq 4,\ -1 \leq u \leq 2$, $k = p^m q + rq + u\ (\operatorname{mod} p^n q)$. Then, for $3 \leq w \leq 5$, $E_1^{w,tq+rq+u,*}$ has no generator which has factors consisting of the above elements, because such a generator will have second degree $(c_n p^{n-1} + \cdots + c_1 p + c_0)q + d\ (\operatorname{mod} p^n q)$ with some $c_i \neq 0$ ($1 \leq i \leq m-1$ or $m < i < n$), where $0 \leq c_l < p$, $l = 0, \cdots, n$, $0 \leq d \leq 5$. Moreover, the second degree $|b_{1,i-1}| = p^i q\ (\operatorname{mod} p^n q)$ for $1 \leq i \leq n$, $|h_{1,i}| = p^i q\ (\operatorname{mod} p^n q)$ for $0 \leq i \leq n$. Then excluding the above factors and factors with second degree $\geq tq + pq$, we know that the only possibly factor of the generator in $E_1^{w,tq+rq+u,*}$ are $a_1, a_0, h_{1,0}, h_{1,n}, h_{1,m}$, $b_{1,n-1}, b_{1,m-1}$.

So, by degree reasons we have

$$E_1^{4,tq+rq+1,*} = 0 \ \text{ for } \ r = 3,4, \quad E_1^{4,tq+rq+u,*} = 0 \ \text{ for } \ r = 2,3,4,\ u = -1,0,$$

$$E_1^{4,tq,*} = Z_p\{b_{1,n-1}b_{1,m-1}\}, \quad E_1^{4,tq+1,*} \cong Z_p\{a_0 h_{1,n} b_{1,m-1}, a_0 h_{1,m} b_{1,n-1}\},$$

$$E_1^{4,tq+2,*} = Z_p\{a_0^2 h_{1,n} h_{1,m}\},$$
$$E_1^{4,tq+2q+1,*} = Z_p\{h_{1,0} a_1 h_{1,n} h_{1,m}\}, \quad E_1^{4,tq+q,*} = Z_p\{h_{1,0} h_{1,n} b_{1,m-1}, h_{1,0} h_{1,m} b_{1,n-1}\},$$
$$E_1^{3,tq+1,*} = Z_p\{a_0 h_{1,n} h_{1,m}\}, \quad E_1^{3,tq,*} = Z_p\{h_{1,n} b_{1,m-1}, h_{1,m} b_{1,n-1}\},$$
$$E_1^{3,tq+q,*} = Z_p\{h_{1,0} h_{1,n} h_{1,m}\}, \quad E_1^{3,tq+2q+1,*} = 0.$$

Note that the differential in the MSS is derivative, that is,

$$d_r(xy) = d_r(x)y + (-1)^s x d_r(y) \quad \text{for } x \in E_1^{s,t,*},\ y \in E_1^{s',t',*}.$$

Moreover, $a_0, h_{1,n}, b_{1,n-1}, h_{1,0}a_1$ are permanent cycles in the MSS which converge to

$$a_0, h_n, b_{n-1}, \tilde{\alpha}_2 \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$$

respectively. Then the differential $d_r E_r^{3,tq+sq+u,*} = 0$ for all $r \geq 1$ and $s = u = 0$ or $s = 1, u = 0$ or $s = 0, u = 1$ or $s = 2, u = 1$. Hence,

$$b_{1,n-1} b_{1,m-1}, a_0 h_{1,n} b_{1,m-1}, a_0 h_{1,m} b_{1,n-1}, h_{1,0} h_{1,n} b_{1,m-1}, h_{1,0} h_{1,m} b_{1,n-1} \in E_r^{4,*,*}$$

do not bound in the MSS and so $b_{n-1}b_{m-1}, a_0 h_n b_{m-1}, a_0 h_m b_{n-1}, h_0 h_n b_{m-1}, h_0 h_m b_{n-1}$ are all nonzero in $\mathrm{Ext}_A^{4,*}(Z_p, Z_p)$. This completes the proof of (1).

Similarly, by degree reasons we have

$$E_1^{5,tq+q+1,*} \cong Z_p\{a_0 h_{1,0} h_{1,n} b_{1,m-1}, a_0 h_{1,0} h_{1,m} b_{1,n-1}, a_1 b_{1,n-1} b_{1,m-1}\},$$
$$E_1^{5,tq+rq+1,*} = 0 \ \text{ for } \ r = 3, 4, \quad E_1^{5,tq+rq,*} = 0 \ \text{ for } \ r = 2, 3,$$
$$E_1^{5,tq+2q+1,*} \cong Z_p\{h_{1,0} a_1 h_{1,n} b_{1,m-1}, h_{1,0} a_1 h_{1,m} b_{1,n-1}\},$$
$$E_1^{5,tq+1,*} = Z_p\{a_0 b_{1,n-1} b_{1,m-1}\}, \quad E_1^{5,tq+2,*} = Z_p\{a_0^2 h_{1,m} b_{1,n-1}, a_0^2 h_{1,n} b_{1,n-1}\},$$
$$E_1^{4,tq+2q+1,*} \cong Z_p\{h_{1,0} a_1 h_{1,n} h_{1,m}\}.$$

The generators in $E_1^{5,tq+q+1,*}$ all die in the MSS since

$$a_0 h_{1,0} h_{1,n} b_{1,m-1} = -d_1(a_1 h_{1,n} b_{1,m-1}), \quad a_0 h_{1,0} h_{1,m} b_{1,n-1} = -d_1(a_1 h_{1,m} b_{1,n-1}),$$
$$d_1(a_1 b_{1,n-1} b_{1,m-1}) = -a_0 h_{1,0} b_{1,n-1} b_{1,m-1} \neq 0 \in E_1^{5,tq+q+1,*},$$

then $\mathrm{Ext}_A^{5,tq+q+1}(Z_p, Z_p) = 0$. Moreover, by the same reason as shown in the proof of (1),

$$d_r E_r^{4,tq+u,*} = 0, \quad d_r E_r^{4,tq+2q+1,*} = 0 \quad \text{for all } \ r \geq 1,\ u = 1, 2.$$

So the generators in $E_1^{5,*,*}$ converges nontrivially in the MSS to

$$\tilde{\alpha}_2 h_n b_{m-1}, \ \ \tilde{\alpha}_2 h_m b_{n-1}, \ \ a_0 b_{n-1} b_{m-1}, \ \ a_0^2 h_m b_{n-1}, \ \ a_0^2 h_n b_{m-1}$$

respectively. For the last result, note that $d_r E_r^{5,tq+2,*} = 0$ for all $r \geq 1$ and so

$$a_0^2 b_{n-1} b_{m-1} \neq 0 \in \mathrm{Ext}_A^{6,tq+2}(Z_p, Z_p).$$

This completes the proof of (2).

Now we consider some spectra related to S, M or K. Let L be the cofibre of $\alpha_1 = j\alpha i : \Sigma^{q-1}S \to S$ given by the cofibration

$$\Sigma^{q-1}S \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^q S. \tag{2.1}$$

Let Y be the cofibre of $i'i : S \to K$ given by the cofibration

$$S \xrightarrow{i'i} K \xrightarrow{\bar{r}} Y \xrightarrow{\epsilon} \Sigma S. \tag{2.2}$$

Y actually is the Toda spectrum $V(1\frac{1}{2})$ and it also is the cofibre of $j\alpha : \Sigma^q M \to \Sigma S$ given by the cofibration

$$\Sigma^q M \xrightarrow{j\alpha} \Sigma S \xrightarrow{\overline{w}} Y \xrightarrow{\bar{u}} \Sigma^{q+1}M. \tag{2.3}$$

This can be seen by the following homotopy commutative (up to sign) diagram of 3×3 Lemma in the stable homotopy category (cf. [9, pp.292–293])

$$\begin{array}{ccccccc}
S & \xrightarrow{i'i} & K & \xrightarrow{j'} & \Sigma^{q+1}M & & \\
\searrow i & \nearrow i' & & \searrow \bar{r} & \nearrow \bar{u} & & \\
 & M & & & Y & & \\
\nearrow \alpha & \searrow j & & \nearrow \overline{w} & \searrow \epsilon & & \\
\Sigma^q M & \xrightarrow{j\alpha} & \Sigma S & \xrightarrow{p} & \Sigma S & &
\end{array}$$

Note that $\alpha_1 \cdot p = p \cdot \alpha_1 = 0$, and then $p = j''\pi$ and $p = \xi i''$ with $\pi \in [\Sigma^q S, L]$ and $\xi \in [L, S]$. Since $\pi_q S = 0$, we have $\pi_q L \cong Z_{(p)}\{\pi\}$. Moreover, $i''\xi i'' = i'' \cdot p = (p \wedge 1_L)i''$, and then $p \wedge 1_L = i''\xi + \lambda\pi j''$ for some $\lambda \in Z_{(p)}$. It follows that $p \cdot j'' = j''(p \wedge 1_L) = \lambda j''\pi \cdot j'' = \lambda p \cdot j''$. Then $\lambda = 1$ and we have

$$p \wedge 1_L = i''\xi + \pi j''. \tag{2.4}$$

By the following commutative diagram of 3×3 Lemma in the stable homotopy category

$$\begin{array}{ccccccccc}
\Sigma^q S & \xrightarrow{p} & \Sigma^q S & \xrightarrow{\alpha_1} & \Sigma S & & \\
\searrow \pi & \nearrow j'' & \searrow i & \nearrow j\alpha & \searrow i'' & & \\
 & L & & \Sigma^q M & & \Sigma^{q+1}L & \\
\nearrow i'' & \searrow \bar{h} & \nearrow \bar{u} & \searrow j & \nearrow \pi & & \\
S & \xrightarrow{\overline{w}} & \Sigma^{-1}Y & \xrightarrow{j\bar{u}} & \Sigma^{q+1}S & &
\end{array}$$

we have a cofibration

$$\Sigma^q S \xrightarrow{\pi} L \xrightarrow{\bar{h}} \Sigma^{-1}Y \xrightarrow{j\bar{u}} \Sigma^{q+1}S \tag{2.5}$$

with $\bar{u}\bar{h} = i \cdot j''$, $\bar{h}i'' = \overline{w}$.

Since $2\alpha ij\alpha = ij\alpha^2 + \alpha^2 ij$ (cf. [7, p.430]), we have $\alpha_1\alpha_1 = 0$ and so there is $\phi \in [\Sigma^{2q-1}S, L]$ and $(\alpha_1)_L \in [\Sigma^{q-1}L, S]$ such that

$$j''\phi = \alpha_1 = (\alpha_1)_L \cdot i''. \tag{2.6}$$

Let W be the cofibre of $\phi : \Sigma^{2q-1}S \to L$. Then W also is the cofibre of $(\alpha_1)_L : \Sigma^{q-1}L \to S$. This can be seen by the commutative diagram of 3×3 Lemma in stable homotopy category

$$\begin{array}{ccccccc}
\Sigma^{2q-1}S & \xrightarrow{\alpha_1} & \Sigma^q S & & \xrightarrow{\alpha_1} & & \Sigma S \\
& \searrow \phi & \nearrow j'' & \searrow i'' & \nearrow (\alpha_1)_L & & \\
& L & & \Sigma^q L & & & \\
& \nearrow i'' & \searrow w & \nearrow u & \searrow j'' & & \\
S & \xrightarrow{wi''} & W & \xrightarrow{j''u} & \Sigma^{2q}S & &
\end{array}$$

That is, we have two cofibrations

$$\Sigma^{2q-1}S \xrightarrow{\phi} L \xrightarrow{w} W \xrightarrow{j''u} \Sigma^{2q}S, \tag{2.7}$$

$$\Sigma^{q-1}L \xrightarrow{(\alpha_1)_L} S \xrightarrow{wi''} W \xrightarrow{u} \Sigma^q L. \tag{2.8}$$

Since $\alpha_1 \cdot (\alpha_1)_L \in [\Sigma^{2q-2}L, S] = 0$ by $\pi_{rq-2}S = 0$ for $r = 2, 3$, we see that there is $\bar{\phi} \in [\Sigma^{2q-1}L, L]$ such that $j''\bar{\phi} = (\alpha_1)_L \in [\Sigma^{q-1}L, S]$ and $\bar{\phi} \cdot i'' \in \pi_{2q-1}L$. Since $\pi_{rq-1}S$ has a unique generator $\alpha_1 = j\alpha i$, $\alpha_2 = j\alpha^2 i$ for $r = 1, 2$ respectively and $j''\phi \cdot p = \alpha_1 \cdot p = 0$, we have $\phi \cdot p = i''\alpha_2$ up to a scalar. That is, $i''_*\pi_{2q-1}S$ also is generated by ϕ and so we know that $\pi_{2q-1}L \cong Z_{p^s}\{\phi\}$ for some $s \geq 1$. Hence, $\bar{\phi}i'' = \lambda\phi$ for some $\lambda \in Z_{(p)}$ and $\lambda\alpha_1 = \lambda j''\phi = j''\bar{\phi}i'' = (\alpha_1)_L i'' = \alpha_1$ so that $\lambda = 1 \pmod p$. Moreover, $(\alpha_1)_L\bar{\phi} \in [\Sigma^{3q-2}L, S] = 0$ since $\pi_{rq-2}S = 0$ for $r = 3, 4$. Then by (2.8), there is $\bar{\phi}_W \in [\Sigma^{3q-1}L, W]$ such that $u\bar{\phi}_W = \bar{\phi}$. Concludingly we have elements $\bar{\phi} \in [\Sigma^{2q-1}L, L]$, $\bar{\phi}_W \in [\Sigma^{3q-1}L, W]$ such that

$$j''\bar{\phi} = (\alpha_1)_L, \quad \bar{\phi}i'' = \lambda\phi \quad (\lambda = 1 \ (\mathrm{mod}\, p)), \quad u\bar{\phi}_W = \bar{\phi}, \quad \pi_{2q-1}L \cong Z_{p^s}\{\phi\}. \tag{2.9}$$

Proposition 2.2 *Let $p \geq 7$. Then up to a mod p nonzero scalar we have*

(1) $\phi \cdot p = i''\alpha_2 = \pi \cdot \alpha_1 \neq 0$, $(\alpha_1)_L \cdot \pi = \alpha_2$, $p \cdot (\alpha_1)_L = \alpha_2 \cdot j'' = (\alpha_1)_L \pi j'' \neq 0$, $[\Sigma^{2q-1}L, L]$ *has a unique generator $\bar{\phi}$ modulo some elements of filtration ≥ 2.*

(2) $\bar{h}\bar{\phi}(p \wedge 1_L) \neq 0 \in [\Sigma^{2q}L, Y]$.

(3) $\bar{h}\tilde{\phi}(\pi \wedge 1_L)(p \wedge 1_L) \neq 0 \in [\Sigma^{3q}L, Y]$, $j''\tilde{\phi}(\pi \wedge 1_L)\pi = j\alpha^3 i \in \pi_{3q-1}S$ *up to a mod p nonzero scalar and* $\bar{h}\tilde{\phi}(\pi \wedge 1_L)\pi \neq 0 \in \pi_{4q}Y$, *where* $\tilde{\phi} \in [\Sigma^{2q-1}L \wedge L, L]$ *such that* $\tilde{\phi}(1_L \wedge i'') = \bar{\phi}$.

(4) $\pi_{4q}Y$ *has a unique generator* $\bar{h}\tilde{\phi}(\pi \wedge 1_L)\pi$ *such that* $\bar{h}\tilde{\phi}(\pi \wedge 1_L)\pi \cdot p = 0$.

Proof (1) Since $j''\phi \cdot p = \alpha_1 \cdot p = 0 = j''\pi \cdot \alpha_1$ and $\pi_{2q-1}S \cong Z_p\{\alpha_2\}$, we have $\phi \cdot p = i''\alpha_2 = \pi \cdot \alpha_1$ up to a scalar. We claim that $\phi \cdot p \neq 0$, which can be shown as follows. Look at the following exact sequence

$$Z_p\{j\alpha^2\} \cong [\Sigma^{2q-1}M, S] \xrightarrow{i''_*} [\Sigma^{2q-1}M, L] \xrightarrow{j''_*} [\Sigma^{q-1}M, S] \xrightarrow{(\alpha_1)_*}$$

induced by (2.1). The right group has a unique generator $j\alpha$ satisfying $(\alpha_1)_* j\alpha = j\alpha i j\alpha = \frac{1}{2}j\alpha\alpha ij \neq 0$. Then the above $(\alpha_1)_*$ is monic, $\mathrm{im} j''_* = 0$ and so $[\Sigma^{2q-1}M, L] \cong Z_p\{i''j\alpha^2\}$. Suppose in contrast that $\phi \cdot p = 0$. Then $\phi \in i^*[\Sigma^{2q-1}M, L]$ so that $\phi = i''j\alpha^2 i$ and so $\alpha_1 = j''\phi = j''i''\alpha_2 = 0$, which is a contradiction. This shows that $\phi \cdot p \neq 0$ and so the above scalar is nonzero (mod p).

The proof of the second result is similar. For the last result, let x be any element in $[\Sigma^{2q-1}L, L]$. Then $j''x \in [\Sigma^{q-1}L, S] \cong Z_{p^s}\{(\alpha_1)_L\}$ for some $s \geq 2$ (similar to the last of (2.9)). Consequently, $j''x = \lambda j''\bar{\phi}$ for some $\lambda \in Z_{p^s}$ and so $x = \lambda\bar{\phi} + i''x'$ with $x' \in [\Sigma^{2q-1}L, S]$. Since $x'i'' \in \pi_{2q-1}S \cong Z_p\{j\alpha^2 i\}$ and $\pi_{3q-1}S \cong Z_p\{j\alpha^3 i\}$, x' is an element of filtration ≥ 2. This shows the result.

(2) Suppose in contrast that $\bar{h}\bar{\phi}(p\wedge 1_L)=0$. Then by (2.5) we have $\bar{\phi}(p\wedge 1_L)=\lambda'\pi\cdot(\alpha_1)_L$ with $\lambda'\in Z_{(p)}$. Note that $\pi\wedge 1_M=(i''\wedge 1_M)\alpha$ since $j''\pi\wedge 1_M=p\wedge 1_M=0$. It follows that $\lambda'(\pi\wedge 1_M)i\cdot(\alpha_1)_L=\lambda'(1_L\wedge i)\pi(\alpha_1)_L=0$. Then $\lambda'(i''\wedge 1_M)\alpha i(\alpha_1)_L=\lambda'(\pi\wedge 1_M)i(\alpha_1)_L=0$ and so $\lambda'\alpha i(\alpha_1)_L\in(\alpha_1\wedge 1_M)[\Sigma^q L,M]$ and $\lambda'\alpha i\alpha_1\in(\alpha_1\wedge 1_M)(i'')^*[\Sigma^q L,M]=0$ by the following exact sequence

$$[\Sigma^{2q}S,M]\xrightarrow{(j'')^*}[\Sigma^q L,M]\xrightarrow{(i'')^*}[\Sigma^q S,M]\xrightarrow{(\alpha_1)^*},$$

where the right group has a unique generator αi satisfying $(\alpha_1)^*\alpha i=\alpha ij\alpha i\neq 0$ so that $(i'')^*[\Sigma^q L,M]=0$. This implies that $\lambda'=0$ and so $\bar{\phi}(p\wedge 1_L)=0$, which contradicts the fact $j''\bar{\phi}(p\wedge 1_L)=p\cdot(\alpha_1)_L\neq 0$ in (1). This shows the result on $\bar{h}\bar{\phi}(p\wedge 1_L)\neq 0$.

(3) Note that $\bar{\phi}(1_L\wedge\alpha_1)\in[\Sigma^{3q-2}L,L]=0$ since $\pi_{rq-2}S=0$ for $r=2,3,4$. Then there is $\tilde{\phi}\in[\Sigma^{2q-1}L\wedge L,L]$ such that $\tilde{\phi}(1_L\wedge i'')=\bar{\phi}$. We first prove that $\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)\neq 0$. For otherwise, if it is zero, then $\bar{\phi}\pi\cdot p=\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)i''=0$ and so $\bar{\phi}\pi\in i^*[\Sigma^{3q-1}M,L]$. However, $(j'')_*[\Sigma^{3q-1}M,L]\subset[\Sigma^{2q-1}M,S]$ which has a unique generator $j\alpha^2$ satisfying $(\alpha_1)_*(j\alpha^2)=j\alpha ij\alpha^2\neq 0$. Then $(j'')_*[\Sigma^{3q-1}M,L]=0$ and so $(\alpha_1)_L\pi=j''\bar{\phi}\pi\in i^*(j'')_*[\Sigma^{3q-1}M,L]=0$, which contradicts the result in (1).

Now suppose in contrast that $\bar{h}\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)=0$. Then, by (2.5), $\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)=\pi\cdot\omega$ with $\omega\in[\Sigma^{2q-1}L,S]$ which satisfies $\omega i''=\lambda_1\alpha_2$ for some $\lambda_1\in Z_p$. It follows that $(i''\wedge 1_M)\alpha i\omega=(1_L\wedge i)\pi\cdot\omega=0$. Then $\alpha i\omega\in(\alpha_1\wedge 1_M)_*[\Sigma^{2q}L,M]$ and so $\lambda_1\alpha i\alpha_2=\alpha i\omega i''\in(\alpha_1\wedge 1_M)_*(i'')^*[\Sigma^{2q}L,M]=(\alpha_1)^*(i'')^*[\Sigma^{2q}L.M]=0$. This shows that $\lambda_1=0$ since $\alpha i\alpha_2=\alpha ij\alpha^2 i\neq 0$. Consequently, $\omega=\lambda_2 j\alpha^3 i\cdot j''$ and $\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)=\lambda_2\pi\cdot j\alpha^3 i\cdot j''$ for some $\lambda_2\in Z_{(p)}$. It follows that $\bar{\phi}\pi\cdot p=\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)i''=0$ and so $\bar{\phi}\pi\in i^*[\Sigma^{3q-1}M,L]$ so that $(\alpha_1)_L\pi=j''\bar{\phi}\pi\in i^*(j'')_*[\Sigma^{3q-1}M,L]=0$. This contradicts the result in (1) on $(\alpha_1)_L\pi\neq 0$.

For the second result, since $\pi\cdot j=i''j\alpha$ by the diagram above (2.5), we have $j''\tilde{\phi}(\pi\wedge 1_L)\pi\cdot j=j''\tilde{\phi}(\pi\wedge 1_L)i''j\alpha=j''\bar{\phi}\pi j\alpha=(\alpha_1)_L\pi j\alpha=\alpha_2 j\alpha=j\alpha^3 ij$ (up to a mod p nonzero scalar). Consequently we have $j''\tilde{\phi}(\pi\wedge 1_L)\pi=j\alpha^3 i$ (up to nonzero scalar) since $\pi_{3q-1}S\cong Z_p\{\alpha_3\}$ so that $p^*\pi_{3q-1}S=0$.

For the last result, we first prove that $\tilde{\phi}(\pi\wedge 1_L)\pi\neq 0$. For otherwise, if it is zero, then $0=\tilde{\phi}(\pi\wedge 1_L)\pi\cdot j=\tilde{\phi}(\pi\wedge 1_L)i''j\alpha=\bar{\phi}\pi j\alpha$ and so $\alpha_2 j\alpha=(\alpha_1)_L\pi j\alpha==j''\bar{\phi}\pi j\alpha=0$ which is a contradiction since $\alpha_2 j\alpha=j\alpha^2 ij\alpha\neq 0\in[\Sigma^{3q-2}M,S]$. Now suppose in contrast that $\bar{h}\tilde{\phi}(\pi\wedge 1_L)\pi=0$. Then, by (2.5) and $\pi_{3q-1}S\cong Z_p\{\alpha_3\}$ we have $\tilde{\phi}(\pi\wedge 1_L)\pi=\lambda\pi\cdot j\alpha^3 i=\lambda i''j\alpha^4 i$ for some $\lambda\in Z_p$ and so $j''\tilde{\phi}(\pi\wedge 1_L)\pi=0$ which contradicts the second result.

(4) Since $(\bar{u})_*\pi_{4q}Y\subset\pi_{3q-1}M$ which has a unique generator $ij\alpha^3 i=ij''\tilde{\phi}(\pi\wedge 1_L)\pi=\bar{u}\bar{h}\tilde{\phi}(\pi\wedge 1_L)\pi$ (up to a nonzero scalar) and $\pi_{4q-1}S\cong Z_p\{j\alpha^4 i\}$ so that $(\overline{w})_*\pi_{4q-1}S=0$, we see that $\pi_{4q}Y$ has a unique generator $\bar{h}\tilde{\phi}(\pi\wedge 1_L)\pi$. Moreover, by (2.4), $\bar{h}\tilde{\phi}(\pi\wedge 1_L)\pi\cdot p=\bar{h}(p\wedge 1_L)\tilde{\phi}(\pi\wedge 1_L)\pi=\bar{h}i''\xi\tilde{\phi}(\pi\wedge 1_L)\pi=\overline{w}j\alpha^4 i=0$. This shows the result.

Proposition 2.3 *Let $p\geq 7,\ n\geq m+2\geq 4,\ tq=p^nq+p^mq$. Then*

$$\mathrm{Ext}_A^{3,tq+q}(H^*L,Z_p)=0,\quad \mathrm{Ext}_A^{3,tq}(H^*L,H^*L)\cong Z_p\{(h_nb_{m-1})',(h_mb_{n-1})'\}$$

which satisfies $(i'')^*(h_nb_{m-1})'=(i'')_*(h_nb_{m-1})$, $(i'')^*(h_mb_{n-1})'=(i'')_*(h_mb_{n-1})$.

Proof Consider the following exact sequence

$$\mathrm{Ext}_A^{3,tq+q}(Z_p,Z_p)\xrightarrow{i''_*}\mathrm{Ext}_A^{3,tq+q}(H^*L,Z_p)\xrightarrow{j''_*}\mathrm{Ext}_A^{3,tq}(Z_p,Z_p)\xrightarrow{(\alpha_1)_*}$$

induced by (2.1). The right group has two generators h_nb_{m-1}, h_mb_{n-1} by [1, Table 8.1] which satisfies

$$(\alpha_1)_*(h_nb_{m-1}) = h_0h_nb_{m-1} \neq 0, \quad (\alpha_1)_*(h_mb_{n-1}) = h_0h_mb_{n-1} \neq 0 \in \mathrm{Ext}_A^{4,tq+q}(Z_p, Z_p)$$

(cf. Proposition 2.1(1)). Then the above $(\alpha_1)_*$ is monic and so $\mathrm{im}\, j''_* = 0$. Moreover, the left group has a unique generator $h_0h_nh_m = (\alpha_1)_*(h_nh_m)$ by [1, Table 8.1], so we have that $\mathrm{im}\, i''_* = 0$ and $\mathrm{Ext}_A^{3,tq+q}(H^*L, Z_p) = 0$. Look at the following exact sequence

$$0 = \mathrm{Ext}_A^{3,tq+q}(H^*L, Z_p) \xrightarrow{(j'')^*} \mathrm{Ext}_A^{3,tq}(H^*L, H^*L) \xrightarrow{(i'')^*} \mathrm{Ext}_A^{3,tq}(H^*L, Z_p) \xrightarrow{(\alpha_1)^*}$$

induced by (2.1). Since $\mathrm{Ext}_A^{3,tq-rq}(Z_p, Z_p) \cong Z_p\{h_nb_{m-1}, h_mb_{n-1}\}$ for $r = 0$ and is zero for $r = 1$ in [1, Table 8.1], we see that the right group has two generators $(i'')_*(h_nb_{m-1})$ and $(i'')_*(h_mb_{n-1})$ whose images under $(\alpha_1)^*$ are zero. So the middle group has two generators as desired.

Proposition 2.4 *Let* $p \geq 7,\ n \geq m+2 \geq 4,\ tq = p^nq + p^mq$. *Then*

(1) $\mathrm{Ext}_A^{5,tq+3q+1}(H^*L, Z_p) \cong Z_p\{\bar{\phi}_*\pi_*(h_nb_{m-1}), \bar{\phi}_*\pi_*(h_mb_{n-1})\}$.

(2) $\mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L) \cong Z_p\{\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*(h_nb_{m-1})', \bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*(h_mb_{n-1})'\}$, *where* $\tilde{\phi} \in [\Sigma^{2q-1}L \wedge L, L]$ *such that* $\tilde{\phi}(1_L \wedge i'') = \bar{\phi} \in [\Sigma^{2q-1}L, L]$ *as in Proposition* 2.2(3).

Proof (1) Consider the following exact sequence

$$\mathrm{Ext}_A^{5,tq+3q+1}(Z_p, Z_p) \xrightarrow{i''_*} \mathrm{Ext}_A^{5,tq+3q+1}(H^*L, Z_p) \xrightarrow{j''_*} \mathrm{Ext}_A^{5,tq+2q+1}(Z_p, Z_p) \xrightarrow{(\alpha_1)_*}$$

induced by (2.1). The left group is zero and the right group has two generators $\tilde{\alpha}_2h_nb_{m-1}$, $\tilde{\alpha}_2h_mb_{n-1}$ by Proposition 2.1(2). Note that $j\alpha\alpha i = (\alpha_1)_L \cdot \pi = j''\bar{\phi} \cdot \pi \in \pi_{2q-1}S$, (cf. Proposition 2.2(1)). Then $\tilde{\alpha}_2h_nb_{m-1} = j_*\alpha_*\alpha_*i_*(h_nb_{m-1}) = j''_*\bar{\phi}_*\pi_*(h_nb_{m-1})$ and $\tilde{\alpha}_2(h_mb_{n-1}) = j''_*\bar{\phi}_*\pi_*(h_mb_{n-1})$ and so the middle group has the two generators as desired.

(2) Look at the exact sequence

$$0 = \mathrm{Ext}_A^{5,tq+4q+1}(H^*L, Z_p) \xrightarrow{(j'')^*} \mathrm{Ext}_A^{5,tq+3q+1}(H^*L, H^*L) \xrightarrow{(i'')^*} \mathrm{Ext}_A^{5,tq+3q+1}(H^*L, Z_p) \xrightarrow{(\alpha_1)^*}$$

induced by (2.1). The left group is zero since $\mathrm{Ext}_A^{5,tq+rq+1}(Z_p, Z_p) = 0$ for $r = 3, 4$ (cf. Proposition 2.1(2)). By (1) and $\bar{\phi} = \tilde{\phi}(1_L \wedge i'')$, the right group has two generators

$$\bar{\phi}_*\pi_*(h_nb_{m-1}) = (i'')^*\tilde{\phi}_*(\pi \wedge 1_L)_*(h_nb_{m-1})', \quad \bar{\phi}_*\pi_*(h_mb_{n-1}) = (i'')^*\tilde{\phi}_*(\pi \wedge 1_L)_*(h_mb_{n-1})'$$

whose image under $(\alpha_1)^*$ is zero. Then the middle group has two generators

$$\tilde{\phi}_*(\pi \wedge 1_L)_*(h_nb_{m-1})', \quad \tilde{\phi}_*(\pi \wedge 1_L)_*(h_mb_{n-1})'.$$

Moreover, by $\mathrm{Ext}_A^{5,tq+rq}(Z_p, Z_p) = 0$ for $r = 2, 3$ in Proposition 2.1(2), we know that

$$\mathrm{Ext}_A^{5,tq+2q}(Z_p, H^*L) = 0.$$

Then, by (2.5), $\mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L) = \bar{h}_*\mathrm{Ext}_A^{5,tq+3q+1}(H^*L, H^*L)$ has the two generators as desired.

Proposition 2.5 *Let* $p \geq 7,\ n \geq m+2 \geq 4,\ tq = p^nq + p^mq$. *Then*

(1) $\mathrm{Ext}_A^{4,tq+3q+1}(H^*Y, H^*L) = 0$, $\quad \mathrm{Ext}_A^{4,tq+4q+2}(H^*Y, Z_p) = 0$.
(2) $\mathrm{Ext}_A^{3,tq+3q+r}(H^*Y, H^*L) = 0$ *for* $r = 0, 1$.

Proof (1) Consider the following exact sequence

$$\mathrm{Ext}_A^{4,tq+3q}(H^*L, H^*L) \xrightarrow{(\bar{h})_*} \mathrm{Ext}_A^{4,tq+3q+1}(H^*Y, H^*L) \xrightarrow{(j\bar{u})_*} \mathrm{Ext}_A^{4,tq+2q-1}(Z_p, H^*L) \xrightarrow{(\pi)_*}$$

induced by (2.5). The left group is zero since $\mathrm{Ext}_A^{4,tq+rq}(Z_p, Z_p) = 0$ for $r = 2, 3, 4$ by Proposition 2.1(1). The right group also is zero since $\mathrm{Ext}_A^{4,tq+rq-1}(Z_p, Z_p) = 0$ for $r = 2, 3$ by Proposition 2.1(1). Then the middle group is zero as desired.

For the second result, look at the following exact sequence

$$\mathrm{Ext}_A^{4,tq+4q+1}(H^*L, Z_p) \xrightarrow{(\bar{h})_*} \mathrm{Ext}_A^{4,tq+4q+2}(H^*Y, Z_p) \xrightarrow{(j\bar{u})_*} \mathrm{Ext}_A^{4,tq+3q}(Z_p, Z_p)$$

induced by (2.5). The left is zero since $\mathrm{Ext}_A^{4,tq+rq+1}(Z_p, Z_p) = 0$ for $r = 3, 4$ by Proposition 2.1(1). The right group also is zero by Proposition 2.1(1). Then the middle group is zero as desired.

(2) Consider the following exact sequence ($r = 0, 1$)

$$\mathrm{Ext}_A^{3,tq+3q+r-1}(H^*L, H^*L) \xrightarrow{(\bar{h})_*} \mathrm{Ext}_A^{3,tq+3q+r}(H^*Y, H^*L) \xrightarrow{(j\bar{u})_*} \mathrm{Ext}_A^{3,tq+2q+r-2}(Z_p, H^*L)$$

induced by (2.5). The left group is zero since $\mathrm{Ext}_A^{3,tq+kq+r-1}(Z_p, Z_p) = 0$ for $k = 2, 3, 4, r = 0, 1$ by [1, Table 8.1] . The right group also is zero since $\mathrm{Ext}_A^{3,tq+kq+r-2}(Z_p, Z_p) = 0$ for $k = 2, 3, r = 0, 1$ by [1, Table 8.1] and so the middle group is zero as desired.

Proposition 2.6 *Let* $p \geq 7$, $n \geq m + 2 \geq 4$, $tq = p^n q + p^m q$. *Then*
(1) $\mathrm{Ext}_A^{3,tq+3q}(H^*W, H^*L) \cong Z_p\{(\bar{\phi}_W)_*(h_n h_m)'\}$, *where* $\bar{\phi}_W \in [\Sigma^{3q-1}L, W]$ *satisfying* $u\bar{\phi}_W = \bar{\phi} \in [\Sigma^{2q-1}L, L]$ *as in* (2.9), $(h_n h_m)' \in \mathrm{Ext}_A^{2,tq}(H^*L, H^*L)$ *such that* $(i'')^*(h_n h_m)' = (i'')_*(h_n h_m) \in \mathrm{Ext}_A^{2,tq}(H^*L, Z_p)$.
(2) $\mathrm{Ext}_A^{2,tq+3q}(H^*Y, H^*L) = 0$, $\quad \mathrm{Ext}_A^{2,tq+q-1}(H^*M, H^*L) = 0$.

Proof (1) Consider the following exact sequence

$$\mathrm{Ext}_A^{3,tq+3q}(H^*L, H^*L) \xrightarrow{w_*} \mathrm{Ext}_A^{3,tq+3q}(H^*W, H^*L) \xrightarrow{(j''u)_*} \mathrm{Ext}_A^{3,tq+q}(Z_p, H^*L) \xrightarrow{\phi_*}$$

induced by (2.7). The left group is zero since $\mathrm{Ext}_A^{3,tq+rq}(Z_p, Z_p) = 0$ for $r = 2, 3, 4$ by [1, Table 8.1]. Since $(i'')^*\mathrm{Ext}_A^{3,tq+q}(Z_p, H^*L) \subset \mathrm{Ext}_A^{3,tq+q}(Z_p, Z_p)$ which has a unique generator $h_0 h_n h_m = (\alpha_1)^*(h_n h_m) = (i'')^*((\alpha_1)_L)^*(h_n h_m)$ and $\mathrm{Ext}_A^{3,tq+2q}(Z_p, Z_p) = 0$ by [1, Table 8.1], we see that the right group has a unique generator

$$((\alpha_L))^*(h_n h_m) = ((\alpha_1)_L)_*(h_n h_m)' = (j''u)_*(\bar{\phi}_W)_*(h_n h_m)'$$

with $(h_n h_m)' \in \mathrm{Ext}_A^{2,tq}(H^*L, H^*L)$ satisfying $(i'')^*(h_n h_m)' = (i'')_*(h_n h_m) \in \mathrm{Ext}_A^{2,tq}(H^*L, Z_p)$. Moreover, $\phi_*((\alpha_1)_L)_*(h_n h_m)' = 0 \in \mathrm{Ext}_A^{4,tq+3q}(H^*L, H^*L)$, so the middle group has a unique generator $(\bar{\phi}_W)_*(h_n h_m)'$ as desired.

(2) Look at the following exact sequences

$$\mathrm{Ext}_A^{2,tq+3q-1}(H^*L, H^*L) \xrightarrow{\bar{h}_*} \mathrm{Ext}_A^{2,tq+3q}(H^*Y, H^*L) \xrightarrow{(j\bar{u})_*} \mathrm{Ext}_A^{2,tq+2q-2}(Z_p, H^*L),$$

$$\mathrm{Ext}_A^{2,tq+q-1}(Z_p, H^*L) \xrightarrow{i_*} \mathrm{Ext}_A^{2,tq+q-1}(H^*M, H^*L) \xrightarrow{j_*} \mathrm{Ext}_A^{2,tq+q-2}(Z_p, H^*L)$$

induced by (2.5) and (1.1) respectively. The upper left group is zero since $\mathrm{Ext}_A^{2,tq+rq-1}(Z_p, Z_p) = 0$ for $r = 2, 3, 4$ and the upper right group also is zero since $\mathrm{Ext}_A^{2,tq+rq-2}(Z_p, Z_p) = 0$ for $r = 2, 3$ (cf. [5]). Then the upper middle group is zero as desired. Similarly, the lower middle group also is zero as desired.

Proposition 2.7 *Let* $p \geq 7$, $n \geq m + 2 \geq 4$, $tq = p^n q + p^m q$. *Then*

$$\mathrm{Ext}_A^{5,tq+2}(H^*M, Z_p) = 0, \quad \mathrm{Ext}_A^{3,tq+q+1}(H^*M \wedge L, Z_p) \cong Z_p\{(i \wedge 1_L)_*\pi_*(h_n h_m)\}.$$

Proof Consider the following exact sequence

$$\mathrm{Ext}_A^{5,tq+2}(Z_p, Z_p) \xrightarrow{i_*} \mathrm{Ext}_A^{5,tq+2}(H^*M, Z_p) \xrightarrow{j_*} \mathrm{Ext}_A^{5,tq+1}(Z_p, Z_p) \xrightarrow{p_*}$$

induced by (1.1). The right group has a unique generator $a_0 b_{n-1} b_{m-1}$ which satisfies

$$p_*(a_0 b_{n-1} b_{m-1}) = a_0^2 b_{n-1} b_{m-1} (\neq 0) \in \mathrm{Ext}_A^{6,tq+2}(Z_p, Z_p)$$

by Proposition 2.1(2). Then $\mathrm{im}\, j_* = 0$. The left group has two generators

$$a_0^2 h_m b_{n-1} = p_*(a_0 h_m b_{n-1}), \quad a_0^2 h_n b_{m-1} = p_*(a_0 h_n b_{m-1})$$

so that $\mathrm{im}\, i_* = 0$. So the middle group is zero as desired.

For the second result, look at the following exact sequence

$$\mathrm{Ext}_A^{3,tq+q+1}(H^*L, Z_p) \xrightarrow{(i \wedge 1_L)_*} \mathrm{Ext}_A^{3,tq+q+1}(H^*M \wedge L, Z_p) \xrightarrow{(j \wedge 1_L)_*} \mathrm{Ext}_A^{3,tq+q}(H^*L, Z_p)$$

induced by (1.1). The right group is zero by Proposition 2.3(1). Since

$$(j'')_* \mathrm{Ext}_A^{3,tq+q+1}(H^*L, Z_p) \subset \mathrm{Ext}_A^{3,tq+1}(Z_p, Z_p) \cong Z_p\{a_0 h_n h_m = (j'')_*\pi_*(h_n h_m)\}$$

and $\mathrm{Ext}_A^{3,tq+q+1}(Z_p, Z_p) = 0$ by [1, Table 8.1], we see that the left group has a unique generator $\pi_*(h_n h_m)$ and so the result follows.

3 Proof of the Main Theorem A

The proof of Theorem A will be done by an argument processing in the Adams resolution of certain spectra related to S which is equivalent to computing the differentials of the ASS. Let

$$\begin{array}{ccccccc} \cdots \xrightarrow{\bar a_2} & \Sigma^{-2}E_2 & \xrightarrow{\bar a_1} & \Sigma^{-1}E_1 & \xrightarrow{\bar a_0} & E_0 = S \\ & \downarrow \bar b_2 & & \downarrow \bar b_1 & & \downarrow \bar b_0 \\ & \Sigma^{-2}KG_2 & & \Sigma^{-1}KG_1 & & KG_0 \end{array}$$

be the minimal Adams resolution of S satisfying

(1) $E_s \xrightarrow{\bar b_s} KG_s \xrightarrow{\bar c_s} E_{s+1} \xrightarrow{\bar a_s} \Sigma E_s$ are cofibrations for all $s \geq 0$ which induce short exact sequences $0 \longrightarrow H^*E_{s+1} \xrightarrow{\bar c_s^*} H^*KG_s \xrightarrow{\bar b_s^*} H^*E_s \longrightarrow 0$ in Z_p-cohomology.

(2) KG_s is a wedge sum of Eilenberg-Maclane spectra of type KZ_p.

(3) $\pi_t KG_s$ are the $E_1^{s,t}$-terms , $(\bar b_s \bar c_{s-1})_* : \pi_t KG_{s-1} \longrightarrow \pi_t KG_s$ are the $d_1^{s-1,t}$-differentials of the ASS and $\pi_t KG_s \cong \mathrm{Ext}_A^{s,t}(Z_p, Z_p)$ (cf. [3, p.180]).

Then, an Adams resolution of arbitrary spectrum V can be obtained by smashing V on the above minimal Adams resolution. We first prove the following lemma.

Lemma 3.1 *Let $p \geq 7$, $m \geq n+2 \geq 4$, $tq = p^n q + p^m q$, $\sigma' = h_m b_{n-1} - h_n b_{m-1}$. Then*

(1) $d_2(h_n h_m) = a_0 \sigma' \in \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p)$, *where* $d_2 : \mathrm{Ext}_A^{2,tq}(Z_p, Z_p) \to \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p)$ *is the differential of the ASS.*

(2) $\bar{c}_3 \cdot h_0 h_n h_m = (1_{E_4} \wedge \alpha_1)\kappa$ *up to a scalar, where* $\kappa \in \pi_{tq+1} E_4$ *such that* $\bar{c}_2 \cdot h_n h_m = \bar{a}_3 \cdot \kappa$ *and* $\bar{b}_4 \cdot \kappa = a_0 \sigma' \in \pi_{tq+1} KG_4 \cong \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p)$ *by (1).*

Proof (1) From [8, Theorem 1.2.14, p.11], $d_2(h_n) = a_0 b_{n-1} \in \mathrm{Ext}_A^{3,p^n q+1}(Z_p, Z_p)$. Then, $d_2(h_n h_m) = d_2(h_n) h_m + (-1)^{1+p^n q} h_n d_2(h_m) = a_0 b_{n-1} h_m - h_n a_0 b_{m-1} = a_0 \sigma'$ as desired.

(2) The d_1-cycle $(1_{KG_3} \wedge i'') h_0 h_n h_m \in \pi_{tq+q}(KG_3 \wedge L)$ represents an element in $\mathrm{Ext}_A^{3,tq+q}(H^* L, Z_p) = 0$ by Proposition 2.3(1), so it is a d_1-boundary and $(\bar{c}_3 \wedge 1_L)(1_{KG_3} \wedge i'') h_0 h_n h_m = 0$ and $\bar{c}_3 \cdot h_0 h_n h_m = (1_{E_4} \wedge \alpha_1) f''$ with $f'' \in \pi_{tq+1} E_4$. It follows that $\bar{a}_3 \cdot (1_{E_4} \wedge \alpha_1) f'' = 0$ and $\bar{a}_3 \cdot f'' = (1_{E_3} \wedge j'') f_2''$ for some $f_2'' \in \pi_{tq+q}(E_3 \wedge L)$. The d_1-cycle $(\bar{b}_3 \wedge 1_L) f_2'' \in \pi_{tq+q} KG_3 \wedge L$ represents an element in $\mathrm{Ext}_A^{3,tq+q}(H^* L, Z_p) = 0$. Then $(\bar{b}_3 \wedge 1_L) f_2'' = (\bar{b}_3 \bar{c}_2 \wedge 1_L) g''$ with $g'' \in \pi_{tq+q}(KG_2 \wedge L)$ and so $f_2'' = (\bar{c}_2 \wedge 1_L) g'' + (\bar{a}_3 \wedge 1_L) f_3''$ for some $f_3'' \in \pi_{tq+q+1} E_4 \wedge L$. It follows that $\bar{a}_3 \cdot f'' = \bar{a}_3 (1_{E_4} \wedge j'') f_3'' + \bar{c}_2 (1_{KG_2} \wedge j'') g'' = \bar{a}_3 (1_{E_4} \wedge j'') f_3'' + \lambda \bar{c}_2 \cdot h_n h_m = \bar{a}_3 (1_{E_4} \wedge j'') f_3'' + \lambda \bar{a}_3 \cdot \kappa$ for some $\lambda \in Z_p$ since $(1_{KG_2} \wedge j'') g'' \in \pi_{tq} KG_2 \cong \mathrm{Ext}_A^{2,tq}(Z_p, Z_p) \cong Z_p\{h_n h_m\}$ (cf. [5]). Hence, $f'' = (1_{E_4} \wedge j'') f_3'' + \lambda \kappa + \bar{c}_3 \cdot g_2''$ for some $g_2'' \in \pi_{tq+1} KG_3$ and so

$$\bar{c}_3 \cdot h_0 h_n h_m = (1_{E_4} \wedge \alpha_1) f'' = \lambda (1_{E_4} \wedge \alpha_1) \kappa.$$

Since $\bar{h}\phi \cdot p = \bar{h} i'' j \alpha^2 i = 0$ by Proposition 2.2(1) and (2.3), (2.5), we have $\bar{h}\phi = (1_Y \wedge j) \alpha_{Y \wedge M} i$ with $\alpha_{Y \wedge M} \in [\Sigma^{2q+1} M, Y \wedge M]$. Let ΣU be the cofibre of $\bar{h}\phi = (1_Y \wedge j) \alpha_{Y \wedge M} i : \Sigma^{2q} S \to Y$ given by the cofibration

$$\Sigma^{2q} S \xrightarrow{\bar{h}\phi} Y \xrightarrow{w_2} \Sigma U \xrightarrow{u_2} \Sigma^{2q+1} S. \tag{3.1}$$

Moreover, $w_2 (1_Y \wedge j) \alpha_{Y \wedge M} = \widetilde{w} \cdot j$ with $\widetilde{w} : \Sigma^{2q} S \to U$ whose cofibre is X given by the cofibtation $\Sigma^{2q} S \xrightarrow{\widetilde{w}} U \xrightarrow{\tilde{u}} X \xrightarrow{j\tilde{\psi}} \Sigma^{2q+1} S$. Then, ΣX also is the cofibre of $\omega = (1_Y \wedge j) \alpha_{Y \wedge M} : \Sigma^{2q} M \to Y$ given by the cofibration

$$\Sigma^{2q} M \xrightarrow{(1_Y \wedge j)\alpha_{Y \wedge M}} Y \xrightarrow{\tilde{u} w_2} \Sigma X \xrightarrow{\tilde{\psi}} \Sigma^{2q+1} M. \tag{3.2}$$

This can be seen by the following commutative diagram of 3×3 Lemma

$$\begin{array}{ccccccc}
\Sigma^{2q} S & \xrightarrow{\bar{h}\phi} & Y & \xrightarrow{\tilde{u} w_2} & \Sigma X & & \\
 \searrow i & & \nearrow \omega \quad \searrow w_2 & & \nearrow \tilde{u} & & \\
 & \Sigma^{2q} M & & \Sigma U & & & \\
 \nearrow \tilde{\psi} & & \searrow j \quad \nearrow \widetilde{w} & & \searrow u_2 & & \\
X & \xrightarrow{j\tilde{\psi}} & \Sigma^{2q+1} S & \xrightarrow{p} & \Sigma^{2q+1} S & &
\end{array}$$

Since $j \bar{u} (\bar{h}\phi) = 0$, then, by (3.1), $j \bar{u} = u_3 w_2$ with $u_3 \in [U, \Sigma^{q+1} S]$. So, the spectrum U in (3.1) also is the cofibre of $w\pi : \Sigma^q S \to W$ given by the cofibration

$$\Sigma^q S \xrightarrow{w\pi} W \xrightarrow{w_3} U \xrightarrow{u_3} \Sigma^{q+1} S. \tag{3.3}$$

This can be seen by the following commutative diagram of 3×3 Lemma

$$\begin{array}{ccccccc}
\Sigma^{-1}Y & & \stackrel{j\bar{u}}{\longrightarrow} & \Sigma^{q+1}S & \stackrel{w\pi}{\longrightarrow} & & \Sigma W \\
& \searrow w_2 & \nearrow u_3 & & \searrow \pi & \nearrow w & \\
& U & & & & \Sigma L & \\
& \nearrow w_3 & \searrow u_2 & & \nearrow \phi & \searrow \bar{h} & \\
W & & \stackrel{j''u}{\longrightarrow} & \Sigma^{2q}S & \stackrel{\bar{h}\phi}{\longrightarrow} & & Y
\end{array}$$

Moreover, by $u_3\widetilde{w} = \alpha_1$, the cofibre of $\tilde{u}w_3 : W \to X$ is $\Sigma^{q+1}L$ given by the cofibration

$$W \stackrel{\tilde{u}w_3}{\longrightarrow} X \stackrel{u''}{\longrightarrow} \Sigma^{q+1}L \stackrel{w'(\pi\wedge 1_L)}{\longrightarrow} \Sigma W, \tag{3.4}$$

where $w' \in [L\wedge L, W]$ such that $w'(1_L \wedge i'') = w$. This can be seen by the following commutative diagram of 3×3 Lemma

$$\begin{array}{ccccccc}
W & & \stackrel{\tilde{u}w_3}{\longrightarrow} & X & \stackrel{j\tilde{\psi}}{\longrightarrow} & & \Sigma^{2q+1}S \\
& \searrow w_3 & \nearrow \tilde{u} & & \searrow u'' & \nearrow j'' & \\
& U & & & & \Sigma^{q+1}L & \\
& \nearrow \widetilde{w} & \searrow u_3 & & \nearrow i'' & \searrow^{w'(\pi\wedge 1_L)} & \\
\Sigma^{2q}S & & \stackrel{\alpha_1}{\longrightarrow} & \Sigma^{q+1}S & \stackrel{w\pi}{\longrightarrow} & & \Sigma W
\end{array}$$

Lemma 3.2 *Let* $\bar{\phi}_W \in [\Sigma^{3q-1}L, W]$ *be the map in* (2.9) *and Proposition* 2.6(1) *which satisfies* $u\bar{\phi}_W = \bar{\phi} \in [\Sigma^{2q-1}L, L]$. *Then*

(1) $\tilde{u}w_3\bar{\phi}_W(p\wedge 1_L) \neq 0 \in [\Sigma^{3q-1}L, X]$.

(2) $\mathrm{Ext}_A^{2,tq+3q-1}(H^*X, H^*L) = 0, \quad \mathrm{Ext}_A^{3,tq+3q}(H^*X, H^*L) = (\tilde{u}w_3)_*\mathrm{Ext}_A^{3,tq+3q}(H^*W, H^*L)$.

Proof (1) Suppose in contrast that $\tilde{u}w_3\bar{\phi}_W(p\wedge 1_L) = 0$. Then by (3.4) and the result on $[\Sigma^{2q-1}L, L]$ in Proposition 2.2(1) we have

$$\bar{\phi}_W(p\wedge 1_L) = \lambda w'(\pi\wedge 1_L)\bar{\phi} \mod F_3[\Sigma^{3q-1}L, W] \tag{3.5}$$

for some $\lambda \in Z_{(p)}$, where $F_3[\Sigma^{3q-1}L, W]$ denotes the subgroup of $[\Sigma^{3q-1}L, W]$ generated by elements of filtration ≥ 3. Moreover, note that $uw'(\pi\wedge 1_L) \in [L, L]$ which has two generators $(p\wedge 1_L)$, $\pi j''$ of filtration 1 (cf. (2.4)). Then $uw'(\pi\wedge 1_L) = \lambda_1(p\wedge 1_L) + \lambda_2\pi j''$ for some $\lambda_1, \lambda_2 \in Z_{(p)}$. It follows by (2.8) that $\lambda_1 p\cdot(\alpha_1)_L + \lambda_2(\alpha_1)_L\pi j'' = 0$ and so we have $\lambda_2 = \lambda_0\lambda_1$, where we use the equation $(\alpha_1)_L\pi j'' = -(\lambda_0)^{-1}p\cdot(\alpha_1)_L$ with nonzero $\lambda_0 \in Z_{(p)}$ (cf. Proposition 2.2(1)). Hence, by composing u on (3.5) we have

$$\bar{\phi}(p\wedge 1_L) = u\bar{\phi}_W(p\wedge 1_L) = \lambda uw'(\pi\wedge 1_L)\bar{\phi} = \lambda\lambda_1\bar{\phi}(p\wedge 1_L) + \lambda\lambda_0\lambda_1\pi j''\bar{\phi} \pmod{F_3[\Sigma^{2q-1}L, L]}$$

and so by (2.5) we have

$$\bar{h}\bar{\phi}(p\wedge 1_L) = \lambda\lambda_1\bar{h}\bar{\phi}(p\wedge 1_L) \pmod{F_3[\Sigma^{2q}L, Y]}.$$

This implies that $\lambda\lambda_1 = 1 \pmod p$ (cf. Remark 3.3 below). Consequently we have $\lambda\lambda_1\lambda_0\pi j''\bar{\phi} = 0 \pmod{F_3[\Sigma^{2q-1}L, L]}$ and by a similar reason as shown in Remark 3.3 below, this implies $\lambda\lambda_1\lambda_0 = 0 \pmod p$, which yields a contradiction.

(2) Consider the following exact sequence

$$\mathrm{Ext}_A^{2,tq+3q}(H^*Y, H^*L) \stackrel{(\tilde{u}w_2)_*}{\longrightarrow} \mathrm{Ext}_A^{2,tq+3q-1}(H^*X, H^*L) \stackrel{(\tilde{\psi})_*}{\longrightarrow} \mathrm{Ext}_A^{2,tq+q-1}(H^*M, H^*L)$$

induced by (3.2). Both sides of group are zero by Proposition 2.6(2) and so the middle group is zero as desired. Look at the following exact sequence

$$\mathrm{Ext}_A^{3,tq+3q}(H^*W, H^*L) \xrightarrow{(\tilde{u}w_3)_*} \mathrm{Ext}_A^{3,tq+3q}(H^*X, H^*L) \xrightarrow{(u'')_*} \mathrm{Ext}_A^{3,tq+2q-1}(H^*L, H^*L)$$

induced by (3.4). The right group is zero since $\mathrm{Ext}_A^{3,tq+rq-1}(Z_p, Z_p) = 0$ for $r = 1, 2, 3$ by [1, Table 8.1]. Then the result follows.

Remark 3.3 We give an explanation for the reason why the scalar in the equation $(1 - \lambda\lambda_1)\bar{h}\bar{\phi}(p\wedge 1_L) = 0 \pmod{F_3[\Sigma^{2q}L, Y]}$ must be zero (mod p). For otherwise , if $1-\lambda\lambda_1 \neq 0 \pmod p$, then $(1 - \lambda\lambda_1)\bar{h}\bar{\phi}(p \wedge 1_L)$ must be represented by some nonzero $x \in \mathrm{Ext}_A^{2,2q+2}(H^*Y, H^*L)$ in the ASS. However, it equals an element of filtration ≥ 3. Then x must be hit by differential and so $x = d_2(x') \in d_2\mathrm{Ext}_A^{0,2q+1}(H^*Y, H^*L) = 0$ since $\mathrm{Ext}_A^{0,2q+1}(H^*Y, H^*L) = \mathrm{Hom}_A^{2q+1}(H^*Y, H^*L) = 0$ by $H^rL \neq 0$ only for $r = 0, q$. This is a contradiction so that $1 - \lambda\lambda_1 = 0 \pmod p$.

Lemma 3.4 *For the map $\kappa \in \pi_{tq+1}E_4$ in Lemma 3.1(2) which satisfies $\bar{a}_4\cdot\kappa = \bar{c}_2\cdot h_nh_m$ and $\bar{b}_4\cdot\kappa = a_0\sigma' \in \pi_{tq+1}KG_4 \cong \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p)$, there exist $f \in \pi_{tq+3}E_6$ and $g \in \pi_{tq+1}(KG_3\wedge M)$ such that*

(A) $(1_{E_4} \wedge i)\kappa = (\bar{c}_3 \wedge 1_M)g + (\bar{a}_4\bar{a}_5 \wedge 1_M)f$,

(B) $(1_{E_6} \wedge (1_Y \wedge j)\alpha_{Y\wedge M})f \cdot (\alpha_1)_L = 0 \in [\Sigma^{tq+4q+2}L, E_6 \wedge Y]$,

where $\alpha_{Y\wedge M} \in [\Sigma^{2q+1}M, Y \wedge M]$ such that $(1_Y \wedge j)\alpha_{Y\wedge M}i = \bar{h}\phi \in \pi_{2q}Y$.

Proof Note that the d_1-cycle $(\bar{b}_4 \wedge 1_M)(1_{KG_4} \wedge i)\kappa \in \pi_{tq+1}KG_4 \wedge M$ represents an element $i_*(a_0\sigma') = i_*p_*(\sigma') = 0 \in \mathrm{Ext}_A^{4,tq+1}(H^*M, Z_p)$ and so it is a d_1-boundary. That is $(\bar{b}_4 \wedge 1_M)(1_{KG_4} \wedge i)\kappa = (\bar{b}_4\bar{c}_3 \wedge 1_M)g$ for some $g \in \pi_{tq+1}KG_3 \wedge M$ and so by $\mathrm{Ext}_A^{5,tq+2}(H^*M, Z_p) = 0$ (cf. Proposition 2.7) we have $(1_{KG_4} \wedge i)\kappa = (\bar{c}_3 \wedge 1_M)g + (\bar{a}_4\bar{a}_5 \wedge 1_M)f$ with $f \in \pi_{tq+3}E_6 \wedge M$. This shows (A).

For the result (B), note from Proposition 2.2(1) that $\phi \cdot p = i''j\alpha^2 i$ up to a nonzero scalar. Then $\bar{h}\phi\cdot p = \bar{h}i''j\alpha^2 i = 0$ and so $\bar{h}\phi = (1_Y \wedge j)\alpha_{Y\wedge M}i$ with $\alpha_{Y\wedge M} \in [\Sigma^{2q+1}M, Y \wedge M]$. Hence, by composing $1_{E_4} \wedge (1_Y \wedge j)\alpha_{Y\wedge M}$ on the equation (A) we have

$$(1_{E_4} \wedge \bar{h}\phi)\kappa = (1_{E_4} \wedge (1_Y \wedge j)\alpha_{Y\wedge M}i)\kappa = (\bar{a}_4\bar{a}_5 \wedge 1_Y)(1_{E_6} \wedge (1_Y \wedge j)\alpha_{Y\wedge M})f, \tag{3.6}$$

where $(1_Y \wedge j)\alpha_{Y\wedge M}$ induces zero homomorphism in Z_p-cohomology so that $(\bar{c}_3 \wedge 1_Y)(1_{KG_3} \wedge (1_Y \wedge j)\alpha_{Y\wedge M})g = 0$.

It follows by composing $(\alpha_1)_L$ on (3.6) that $(\bar{a}_4\bar{a}_5 \wedge 1_Y)(1_{E_6} \wedge (1_Y \wedge j)\alpha_{Y\wedge M})f \cdot (\alpha_1)_L = (1_{E_4} \wedge \bar{h})(\kappa \wedge 1_L)\phi \cdot (\alpha_1)_L = 0$ since $\phi \cdot (\alpha_1)_L \in [\Sigma^{3q-2}L, L] = 0$ by $\pi_{rq-2}S = 0$ for $r = 2, 3, 4$. Hence we have

$$(\bar{a}_5 \wedge 1_Y)(1_{E_6} \wedge (1_Y \wedge j)\alpha_{Y\wedge M})f \cdot (\alpha_1)_L = (\bar{c}_4 \wedge 1_Y)g_1 = 0,$$

where the d_1-cycle $g_1 \in [\Sigma^{tq+3q+1}L, KG_4\wedge Y]$ represents an element in $\mathrm{Ext}_A^{4,tq+3q+1}(H^*Y, H^*L) = 0$ (cf. Proposition 2.5(1)) so that it is a d_1-boundary and so $(\bar{c}_4 \wedge 1_Y)g_1 = 0$. Briefly write $(1_Y\wedge j)\alpha_{Y\wedge M} = \omega$ and let V be the cofibre of $(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L) = \omega\cdot(\alpha_1)_L : \Sigma^{3q-1}M\wedge L \to Y$ given by the cofibration

$$\Sigma^{3q-1}M \wedge L \xrightarrow{(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L)} Y \xrightarrow{w_4} V \xrightarrow{u_4} \Sigma^{3q}M \wedge L. \tag{3.7}$$

It follows that $(\bar{a}_5\wedge 1_Y)(1_{E_6}\wedge(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L)(f\wedge 1_L) = (\bar{a}_5\wedge 1_Y)(1_{E_6}\wedge(1_Y\wedge j)\alpha_{Y\wedge M})f\cdot(\alpha_1)_L = 0$. Then by (3.7) we have $(\bar{a}_5\wedge 1_{M\wedge L})(f\wedge 1_L) = (1_{E_5}\wedge u_4)f_2$ for some $f_2 \in [\Sigma^{tq+3q+2}L, E_5\wedge V]$. It follows that $(\bar{b}_5\wedge 1_V)(1_{E_5}\wedge u_4)f_2 = 0$ and so

$$(\bar{b}_5\wedge 1_V)f_2 = (1_{KG_5}\wedge w_4)g_2 \tag{3.8}$$

for some $g_2 \in [\Sigma^{tq+3q+2}L, KG_5\wedge Y]$. Consequently, $(\bar{b}_6\bar{c}_5\wedge 1_V)(1_{KG_5}\wedge w_4)g_2 = 0$ and so $(\bar{b}_6\bar{c}_5\wedge 1_Y)g_2 \in (1_{KG_6}\wedge(1_Y\wedge(\alpha_1)_L(\omega\wedge 1_L))_*[\Sigma^*L, KG_6\wedge M\wedge L] = 0$. That is, g_2 is a d_1-cycle and it represents an element $[g_2] \in \mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L)$ which has two generators stated in Proposition 2.4(2) so that

$$[g_2] = \bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*(\lambda_1[h_m b_{n-1}\wedge 1_L] + \lambda_2[h_n b_{m-1}\wedge 1_L]) \tag{3.9}$$

for some $\lambda_1, \lambda_2 \in Z_p$. By (3.8) we know that $(w_4)_*[g_2] \in E_2^{5,tq+3q+2}(V) = \mathrm{Ext}_A^{5,tq+3q+2}(H^*V, H^*L)$ is a permanent cycle in the ASS. However, $(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L)$ is a map of filtration 2, then the cofibration (3.7) induces a short exact sequence in Z_p-cohomology which is split as A-modules, that is, it induces a split exact sequence in E_1-term of the ASS:

$$E_1^{5,*}(Y) \xrightarrow{(w_4)_*} E_1^{5,*}(V) \xrightarrow{(u_4)_*} E_1^{5,*-3q}(M\wedge L).$$

Consequently, it induces a split exact sequence in E_r-term of the ASS:

$$E_r^{5,*}(Y) \xrightarrow{(w_4)_*} E_r^{5,*}(V) \xrightarrow{(u_4)_*} E_r^{5,*-3q}(M\wedge L) \tag{3.10}$$

for all $r \geq 2$. Hence, the fact that $d_r((w_4)_*[g_2]) = 0$ implies $d_r([g_2]) = 0$ for all $r \geq 2$. That is, (3.8) implies that $[g_2]$ is a permanent cycle in the ASS. By the vanishing of the d_2-differential we have $(\lambda_1+\lambda_2)\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*[a_0 b_{n-1}b_{m-1}\wedge 1_L] = d_2[g_2] = 0$ and then we have $\lambda_1+\lambda_2 = 0$, where $\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*[a_0 b_{n-1}b_{m-1}\wedge 1_L] \neq 0 \in \mathrm{Ext}_A^{7,tq+3q+3}(H^*Y, H^*L)$ since $\bar{h}\tilde{\phi}(\pi\wedge 1_L)(p\wedge 1_L)(\neq 0) \in [\Sigma^{3q}L, Y]$ by Proposition 2.2(3). That is, (3.9) becomes $[g_2] = \lambda_1\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*[\sigma'\wedge 1_L]$. Now we consider the cases that λ_1 is nonzero and zero separately.

If $\lambda_1 \neq 0$, (3.8) implies that $[g_2]$ and so $\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*[\sigma'\wedge 1_L] \in E_2^{5,tq+3q+2}(Y) = \mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L)$ is a permanent cycle in the ASS. Moreover, by $(\bar{a}_5\wedge 1_Y)(1_{E_6}\wedge(1_Y\wedge j)\alpha_{Y\wedge M})f\cdot(\alpha_1)_L = 0$ we have

$$(1_{E_6}\wedge(1_Y\wedge j)\alpha_{Y\wedge M})f\cdot(\alpha_1)_L = (\bar{c}_5\wedge 1_Y)g_3$$

with d_1-cycle $g_3 \in [\Sigma^{tq+3q+2}L, KG_5\wedge Y]$ which represents an element $[g_3] \in \mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L)$ so that $[g_3] = \bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*(\lambda_3[h_m b_{n-1}\wedge 1_L] + \lambda_4[h_n b_{m-1}\wedge 1_L])$ for some $\lambda_3, \lambda_4 \in Z_p$. By the above equation and the fact that $(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L)$ has filtration 2, we know that the differential $d_2([g_3]) = 0$ and so by a similar argument as shown above we have $\lambda_3+\lambda_4 = 0$. That is, $[g_3] = \lambda_3\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*[\sigma'\wedge 1_L]$ and so we have

$$(1_{E_6}\wedge(1_Y\wedge(\alpha_1)_L)(\omega\wedge 1_L))(f\wedge 1_L) = (\bar{c}_5\wedge 1_Y)g_3 = 0$$

which shows the result.

If $\lambda_1 = 0$, then $g_2 = (\bar{b}_5\bar{c}_4\wedge 1_Y)g_4$ for some $g_4 \in [\Sigma^{tq+3q+2}L, KG_4\wedge Y]$ and (3.8) becomes $(\bar{b}_5\wedge 1_V)f_2 = (\bar{b}_5\bar{c}_4\wedge 1_V)(1_{KG_4}\wedge w_4)g_4$. Consequently we have $f_2 = (\bar{c}_4\wedge 1_V)(1_{KG_4}\wedge w_4)g_4 +$

$(\bar{a}_5 \wedge 1_V)f_3$ for some $f_3 \in [\Sigma^{tq+3q+3}L, E_6 \wedge V]$ and so $(\bar{a}_5 \wedge 1_{M\wedge L})(f \wedge 1_L) = (1_{E_5} \wedge u_4)f_2 = (\bar{a}_5 \wedge 1_{M\wedge L})(1_{E_6} \wedge u_4)f_3$. It follows that $(f \wedge 1_L) = (1_{E_6} \wedge u_4)f_3 + (\bar{c}_5 \wedge 1_{M\wedge L})g_5$ for some $g_5 \in [\Sigma^{tq+3q+3}L, KG_5 \wedge M \wedge L]$ and so by (3.7) we have $(1_{E_6} \wedge (1_Y \wedge (\alpha_1)_L)(\omega \wedge 1_L))(f \wedge 1_L) = (\bar{c}_5 \wedge 1_Y)(1_{KG_5} \wedge (1_Y \wedge (\alpha_1)_L)(\omega \wedge 1_L))g_5 = 0$ since $(\alpha_1)_L$ induces zero homomorphism in Z_p-cohomology.

Proof of Theorem A We will continue the argument in Lemma 3.4. Note that the spectrum V in (3.7) also is the cofibre of $(1_M \wedge wi'')\tilde{\psi} : X \to \Sigma^{2q}M \wedge W$ given by the cofibration

$$X \xrightarrow{(1_M \wedge wi'')\tilde{\psi}} \Sigma^{2q}M \wedge W \xrightarrow{w_5} V \xrightarrow{u_5} \Sigma X. \tag{3.11}$$

This can be seen by the following commutative diagram of 3×3 Lemma

$$\begin{array}{ccccccccc}
\Sigma^{3q-1}M \wedge L & & \longrightarrow & & Y & & \xrightarrow{\tilde{u}w_2} & & \Sigma X \\
& \searrow^{1_M\wedge(\alpha_1)_L} & & \nearrow \omega & & \searrow w_4 & & \nearrow u_5 & & \searrow \tilde{\psi} \\
& & \Sigma^{2q}M & & & & V & & & & \Sigma^{2q+1}M \\
& \nearrow \tilde{\psi} & & \searrow^{1_M\wedge wi''} & & \nearrow w_5 & & \searrow u_4 & & \nearrow_{1_M\wedge(\alpha_1)_L} \\
X & & \longrightarrow & & \Sigma^{2q}M \wedge W & & \xrightarrow{1_M\wedge u} & & \Sigma^{3q}M \wedge L
\end{array}$$

It follows from Lemma 3.4(B) and (3.7) that $f \wedge 1_L = (1_{E_6} \wedge u_4)f_5$ for some $f_5 \in [\Sigma^{tq+3q+3}L, E_6 \wedge V]$ and so by Lemma 3.4(A) we have

$$\begin{aligned}(\bar{a}_4\bar{a}_5 \wedge 1_{M\wedge L})(1_{E_6} \wedge u_4)f_5 &= (\bar{a}_4\bar{a}_5 \wedge 1_{M\wedge L})(f \wedge 1_L) \\ &= (1_{E_4} \wedge i \wedge 1_L)(\kappa \wedge 1_L) - (\bar{c}_3 \wedge 1_{M\wedge L})(g \wedge 1_L).\end{aligned} \tag{3.12}$$

Consequently, $(\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5 \wedge 1_{M\wedge L})(1_{E_6} \wedge u_4)f_5 = 0$ and so $(\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5 \wedge 1_V)f_5 = (1_{E_2} \wedge w_4)f_6$ for some $f_6 \in [\Sigma^{tq+3q-1}L, E_2 \wedge Y]$. It follows that $(\bar{b}_2 \wedge 1_V)(1_{E_2} \wedge w_4)f_6 = 0$. Then $(\bar{b}_2 \wedge 1_Y)f_6 = 0$ and by $\mathrm{Ext}_A^{3+r,tq+3q+r}(H^*Y, H^*L) = 0$ for $r = 0, 1$ (cf. Proposition 2.5) we have $(\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5 \wedge 1_V)f_5 = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_V)(1_{E_5} \wedge w_4)f_7$ for some $f_7 \in [\Sigma^{tq+3q+2}L, E_5 \wedge Y]$. It follows that

$$(\bar{a}_3\bar{a}_4\bar{a}_5 \wedge 1_V)f_5 = (\bar{a}_3\bar{a}_4 \wedge 1_V)(1_{E_5} \wedge w_4)f_7 + (\bar{c}_2 \wedge 1_V)g_6 \tag{3.13}$$

with d_1-cycle $g_6 \in [\Sigma^{tq+3q}L, KG_2 \wedge V]$ which represents an element

$$[g_6] \in \mathrm{Ext}_A^{2,tq+3q}(H^*V, H^*L).$$

Note that the d_1-cycle $(\bar{b}_5 \wedge 1_Y)f_7 \in [\Sigma^{tq+3q+2}L, KG_5 \wedge Y]$ represents an element

$$[(\bar{b}_5 \wedge 1_Y)f_7] \in \mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L)$$

which has two generators stated in Proposition 2.4(2). Then

$$[(\bar{b}_5 \wedge 1_Y)f_7] = \lambda'\bar{h}_*\tilde{\phi}_*(\pi \wedge 1_L)_*[h_m b_{n-1} \wedge 1_L] + \lambda''\bar{h}_*\tilde{\phi}_*(\pi \wedge 1_L)_*[h_n b_{m-1} \wedge 1_L]$$

for some $\lambda', \lambda'' \in Z_p$. By the vanishing of the differential

$$0 = d_2[(\bar{b}_5 \wedge 1_Y)f_7] = (\lambda' + \lambda'')\bar{h}_*\tilde{\phi}_*(\pi \wedge 1_L)_*[a_0 b_{n-1} b_{m-1} \wedge 1_L]$$

we have $\lambda' + \lambda'' = 0$ since $\bar{h}\tilde{\phi}(\pi \wedge 1_L)(p \wedge 1_L) \neq 0 \in [\Sigma^{3q}L, Y]$ by Proposition 2.2(3). Hence we have

$$[(\bar{b}_5 \wedge 1_Y)f_7] = \lambda'\bar{h}_*\tilde{\phi}_*(\pi \wedge 1_L)_*[\sigma' \wedge 1_L] \in \mathrm{Ext}_A^{5,tq+3q+2}(H^*Y, H^*L). \tag{3.14}$$

We claim that the scalar λ' in (3.14) is zero. This can be proved as follows.

The equation (3.13) means that the second order differential of the ASS $d_2[g_6] = 0 \in E_2^{4,tq+3q+1}(L,V) = \mathrm{Ext}_A^{4,tq+3q+1}(H^*V, H^*L)$ so that $[g_6] \in E_3^{2,tq+3q}(L,V)$ and the third order differential

$$d_3[g_6] = (w_4)_*[(\bar{b}_5 \wedge 1_Y)f_7] \in E_3^{5,tq+3q+2}(L,V). \tag{3.15}$$

Note that

$$(\omega \wedge 1_L)(1_M \wedge (\alpha_1)_L)(i \wedge 1_L)\pi = (1_Y \wedge j)\alpha_{Y\wedge M} i(\alpha_1)_L\pi = \bar{h}\phi(\alpha_1)_L\pi = 0$$

since $\phi(\alpha_1)_L \in [\Sigma^{3q-2}L, L] = 0$ by $\pi_{rq-2}S = 0$ for $r = 2,3,4$. Then, by (3.7), $(i \wedge 1_L)\pi = u_4\tau$ with $\tau \in [\Sigma^{4q}S, V]$ which has filtration 1. Moreover, $u_4\tau \cdot p = (i \wedge 1_L)\pi \cdot p = 0$. Then, by Proposition 2.2(4), $\tau \cdot p = \tilde{\lambda} w_4 \bar{h}\tilde{\phi}(\pi \wedge 1_L)\pi$ for some $\tilde{\lambda} \in Z_{(p)}$. The scalar $\tilde{\lambda}$ must be zero (mod p) since the left-hand side has filtration 2 and the right-hand side has filtration 3 (cf. Remark 3.3 and $\mathrm{Ext}_A^{0,4q+1}(H^*V, Z_p) = 0$ by $\mathrm{Ext}_A^{0,4q+1}(H^*Y, Z_p) = 0 = \mathrm{Ext}_A^{0,q+1}(H^*M \wedge L, Z_p)$). Consequently, by Proposition 2.2(4), $\tau \cdot p = 0$ and so $\tau = \bar{\tau} i$ with $\bar{\tau} \in [\Sigma^{4q}M, V]$. Since

$$(u_4)_*(\pi)^*[g_6] \in \mathrm{Ext}_A^{3,tq+q+1}(H^*M \wedge L, Z_p) \cong Z_p\{(i \wedge 1_L)_*(\pi)_*(h_n h_m)\}$$

(cf. Proposition 2.7), we have

$$(u_4)_*\pi^*[g_6] = \lambda_0 (i \wedge 1_L)_*\pi_*(h_n h_m) = \lambda_0 (u_4)_*(\bar{\tau} i)_*(h_n h_m)$$

for some $\lambda_0 \in Z_p$ and so by (3.7) we have

$$\pi^*[g_6] = \lambda_0 \bar{\tau}_* i_*(h_n h_m) \in \mathrm{Ext}_A^{3,tq+3q+1}(H^*V, Z_p)$$

since $\mathrm{Ext}_A^{3,tq+3q+1}(H^*Y, H^*L) = 0$ (cf. Proposition 2.5(1)). Recall from Lemma 3.1(1) that

$$d_2(h_n h_m) = a_0\sigma' = p_*(\sigma') \in \mathrm{Ext}_A^{4,tq+1}(Z_p, Z_p).$$

Then $d_2 i_*(h_n h_m) = 0$ and so $i_*(h_n h_m) \in E_3^{4,tq+1}(S, M)$. Moreover,

$$E_2^{5,tq+2}(S,M) = \mathrm{Ext}_A^{5,tq+2}(H^*M, Z_p) = 0$$

by Proposition 2.7. Then the E_3-term $E_3^{5,tq+2}(S,M) = 0$ so that the third order differential

$$d_3 i_*(h_n h_m) \in E_3^{5,tq+2}(S,M) = 0.$$

Since $\pi^*[g_6] = \lambda_0(\bar{\tau})_* i_*(h_n h_m) \in E_2^{3,tq+4q+1}(S,V)$, we have

$$\pi^*[g_6] = \lambda_0 \bar{\tau}_*(i_*(h_n h_m)) \in E_3^{3,tq+4q+1}(S,V)$$

and so

$$d_3\pi^*[g_6] = \lambda_0 d_3(\bar{\tau})_*(i_*(h_n h_m)) = \lambda_0(\bar{\tau})_* d_3(i_*(h_n h_m)) = 0 \in E_3^{6,tq+4q+3}(S,V).$$

It follows from (3.15) that $(w_4)_*\pi^*[(\bar{b}_5 \wedge 1_Y)f_7] = d_3\pi^*[g_6] = 0 \in E_3^{6,tq+4q+2}(S,V)$. Moreover, by the split exact sequence (3.10) we have $\pi^*[(\bar{b}_5 \wedge 1_Y)f_7] = 0 \in E_3^{6,tq+4q+3}(S,Y)$. Consequently, in the E_2-term, $\pi^*[(\bar{b}_5 \wedge 1_Y)f_7]$ must be a d_2-boundary, that is

$$\pi^*[(\bar{b}_5 \wedge 1_Y)f_7] \in d_2 E_2^{4,tq+4q+2}(S,Y) = d_2\mathrm{Ext}_A^{4,tq+4q+2}(H^*Y, Z_p) = 0$$

by Proposition 2.5(1) and so, by (3.14), $\lambda'\bar{h}_*\tilde{\phi}_*(\pi\wedge 1_L)_*\pi_*(\sigma')=0$. This implies that the scalar λ' is zero (cf. Proposition 2.2(4)) which shows the above claim.

Hence, (3.13) becomes

$$(\bar{a}_3\bar{a}_4\bar{a}_5\wedge 1_V)f_5=(\bar{a}_3\bar{a}_4\bar{a}_5\wedge 1_V)(1_{E_6}\wedge w_4)f_8+(\bar{c}_2\wedge 1_V)g_6$$

with $f_8\in[\Sigma^{tq+3q+3}L,E_6\wedge Y]$. It follows by composing $1_{E_3}\wedge u_5$ that

$$(\bar{a}_3\bar{a}_4\bar{a}_5\wedge 1_{Y\wedge W})(1_{E_6}\wedge u_5)f_5=(\bar{a}_3\bar{a}_4\bar{a}_5\wedge 1_X)(1_{E_6}\wedge\tilde{u}w_2)f_8$$

(cf. the diagram above (3.12)), this is because $(\bar{c}_2\wedge 1_X)(1_{KG_2}\wedge u_5)g_6=0$ by the fact that $(1_{KG_2}\wedge u_5)g_6\in[\Sigma^{tq+3q-1}L,KG_2\wedge X]$ represents an element in $\mathrm{Ext}_A^{2,tq+3q-1}(H^*X,H^*L)=0$ (cf. Lemma 3.2(2)). Consequently we have

$$(\bar{a}_4\bar{a}_5\wedge 1_X)(1_{E_6}\wedge u_5)f_5=(\bar{a}_4\bar{a}_5\wedge 1_X)(1_{E_6}\wedge\tilde{u}w_2)f_8+(\bar{c}_3\wedge 1_X)g_7 \tag{3.16}$$

with d_1-cycle $g_7\in[\Sigma^{tq+3q+1}L,KG_3\wedge X]$ which represents an element in $\mathrm{Ext}_A^{3,tq+3q}(H^*X,H^*L)$.

Now we prove $(\bar{c}_3\wedge 1_X)g_7=0$ as follows. By Lemma 3.2(2) and Proposition 2.6(1),

$$[g_7]=\lambda_3(\tilde{u}w_3)_*(\bar{\phi}_W)_*[h_nh_m\wedge 1_L]$$

and the equation (3.16) means the second order differential $d_2[g_7]=0$. Since

$$d_2(h_nh_m)=a_0\sigma'=p_*(\sigma')\in\mathrm{Ext}_A^{4,tq+1}(Z_p,Z_p)$$

by Lemma 3.1(1), we have

$$\lambda_3(\tilde{u}w_3)_*(\bar{\phi}_W)_*(p\wedge 1_L)_*[\sigma'\wedge 1_L]=d_2[g_7]=0\in\mathrm{Ext}_A^{5,tq+3q+1}(H^*X,H^*L).$$

By Lemma 3.2(1), this implies $\lambda_3=0$ and so g_7 is a d_1-boundary so that $(\bar{c}_3\wedge 1_X)g_7=0$.

Consequently, (3.16) becomes

$$(\bar{a}_4\bar{a}_5\wedge 1_{Y\wedge W})(1_{E_6}\wedge u_5)f_5=(\bar{a}_4\bar{a}_5\wedge 1_X)(1_{E_6}\wedge\tilde{u}w_2)f_8$$

and so by (3.2) and the diagram above (3.12),

$$(\bar{a}_4\bar{a}_5\wedge 1_M)(1_{E_6}\wedge(1_M\wedge(\alpha_1)_L)u_4)f_5=(\bar{a}_4\bar{a}_5\wedge 1_M)(1_{E_6}\wedge\tilde{\psi}u_5)f_5=0.$$

Moreover, by composing $(1_{E_4}\wedge 1_M\wedge(\alpha_1)_L)$ on (3.12) we have

$$\begin{aligned}(1_{E_4}\wedge i)\kappa\cdot(\alpha_1)_L&=(1_{E_4}\wedge 1_M\wedge(\alpha_1)_L)(1_{E_4}\wedge i\wedge 1_L)(\kappa\wedge 1_L)\\&=(\bar{a}_4\bar{a}_5\wedge 1_M)(1_{E_6}\wedge(1_M\wedge(\alpha_1)_L)u_4)f_5=0.\end{aligned}$$

It follows that

$$\kappa\cdot(\alpha_1)_L=(1_{E_4}\wedge p)f_9 \tag{3.17}$$

with $f_9\in[\Sigma^{tq+q}L,E_4]$. Recall that $\bar{b}_6\cdot\kappa=a_0\sigma'=p_*(\sigma')\in\mathrm{Ext}_A^{4,tq+q}(Z_p,Z_p)$. Then $\kappa\cdot(\alpha_1)_L$ lifts to a map $\tilde{f}\in[\Sigma^{tq+q+1}L,E_5]$ such that $\bar{b}_5\cdot\tilde{f}$ represents

$$p_*((\alpha_1)_L)_*[\sigma'\wedge 1_L]\neq 0\in\mathrm{Ext}_A^{5,tq+q+1}(Z_p,H^*L)$$

(cf. Proposition 2.2(1)). Then, by (3.17),

$$p_*[\bar{b}_4 \cdot f_9] = p_*((\alpha_1)_L)_*[\sigma' \wedge 1_L]$$

and so $[\bar{b}_4 \cdot f_9] \in \mathrm{Ext}_A^{4,tq+q}(Z_p, H^*L)$ must be equal to $((\alpha_1)_L)_*[\sigma' \wedge 1_L]$ since the location group has two generator $((\alpha_1)_L)_*[h_m b_{n-1} \wedge 1_L]$ and $((\alpha_1)_L)_*[h_n b_{m-1} \wedge 1_L]$ by $\mathrm{Ext}_A^{4,tq+q}(Z_p, Z_p) \cong Z_p\{h_0 h_n b_{m-1}, h_0 h_m b_{n-1}\}$ and $\mathrm{Ext}_A^{4,tq+2q}(Z_p, Z_p) = 0$ in Proposition 2.1(1). Write $\xi_{n,4} = f_9 i''$. Then

$$\kappa \cdot \alpha_1 = (1_{E_4} \wedge p)\xi_{n,4} \tag{3.18}$$

with $\bar{b}_4 \cdot \xi_{n,4} = h_0\sigma' \in \mathrm{Ext}_A^{4,tq+q}(Z_p, Z_p)$ and so by Lemma 3.1(2) we have

$$(\bar{c}_2 \wedge 1_M)(1_{KG_3} \wedge i)h_0 h_n h_m = (1_{E_4} \wedge i)\kappa \cdot \alpha_1 = 0.$$

This shows the second result of the theorem. Moreover, by (3.18) and Lemma 3.1(2),

$$\bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3(1_{E_4} \wedge p)\xi_{n,4} = 0,$$

this shows that $\xi_n = \bar{a}_0\bar{a}_1\bar{a}_2\bar{a}_3 \cdot \xi_{n,4} \in \pi_{tq+q-4}S$ is a map of order p which is represented by $h_0\sigma' \in \mathrm{Ext}_A^{4,tq+q}(Z_p, Z_p)$ in the ASS.

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