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The Convergence of $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$**

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Abstract This paper computes the Thom map on γ_2 and proves that it is represented by $2b_{2,0}h_{1,2}$ in the ASS. The authors also compute the higher May differential of $b_{2,0}$, from which it is proved that $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ for $2 \le s < p-1$ are permanent cycles in the ASS.

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1 Introduction

Let p be a prime and let S be the sphere spectrum localized at p. To determine the stable homotopy groups π_*S is one of the central problems in stable homotopy. One of the main tools to reach it is the Adams spectral sequence (ASS) with E_2-term $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(Z/p, Z/p) \Rightarrow \pi_{t-s}S$, where $\mathrm{Ext}_A^{s,t}(Z/p, Z/p)$ denotes the cohomology of the Steenrod algebra A.

From [5] we know that for odd prime p, $\mathrm{Ext}_A^{1,*}(Z/p, Z/p)$ is the Z/p-module generated by a_0 and h_i for $i \ge 0$. $\mathrm{Ext}_A^{2,*}(Z/p, Z/p)$ is the Z/p-module generated by a_1h_0, a_0^2, a_0h_i $(i > 0)$, g_i $(i \ge 0)$, k_i $(i \ge 0)$, b_i $(i \ge 0)$ and h_ih_j $(j \ge i+2 \ge 2)$. The Ext groups $\mathrm{Ext}_A^{3,*}(Z/p, Z/p)$ were detected in [2].

If a family of homology elements x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of homotopy elements f_i in π_*S. In this case we say that the homotopy element f_i is represented by x_i in the ASS. So far, not so many families of homotopy elements were detected. For example, a family $\zeta_n \in \pi_*S$ for $n \ge 1$ was detected in [3] which is represented by $h_0b_n \in \mathrm{Ext}_A^{3,*}(Z/p, Z/p)$ in the ASS. In this paper, we detect a family of homotopy elements which has filtration $s+3$.

Let M denote the mod p Moore spectrum and $v_1 : \Sigma^q M \to M$ denote the Adams map which is known to exist for $p > 2$ (cf. [9]). Then we have the cofibre sequence

$$\Sigma^q M \xrightarrow{v_1} M \longrightarrow V(1),$$

where $q = 2(p-1)$ and $V(1)$ is known to be the Smith-Toda spectrum. For $p > 3$, there exists the Smith-Toda map $v_2 : \Sigma^{(p+1)q}V(1) \to V(1)$, and its cofibre is denoted by $V(2)$. For $p \ge 7$, there exists the Smith-Toda map $v_3 : \Sigma^{(p^2+p+1)q}V(2) \to V(2)$ (cf. [9]). From [10] we know that

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the following composition of maps denoted by γ_s in π_*S,

$$S^{s\cdot(p^2+p+1)q} \xrightarrow{i} \Sigma^{s\cdot(p^2+p+1)q}V(2) \xrightarrow{v_3^s} V(2) \xrightarrow{j} S^{(p+2)q+3},$$

is represented in the ASS by $\tilde{\gamma}_s = \frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$ for $s \geq 3$ and $\not\equiv 0, 1, 2 \bmod p$.

In [4], Lin detected a new family in the stable homotopy groups of spheres and proved the following theorem:

Theorem 1.1 (See [4, Theorem A]) *Let $p \geq 5$, $n \geq 3$. Then*

(1) *$i_*(h_1h_n) \in \mathrm{Ext}_A^{2,(p+p^n)q}(H^*M, Z/p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi_n \in \pi_{(p+p^n)q-2}M$.*

(2) *For $\xi_n \in \pi_{(p+p^n)q-2}M$ obtained in (1), $j\xi_n \in \pi_{(p+p^n)q-3}S$ is a nontrivial element of order p represented (up to nonzero scalar) by $(b_0h_n - h_1b_{n-1}) \in \mathrm{Ext}_A^{3,(p+p^n)q}(Z/p, Z/p)$ in the ASS.*

In this paper we will prove that γ_2 is represented by $\tilde{\gamma}_2 = 2b_{2,0}h_{1,2} \in \mathrm{Ext}_A^{3,*}(Z/p, Z/p)$ in the ASS and from which we prove

Theorem A *Let $p \geq 7$ be an odd prime and $n \geq 4$. Then for $3 \leq s < p-1$,*

$$\tilde{\gamma}_s(b_0h_n - h_1b_{n-1}) \in \mathrm{Ext}_A^{s+3,q((s-2)+sp+sp^2+p^n)+(s-3)}(Z/p, Z/p)$$

and

$$\tilde{\gamma}_2(b_0h_n - h_1b_{n-1}) \in \mathrm{Ext}_A^{6,q(2p+2p^2+p^n)}(Z/p, Z/p)$$

are permanent cycles in the ASS, and then they converge to nontrivial elements in π_*S, where $q = 2(p-1)$.

2 The Representation of γ_2 in the ASS

In this section we will consider some of the Thom maps

$$\Phi : \mathrm{Ext}_{BP_*BP}^{s,*}(BP_*, BP_*K) \longrightarrow \mathrm{Ext}_{A^*}^{s,*}(Z/p, H_*K) = \mathrm{Ext}_A^{s,*}(H^*K, Z/p)$$

for the spectrum $K = V(1)$, M and S, from which we get the representation of γ_2 in the classical ASS. The result might be known to many people, but we had not seen it appearing. Here we give a brief proof of the result.

Consider the Brown-Peterson ring spectrum BP at a prime p with $BP_* = \pi_*BP = Z_{(p)}[v_1, v_2, \cdots, v_n, \cdots]$ (cf. [7, Chapters 4, 5]). There is the Thom map $\Phi : BP \to KZ/p$ which induces

$$\Phi : \pi_*(BP) = BP_* \longrightarrow \pi_*(KZ/p) = Z/p$$

and

$$\Phi : \pi_*(BP \wedge BP) = BP_*BP \longrightarrow \pi_*(KZ/p \wedge KZ/p) = A^*,$$

where $BP_*BP = BP_*[t_1, t_2, \cdots, t_n, \cdots]$, A^* is the dual of the Steenrod algebra A,

$$A^* = Z/p\,[\xi_1, \xi_2, \cdots, \xi_n, \cdots] \otimes E[\tau_0, \tau_1, \cdots, \tau_n, \cdots].$$

The Thom map sends v_i to 0 and t_n to $\tilde{t}_n$, where $\tilde{t}_n$ is the Hopf conjugate of Milnor's ξ_n. Applying the cobar construction, we get the Thom map

$$\Phi : \mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*K) \longrightarrow \mathrm{Ext}^{s,*}_{A^*}(Z/p, H_*K).$$

To compute the Ext groups $\mathrm{Ext}^{s,t}_A(Z/p, Z/p)$, we have the May spectral sequence (MSS) (cf. [6] and [7, Chapter 3]) whose $E_0^{s,t,*}$-term is the filtrated cobar complex and the E_1-term is the cohomology of $E_0^{s,t,*}$,

$$E_1^{s,t,*} = E[h_{i,j} \mid i > 0, j \geq 0] \otimes P[b_{i,j} \mid i > 0, j \geq 0] \otimes P[a_i \mid i \geq 0],$$

where $h_{i,j}$ is the cohomology class represented by $\xi_i^{p^j}$, $b_{i,j}$ is the cohomology class represented by $\sum\limits_{k=1}^{p-1} \binom{p}{k}/p\,(\xi_i^{kp^j} \otimes \xi_i^{(p-k)p^j})$ and a_i is the cohomology class represented by τ_i. Thus the homological dimensions of a_i, $h_{i,j}$ and $b_{i,j}$ are 1, 1 and 2 respectively. The inner degrees are

$$\begin{aligned} \deg(a_i) &= 2(p-1)(1 + p + \cdots + p^{i-1}) + 1, \\ \deg(h_{i,j}) &= 2(p-1)(p^j + p^{j+1} + \cdots + p^{i+j-1}), \\ \deg(b_{i,j}) &= 2(p-1)(p^{j+1} + p^{j+2} + \cdots + p^{i+j}). \end{aligned} \tag{2.1}$$

The May filtration M's are $M(a_i) = 2i + 1$, $M(h_{i,j}) = 2i - 1$ and $M(b_{i,j}) = p(2i - 1)$. For the May differentials, we have $d_r : E_r^{s,t,M} \to E_r^{s+1,t,M-r}$, where M denotes the May filtration. If $x \in E_r^{s,t,*}$, $y \in E_r^{s',t',*}$, then $x \cdot y = (-1)^{(s+t)(s'+t')} y \cdot x$ and $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{s+t} x \cdot d_r(y)$. For the first May differential, we have

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,j+k} h_{k,j}, \quad d_1(a_i) = \sum_{0 \leq k < i} h_{i-k,k} a_k, \quad d_1(b_{i,j}) = 0.$$

To compute the Ext groups $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*K)$, we have the chromatic spectral sequence (cf. [7]), induced by the short exact sequences

$$0 \longrightarrow N_n^s \xrightarrow{j} M_n^s \xrightarrow{k} N_n^{s+1} \longrightarrow 0, \tag{2.2}$$

where $N_n^0 = BP_*/I_n$, I_n is the ideal of BP_* generated by $\{p, v_1, \cdots v_{n-1}\}$ with $I_0 = 0$. M_n^s is inductively defined by $M_n^s = v_{n+s}^{-1} N_n^s$.

From [7], we jot down some of the structure maps $\eta_R : BP_* \to BP_*BP$ and $\Delta : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$,

$$\begin{aligned} \eta_R(v_1) &= v_1 + pt_1, \\ \eta_R(v_2) &= v_2 + v_1 t_1^p - v_1^p t_1 && \mod (p), \\ \eta_R(v_3) &= v_3 + v_2 t_1^{p^2} + v_1 t_2^p - t_1 \eta_R(v_2^p) + v_1^2 V && \mod (p, v_1^{p^2}), \\ \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1, \\ \Delta(t_2) &= t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1^p - v_1 \bar{b}_{1,0} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} pv_1V &= v_2^p + v_1^p t_1^{p^2} - v_1^{p^2} t_1^p - (v_2 + v_1 t_1^p - v_1^p t_1)^p, \\ p\bar{b}_{1,0} &= (t_1 \otimes 1 + 1 \otimes t_1)^p - (t_1^p \otimes 1 + 1 \otimes t_1^p). \end{aligned}$$

Thus we have

$$d(v_3^s) = \eta_R(v_3^s) - v_3^s \equiv 0 \mod (p, v_1, v_2),$$

and in the Ext groups we have

$$\begin{aligned} v_3^s/v_2 &\in \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, N_2^1), \quad &\gamma_s'' &= \delta_1''(v_3^s/v_2) \in \mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*V(1)), \\ v_3^s/v_1v_2 &\in \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, N_1^2), \quad &\gamma_s' &= \delta_1'\delta_2'(v_3^s/v_1v_2) \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*M), \\ v_3^s/pv_1v_2 &\in \mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, N_0^3), \quad &\gamma_s &= \delta_1\delta_2\delta_3(v_3^s/pv_1v_2) \in \mathrm{Ext}^{3,*}_{BP_*BP}(BP_*, BP_*), \end{aligned}$$

where

$$\begin{aligned} \delta_1'' &: \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, N_2^1) \longrightarrow \mathrm{Ext}^{*+1,*}_{BP_*BP}(BP_*, BP_*V(1)), \\ \delta_s' &: \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, N_1^s) \longrightarrow \mathrm{Ext}^{*+1,*}_{BP_*BP}(BP_*, N_1^{s-1}), \\ \delta_s &: \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, N_0^s) \longrightarrow \mathrm{Ext}^{*+1,*}_{BP_*BP}(BP_*, N_0^{s-1}) \end{aligned}$$

are the connecting homomorphisms induced by (2.2).

Theorem 2.1 *For the Thom map $\Phi : \mathrm{Ext}^{s,*}_{BP_*BP}(BP_*K, BP_*) \to \mathrm{Ext}^{s,*}_{A}(H_*K, Z/p)$ we have*

$$\begin{aligned} \Phi(\gamma_1'') &= i_*''(-h_2) \in \mathrm{Ext}^{1,*}_{A^*}(Z/p, H_*V(1)), \\ \Phi(\gamma_2') &= i_*'(2h_{2,1}h_{1,2}) \in \mathrm{Ext}^{2,*}_{A^*}(Z/p, H_*M), \\ \Phi(\gamma_2) &= 2b_{2,0}h_{1,2} \in \mathrm{Ext}^{3,*}_{A^*}(Z/p, Z/p), \end{aligned}$$

where $h_{2,1}h_{1,2}$ represents the generator $k_1 \in \mathrm{Ext}^{2,*}_{A}(Z/p, Z/p)$ (cf. [5]), and i_*'', i_*' are the maps induced by $i'' : S \to V(1)$ and $i' : S \to M$ respectively.

Proof From (2.3) we see that

$$\begin{aligned} d(v_3) &\equiv v_2(t_1^{p^2} - v_2^{p-1}t_1) &\mod (p, v_1), \\ d(v_3^2) &\equiv 2v_2v_3(t_1^{p^2} - v_2^{p-1}t_1) + v_2^2(t_1^{p^2} - v_2^{p-1}t_1)^2 &\mod (p, v_1). \end{aligned}$$

Thus we have

$$\begin{aligned} &\gamma_1'' = \delta_1''(v_3/v_2) = (t_1^{p^2} - v_2^{p-1}t_1), \\ &\delta_2'(v_3^2/v_1v_2) = (2v_3(t_1^{p^2} - v_2^{p-1}t_1) + v_2(t_1^{p^2} - v_2^{p-1}t_1)^2)/v_1, \\ &\delta_3(v_3^2/pv_1v_2) = (2v_3(t_1^{p^2} - v_2^{p-1}t_1) + v_2(t_1^{p^2} - v_2^{p-1}t_1)^2)/pv_1. \end{aligned}$$

By the Thom map, we see that $\Phi(\gamma_1'') = \tilde{t}_1^{p^2}$ which represents $i_*''(-h_2) \in \mathrm{Ext}^{1,*}_{A^*}(Z/p, H_*V(1))$. The first follows.

Similarly from (2.3), we see that mod $(p, v_1^3, v_1^2v_2, v_1v_2^2)$,

$$d(2v_3(t_1^{p^2} - v_2^{p-1}t_1) + v_2(t_1^{p^2} - v_2^{p-1}t_1)^2) \equiv 2v_1t_2^p \otimes t_1^{p^2} + v_1t_1^p \otimes t_1^{2p^2}.$$

Thus, we have

$$\gamma_2' = \delta_1'\delta_2'(v_3^2/v_1v_2) = 2t_2^p \otimes t_1^{p^2} + t_1^p \otimes t_1^{2p^2} + v_1^2x + v_1v_2y + v_2^2z,$$
$$\delta_2\delta_3(v_3^2/pv_1v_2) = (2t_2^p \otimes t_1^{p^2} + t_1^p \otimes t_1^{2p^2} + v_1^2x + v_1v_2y + v_2^2z)/p,$$

and the Thom map sends γ_2' to $2\tilde{t}_2^p \otimes \tilde{t}_1^{p^2} + \tilde{t}_1^p \otimes \tilde{t}_1^{2p^2}$, which represents $i_*'(h_{2,1}h_{1,2})$. The second follows.

From $d(v_1^2) \equiv d(v_1v_2) \equiv d(v_2^2) \equiv 0 \bmod (p^2, v_1, v_2)$ we see that

$$\begin{aligned} &d(2t_2^p \otimes t_1^{p^2} + t_1^p \otimes t_1^{2p^2} + v_1^2x + v_1v_2y + v_2^2z) \\ \equiv\; &-2p(\bar{b}_{2,0} \otimes t_1^{p^2} - t_2^p \otimes \bar{b}_{1,1} + \bar{b}_{1,0} \otimes t_1^{2p^2} + \cdots) + p(v_1\tilde{x} + v_2\tilde{y}) \mod (p^2), \end{aligned}$$

where

$$\begin{aligned} p\bar{b}_{2,0} &= (t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2)^p - t_2^p \otimes 1 - t_1^p \otimes t_1^{p^2} - 1 \otimes t_2^p, \\ p\bar{b}_{1,1} &= (t_1 \otimes 1 + 1 \otimes t_1)^{p^2} - t_1^{p^2} \otimes 1 - 1 \otimes t_1^{p^2}, \\ p\bar{b}_{1,0} &= (t_1 \otimes 1 + 1 \otimes t_1)^p - t_1^p \otimes 1 - 1 \otimes t_1^p. \end{aligned} \tag{2.4}$$

Thus

$$\gamma_2 = \delta_1\delta_2\delta_3(v_3^2/pv_1v_2) = -2\bar{b}_{2,0} \otimes t_1^{p^2} + 2t_2^p \otimes \bar{b}_{1,1} + \cdots$$

and the Thom map sends it to $-2\bar{b}_{2,0} \otimes \tilde{t}_1^{p^2} + 2\tilde{t}_2^p \otimes \bar{b}_{1,1} + \cdots$ which represents $2b_{2,0}h_{1,2}$ in the MSS. The third follows.

Corollary 2.1 *For $p \geq 7$, the homotopy class γ_2 is represented by $\tilde{\gamma}_2 \in \mathrm{Ext}_A^{3,*}(Z/p, Z/p)$ in the classical ASS and $\tilde{\gamma}_2$ is represented by $2b_{2,0}h_{1,2}$ in the MSS.*

3 The $E_1^{s,t,M}$-Term of the MSS with Specialized s and t

Consider the MSS $E_1^{s,t,M} \Rightarrow \mathrm{Ext}_A^{s,t}(Z/p, Z/p)$, whose E_1-term is

$$E_1^{s,t,*} = E[h_{i,j} \mid i > 0, j \geq 0] \otimes P[b_{i,j} \mid i > 0, j \geq 0] \otimes P[a_i \mid i \geq 0].$$

The generators of $E_1^{s,t,*}$ are denoted by monomials

$$g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_l) \cdot (z_1 \cdots z_m), \tag{3.1}$$

where x_i is of the elements a_i, y_i is of the elements $h_{i,j}$ and z_i is of $b_{i,j}$. In this section, we will compute some $E_1^{s,t,*}$ with specialized s and t, from which we will prove the Theorem A in the next section.

To compute $E_1^{s,t,*}$ with specialized s and t, denote t by $\bar{t}+b$ where $\bar{t}=2(p-1)(\bar{c}_0+\bar{c}_1p+\cdots+\bar{c}_np^n)$ with $0\le\bar{c}_i<p$ and $0<\bar{c}_n<p$. Thus $b\equiv t \bmod 2(p-1)$. If a monomial $x_1\cdots x_b\cdot y_1\cdots y_l\cdot z_1\cdots z_m$ of the form (3.1) is a generator of $E_1^{s,t,*}$, then $b+l+2m=s$.

Notice from (2.1), the inner degrees of x_i, y_i and z_i could be uniquely expressed as

$$\begin{aligned}\deg(x_i)&=2(p-1)(x_{i,0}+x_{i,1}p+\cdots+x_{i,n}p^n)+1,\\ \deg(y_i)&=2(p-1)(y_{i,0}+y_{i,1}p+\cdots+y_{i,n}p^n),\\ \deg(z_i)&=2(p-1)(0+z_{i,1}p+\cdots+z_{i,n}p^n),\end{aligned}$$

where the number sequence $(x_{1,0},x_{1,1},\cdots,x_{1,n})$ is the form of $(1,\cdots1,0,\cdots,0)$, the sequences $(y_{i,0},y_{i,1},\cdots,y_{i,n})$ and $(0,z_{i,1},\cdots,z_{i,n})$ are the form of $(0,\cdots,0,1,\cdots,1,0,\cdots,0)$. Thus a generator of the form (3.1) determines a matrix

$$\begin{pmatrix} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{l,0} & 0 & \cdots & 0\\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{l,1} & z_{1,1} & \cdots & z_{m,1}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{l,n} & z_{1,n} & \cdots & z_{m,n}\end{pmatrix}. \tag{3.2}$$

And from the property of the p-adic number, we have

$$\begin{cases}\sum\limits_{1\le i\le b}x_{i,0}+\sum\limits_{1\le i\le l}y_{i,0}=\bar{c}_0+k_1p=c_0,\\ \sum\limits_{1\le i\le b}x_{i,1}+\sum\limits_{1\le i\le l}y_{i,1}+\sum\limits_{1\le i\le m}z_{i,1}=\bar{c}_1+k_2p-k_1=c_1,\\ \qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\ \sum\limits_{1\le i\le b}x_{i,n-1}+\sum\limits_{1\le i\le l}y_{i,n-1}+\sum\limits_{1\le i\le m}z_{i,n-1}=\bar{c}_{n-1}+k_np-k_{n-1}=c_{n-1},\\ \sum\limits_{1\le i\le b}x_{i,n}+\sum\limits_{1\le i\le l}y_{i,n}+\sum\limits_{1\le i\le m}z_{i,n}=\bar{c}_n-k_n=c_n.\end{cases} \tag{3.3}$$

From the commutativity of $E_1^{s,t,*}$, the monomial of the form (3.1) is arranged in the following way:

(a) If $i>j$, we put a_i on the left side of a_j;

(b) We put $h_{i,j}$ on the left side of $h_{m,k}$ if $j<k$;

(c) If $i>k$, we put $h_{i,j}$ on the left side of $h_{k,j}$;

(d) Apply the same rules (b) and (c) to $b_{i,j}$.

Thus the entries of matrix (3.2) are 0 or 1, and satisfy

$$\begin{cases}(1)\ x_{1,j}\ge x_{2,j}\ge\cdots\ge x_{b,j},\ x_{i,0}\ge x_{i,1}\ge\cdots\ge x_{i,n}\ \text{for } i\le b,\ j\le n;\\ (2)\ \text{if } y_{i,j}\ne0 \text{ and } y_{i,j-1}=0 \text{ then for all } k<j,\ y_{i,k}=0;\\ (3)\ \text{if } y_{i,j}\ne0 \text{ and } y_{i,j+1}=0 \text{ then for all } k>j,\ y_{i,k}=0;\\ (4)\ y_{1,0}\ge y_{2,0}\ge\cdots\ge y_{l,0};\\ (5)\ \text{if } y_{i,0}=y_{i+1,0},\ y_{i,1}=y_{i+1,1},\ \cdots\ y_{i,j}=y_{i+1,j},\ \text{then } y_{i,j+1}\ge y_{i+1,j+1};\\ (6)\ \text{Apply the same rules (2)–(5) to } z_{i,j}.\end{cases} \tag{3.4}$$

It is easy to see that a matrix solution of (3.3) satisfying (3.4) determines a unique generator of the form $g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_l) \cdot (z_1 \cdots z_m) \in F_1^{b+l+2m,\bar{t}+b,*}$, where

$$F_1^{s,\bar{t}+b,*} = P[h_{i,j} \mid i > 0, j \geq 0] \otimes P[b_{i,j} \mid i > 0, j \geq 0] \otimes P[a_i \mid i \geq 0].$$

The E_1-term of the MSS is $F_1^{s,\bar{t}+b,*}/\{h_{i,j}^2\}$.

Remark 3.1 The matrix solution of (3.3) does not always induce a generator of $E_1^{s,t,*}$.

(1) If two columns in the $y_{i,j}$ part are the same, it deduces $h_{i,j}^2$ in the monomial.

(2) If all the entries of a column are 0 in the $y_{i,j}$ or $z_{i,j}$ parts, it deduces none. But it deduces a_0 while in the $x_{i,j}$ part.

Proposition 3.1 *Let $p > 5$ be an odd prime and $t = 2(p-1)(2p + 2p^2 + p^n)$ with $n \geq 4$. Then the E_1-terms of the MSS $E_1^{6-r,t-r+1,M}$ are zero except for $r = 1$, which are the Z/p modules generated by*

$$\begin{array}{ll}
b_{2,0}^2 h_{1,n} & \text{with } M = 6p+1; \\
b_{2,0}h_{2,1}b_{1,n-1} & \text{with } M = 4p+3; \\
b_{2,0}h_{1,2}h_{1,1}h_{1,n} & \text{with } M = 3p+3; \\
\left\{\begin{matrix} h_{2,1}h_{1,2}h_{1,1}b_{1,n-1} & h_{2,1}h_{1,2}h_{1,n}b_{1,0} \\ h_{2,1}h_{1,1}h_{1,n}b_{1,1} & \end{matrix}\right\} & \text{with } M = p+5.
\end{array}$$

Proof Suppose we have a generator of the form (3.1) $g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_l) \cdot (z_1 \cdots z_m)$ in $E_1^{6-r,t-r+1,M}$. Then from $b + l + 2m = 6 - r$ and $b \equiv -r + 1 \bmod 2(p-1)$ we would have that $b = 2(p-1) - r + 1 \geq p$ for $r > 1$. Thus $E_1^{6-r,2(p-1)(2p+2p^2+p^n)-r+1,M} = 0$ for $r > 1$.

For $r = 1$, we see that $b = 0$ from $6 - r = 5$, $t-r+1 = 2(p-1)(2p+2p^2+p^n)$ and $b \equiv t-r+1 \bmod 2(p-1)$. Suppose we have a generator of the form (3.1) $g = (y_1 \cdots y_l) \cdot (z_1 \cdots z_m)$ in $E_1^{5,2(p-1)(2p+2p+p^n),*}$. Then we see from $l + 2m = 5$ that $(0, l, m)$ could be $(0, 1, 2)$, $(0, 3, 1)$ or $(0, 5, 0)$.

For $(0, l, m) = (0, 1, 2)$, the corresponding equation (3.3) becomes

$$\begin{cases}
y_{1,0} & & & = 0 + k_1 p & = c_0, \\
y_{1,1} & + z_{1,1} & + z_{2,1} & = 2 + k_2 p - k_1 & = c_1, \\
y_{1,2} & + z_{1,2} & + z_{2,2} & = 2 + k_3 p - k_2 & = c_2, \\
y_{1,3} & + z_{1,3} & + z_{2,3} & = 0 + k_4 p - k_3 & = c_3, \\
 & \vdots & & \vdots & \\
y_{1,n-1} & + z_{1,n-1} & + z_{2,n-1} & = 0 + k_n p - k_{n-1} & = c_{n-1}, \\
y_{1,n} & + z_{1,n} & + z_{2,n} & = 1 - k_n & = c_n.
\end{cases}$$

To solve it, we firstly determine that the carrying numbers $k_1, k_2, \cdots, k_n$ are 0 from all of $y_{1,j}$ and $z_{i,j}$ being 0 or 1. Then applying the rule (3.4), list the matrix entries row by row so that the sum of rows are $c_0 = 0$, $c_1 = 2$, $c_2 = 2$, $c_3 = 0$, $\cdots$, $c_{n-1} = 0$ and $c_n = 1$ respectively.

Indeed, from $c_0 = c_3 = \cdots = c_{n-1} = 0$ we could determine that all the entries in the 1^{st} and the $4^{\text{th}} \sim (n-1)^{\text{th}}$ rows are 0. Then from $c_n = 1$ we see that the last row is one of $(0|0\ 1)$, (0|1 0) or (1|0 0), here we use $(*|*\ *)$ to distinguish the $y_{i,j}$ part from $z_{i,j}$ part. Yet we see that the last row could not be (0|1 0). If so, then from $z_{1,n}=1, z_{1,n-1}=0$ and rule (3.4) (2), we see that $z_{1,1}=0$. Now $z_{1,0}=z_{2,0}=0$, $z_{1,1}=0$, applying rule (3.4) (5) we see that $0=z_{1,1}\geq z_{1,2}$, this contradicts $y_{1,1}+z_{1,1}+z_{1,2}=2$. From the analysis above, we get two matrix solutions

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)\begin{array}{c} 0 \\ 2 \\ 2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array} \qquad \left(\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)\begin{array}{c} 0 \\ 2 \\ 2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}$$

which induce respectively the generators

$$h_{2,1}b_{2,0}b_{1,n-1} \quad \text{and} \quad h_{1,n}b_{2,0}^2.$$

Similarly solving the corresponding equation (3.3) for $(0,l,m)=(0,3,1)$, we get the generators

$$h_{1,2}h_{1,1}h_{1,n}b_{2,0}, \quad h_{2,1}h_{1,1}h_{1,n}b_{1,1}, \quad h_{2,1}h_{1,2}h_{1,n}b_{1,0}, \quad h_{2,1}h_{1,2}h_{1,1}b_{1,n-1}$$

in $E_1^{5,t,*}$. Solving the corresponding equation (3.3) for $(0,l,m)=(0,5,0)$, we get the generator $h_{1,1}^2h_{1,2}^2h_{1,n}\in F_1^{5,t,*}$. But it is sent to 0 in $E_1^{5,t,*}$.

Compute the May filtrations of the generators given above, we get the proposition.

Proposition 3.2 *Let $3\leq s<p-1$ be a positive integer, $t_s=2(p-1)((s-2)+sp+sp^2+p^n)+s-3$ with $n\geq 4$. Then the E_1-term of the MSS $E_1^{s-r+3,t_s-r+1,*}$ is zero except for $r=1$. $E_1^{s+2,2(p-1)((s-2)+sp+sp^2+p^n)+s-3,M}$ are the Z/p-modules generated by*

$$\begin{aligned} &a_3^{s-3}h_{3,0}h_{2,1}h_{1,n}b_{2,0} && \text{with } M=7s+3p-12,\\ &a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,n} && \text{with } M=7s-10. \end{aligned}$$

Proof For $r=1$, $t_s-r+1=2(p-1)((s-2)+sp+sp^2+p^n)+s-3$, suppose we have a generator $g=(x_1\cdots x_b)\cdot(y_1\cdots y_l)\cdot(z_1\cdots z_m)$ of the form (3.1) in $E_1^{s+2,t_s,*}$. Then from $b+l+2m=s+2<p+1$ and $b\equiv t_s \bmod 2(p-1)$, we see that $b=s-3$ and then (b,l,m) is one of $(s-3,5,0)$, $(s-3,3,1)$ and $(s-3,1,2)$.

For $(b,l,m)=(s-3,5,0)$, the corresponding equation (3.3) becomes

$$\begin{cases} x_{1,0}+\cdots+x_{s-3,0}+y_{1,0}+\cdots+y_{5,0}=s-2+k_1p=c_0,\\ x_{1,1}+\cdots+x_{s-3,1}+y_{1,1}+\cdots+y_{5,1}=s+k_2p-k_1=c_1,\\ x_{1,2}+\cdots+x_{s-3,2}+y_{1,2}+\cdots+y_{5,2}=s+k_3p-k_2=c_3,\\ x_{1,3}+\cdots+x_{s-3,3}+y_{1,3}+\cdots+y_{5,3}=0+k_4p-k_3=c_3,\\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ x_{1,n-1}+\cdots+x_{s-3,n-1}+y_{1,n-1}+\cdots+y_{5,n-1}=0+k_np-k_{n-1}=c_{n-1},\\ x_{1,n}+\cdots+x_{s-3,n}+y_{1,n}+\cdots+y_{5,n}=1-k_n=c_n. \end{cases} \tag{3.5}$$

To solve the equation (3.5), noticing that each row has $s+2$ entries, we firstly determine that $k_1 = k_2 = \cdots = k_n = 0$ for $s < p-2$. In the case $s = p-2$, notice that each row has p entries which may cause carry on the fourth row. Thus the carrying numbers $k_1, k_2, \cdots k_n$ have another possibility $k_1 = k_2 = k_3 = 0$, $k_4 = \cdots = k_n = 1$.

For $k_1 = k_2 = \cdots = k_n = 0$, similarly to the proof of Proposition 3.1, we see that all the entries in the $4^{\text{th}} \sim (n-1)^{\text{th}}$ rows are 0, and then the last row is $(0\ \cdots\ 0 \mid 0\ 0\ 0\ 0\ 1)$ or $(0\ \cdots\ 0 \mid 0\ 0\ 0\ 1\ 0)$.

(1) The last row is $(0\ \cdots\ 0 \mid 0\ 0\ 0\ 1\ 0)$. In this case the equation (3.5) has a solution, but it deduces none by Remark 3.1(2).

(2) The last row is $(0\ \cdots\ 0 \mid 0\ 0\ 0\ 0\ 1)$. In this case the equation (3.5) has 6 solutions, and they deduce the following generators in $F_1^{s+2,t_s,*}$:

$$\begin{array}{lll} a_3^{s-4}a_0h_{3,0}^2h_{2,1}^2h_{1,n}, & a_3^{s-3}h_{1,0}h_{2,1}^3h_{1,n}, & a_3^{s-4}a_1h_{3,0}h_{2,1}^3h_{1,n}, \\ a_3^{s-3}h_{3,0}h_{2,1}h_{1,1}h_{1,2}h_{1,n}, & a_3^{s-3}h_{20}h_{21}^2h_{1,2}h_{1,n}, & a_3^{s-4}a_2h_{3,0}h_{2,1}^2h_{1,2}h_{1,n}. \end{array}$$

But only the generator $a_3^{s-3}h_{3,0}h_{2,1}h_{1,1}h_{1,2}h_{1,n}$ is sent to a nonzero generator of $E_1^{s+2,t_s,*}$.

In the case $k_1 = k_2 = k_3 = 0$, $k_4 = \cdots = k_n = 1$, notice that the fourth equation is

$$x_{1,3} + \cdots + x_{p-5,3} + y_{1,3} + \cdots + y_{5,3} = 0 + p = p.$$

Thus the fourth row must be $(1\ \cdots\ 1 \mid 1\ 1\ 1\ 1\ 1)$, from which we see that the beginning four rows are

$$\begin{array}{cl} \left(\begin{array}{ccc|ccccc} 1 & \cdots & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) & \begin{array}{l} p-4 \\ p-2 \\ p-2 \\ p. \end{array} \end{array}$$

Then since the sum of the fifth row is $0 + p - 1$, we get 4 solutions, which deduce the following generators of $F_1^{p,t_{p-2},*}$:

$$\begin{array}{ll} a_n^{p-5}h_{n,0}h_{n-1,1}^2h_{n-3,3}h_{1,3}, & a_n^{p-5}h_{n,0}h_{n-1,1}h_{3,1}h_{n-3,3}^2, \\ a_n^{p-5}h_{4,0}h_{n-1,1}^2h_{n-3,3}^2, & a_n^{p-6}a_4h_{n,0}h_{n-1,1}^2h_{n-3,3}^2. \end{array}$$

But they are all sent to zero in $E_1^{p,t_{p-2},*}$.

For $(b, l, m) = (s-3, 3, 1)$, the corresponding equation (3.3) becomes

$$\begin{cases} \sum\limits_{1\le i\le s-3} x_{i,0} + y_{1,0} + y_{2,0} + y_{3,0} = s - 2 + k_1p = c_0, \\ \sum\limits_{1\le i\le s-3} x_{i,1} + y_{1,1} + y_{2,1} + y_{3,1} + z_{1,1} = s + k_2p - k_1 = c_1, \\ \sum\limits_{1\le i\le s-3} x_{i,2} + y_{1,2} + y_{2,2} + y_{3,2} + z_{1,2} = s + k_3p - k_2 = c_3, \\ \sum\limits_{1\le i\le s-3} x_{i,3} + y_{1,3} + y_{2,3} + y_{3,3} + z_{1,3} = 0 + k_2p - k_1 = c_3, \\ \qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \sum\limits_{1\le i\le s-3} x_{i,n-1} + y_{1,n-1} + y_{2,n-1} + y_{3,n-1} + z_{1,n-1} = 0 + k_np - k_{n-1} = c_{n-1}, \\ \sum\limits_{1\le i\le s-3} x_{i,n} + y_{1,n} + y_{2,n} + y_{3,n} + z_{1,n} = 1 - k_n = c_n. \end{cases}$$

Similarly we determine that $k_1 = k_2 = \cdots = k_n = 0$. Then all the entries in the $4^{\text{th}} \sim (n-1)^{\text{th}}$ rows are 0, and the last row is $(0\ \cdots\ 0 \mid 0\ 0\ 0 \mid 1)$ or $(0\ \cdots\ 0 \mid 0\ 0\ 1 \mid 0)$, from which we get the following two solutions

$$\left(\begin{array}{ccc|ccc|c} 1 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} s-2 \\ s \\ s \\ 0 \\ \vdots \\ 0 \\ 1 \end{array} \qquad \left(\begin{array}{ccc|ccc|c} 1 & \cdots & 1 & 1 & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{c} s-2 \\ s \\ s \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}$$

and they deduce the generators $a_3^{s-3}h_{3,0}h_{2,1}^2 b_{1,n-1}$ and $a_3^{s-3}h_{3,0}h_{2,1}h_{1,n}b_{2,0}$ of $F_1^{s+2,t_s,*}$ respectively. But the former one is sent to zero in $E_1^{s+2,t_s,*}$.

For $(b,l,m) = (s-3,1,2)$, the corresponding equation (3.3) has no solution.

For $r > 1$, $t_s - r + 1 = 2(p-1)((s-2)+sp+sp^2+p^n)+s-r-2$, suppose we have a generator $g = (x_1 \cdots x_b) \cdot (y_1 \cdots y_l) \cdot (z_1 \cdots z_m)$ of the form (3.1) in $E_1^{s-r+3,t_s-r+1,*}$. Similarly we see that $b = s-r-2$, and (b,l,m) is the one in $(s-r-2,1,2)$, $(s-r-2,3,1)$ and $(s-r-2,5,0)$. Considering the corresponding equations (3.3), we see that they all have no solution except for $r = 2$ and $(b,l,m) = (s-r-2,5,0) = (s-4,5,0)$. In this case, the corresponding equation (3.5) has one solution which induces the generator $a_3^{s-4}h_{3,0}^2 h_{2,1}^2 h_{1,n}$ of $F_1^{s+1,t_s-1,*}$.

Computing the May filtrations of the generators above we get the proposition.

4 The Proof of Theorem A

From [4, 10] and Corollary 2.1, we see that the homotopy elements γ_2, γ_s for $3 \le s < p$ and $j\xi_n$ are represented by $2b_{2,0}h_{1,2}$, $\tilde{\gamma}_s$ and $b_0h_n - h_1b_{n-1}$ respectively. Thus $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ and $2b_{2,0}h_{1,2}(b_0h_n - h_1b_{n-1})$ are permanent cycles if:

(1) $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ and $2b_{2,0}h_{1,2}(b_0h_n - h_1b_{n-1})$ are not zero in the Ext groups. This could be proved by showing that no May differentials hit $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ and $2b_{2,0}h_{1,2}(b_0h_n - h_1b_{n-1})$ in the MSS.

(2) No Adams differentials hit $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ and $2b_{2,0}h_{1,2}(b_0h_n - h_1b_{n-1})$. This could be proved by showing that

$$\operatorname{Ext}_A^{s-r+3,\bar{t}_s+s-r-2}(Z/p, Z/p) = 0 \quad \text{for } r \ge 2,$$
$$\operatorname{Ext}_A^{6-r,2(p-1)(2p+2p^2+p^n)-r+1}(Z/p, Z/p) = 0 \quad \text{for } r \ge 2.$$

This was done in Proposition 3.1 and Proposition 3.2 by showing that the corresponding E_1-terms of the MSS are zero.

Lemma 4.1 *In the cobar complex $\Omega^2(A)$ we have the cochain $\tilde{b}_{2,0}$ such that*

$$d(\tilde{b}_{2,0}) = -\,\tilde{b}_{1,1} \otimes \xi_1^p + \xi_1^{p^2} \otimes \tilde{b}_{1,0},$$

where

$$\begin{aligned}
p\,\tilde{b}_{2,0} &= (\xi_2 \otimes 1 + \xi_1^p \otimes \xi_1 + 1 \otimes \xi_2)^p - \xi_2^p \otimes 1 - \xi_1^{p^2} \otimes \xi_1^p - 1 \otimes \xi_2^p,\\
p\,\tilde{b}_{1,0} &= (\xi_1 \otimes 1 + 1 \otimes \xi_1)^p - \xi_1^p \otimes 1 - 1 \otimes \xi_1^p,\\
p\,\tilde{b}_{1,1} &= (\xi_1 \otimes 1 + 1 \otimes \xi_1)^{p^2} - \xi_1^{p^2} \otimes 1 - 1 \otimes \xi_1^{p^2}.
\end{aligned}$$

Thus in the MSS we have $d_{2p-1}(b_{2,0}) = -b_{1,1}h_{1,1} + h_{1,2}b_{1,0}$.

Proof In the cobar complex for BP_*BP, consider the differential $d(c(t_2^p))$, where $c : BP_*BP \to BP_*BP$ is the conjugation of BP_*BP. Then from the commutativity of the following diagram

$$\begin{tikzcd}
BP_*BP \arrow[r, "\Delta"] \arrow[d, "c"] & BP_*BP \otimes_{BP_*} BP_*BP \arrow[d, "(c\otimes c)\cdot T"] \\
BP_*BP \arrow[r, "\Delta"] & BP_*BP \otimes_{BP_*} BP_*BP
\end{tikzcd}$$

and (2.3), we see that

$$\begin{aligned}
d(c(t_2^p)) &\equiv -\,p\,\tilde{b}_{2,0} - c(t_1^{p^2}) \otimes c(t_1^p) && \bmod (p^2, p\,v_1, v_1^p),\\
d(d(c(t_2^p))) &\equiv -\,p\,d(\tilde{b}_{2,0}) - d(c(t_1^{p^2}) \otimes c(t_1^p)) \equiv 0 && \bmod (p^2, p\,v_1, v_1^p).
\end{aligned}$$

Thus we have $d(\tilde{b}_{2,0}) = \tilde{b}_{1,1} \otimes c(t_1^p) - c(t_1^{p^2}) \otimes \tilde{b}_{1,0}$ mod (p, v_1), where

$$\begin{aligned}
p\,\tilde{b}_{2,0} &= (c(t_2) \otimes 1 + c(t_1^p) \otimes c(t_1) + 1 \otimes c(t_2))^p - c(t_2^p) \otimes 1 - c(t_1^{p^2}) \otimes c(t_1^p) - 1 \otimes c(t_2^p),\\
p\,\tilde{b}_{1,1} &= (c(t_1) \otimes 1 + 1 \otimes c(t_1))^{p^2} - c(t_1^{p^2}) \otimes 1 - 1 \otimes c(t_1^{p^2}),\\
p\,\tilde{b}_{1,0} &= (c(t_1) \otimes 1 + 1 \otimes c(t_1))^p - c(t_1^p) \otimes 1 - 1 \otimes c(t_1^p).
\end{aligned}$$

Applying the Thom map, we get the lemma from $\Phi \cdot d = -d \cdot \Phi$.

Proposition 4.1 *For $n \geq 4$, no May differentials hit $2b_{2,0}h_{1,2}(b_0h_n - h_1b_{n-1})$ in the MSS, and then $\tilde{\gamma}_2(b_0h_n - h_1b_{n-1})$ is a permanent cycle in the ASS.*

Proof Consider the May differential $d_r : E_r^{5,*,M+r} \to E_r^{6,*,M}$, we see that $\tilde{\gamma}_2(b_0h_n - h_1b_{n-1})$ is represented by $2b_{2,0}h_{1,2}(b_{1,0}h_{1,n} - h_{1,1}b_{1,n-1})$ in $E_r^{6,*,M}$ with May filtration $M = 4p+2$. From Proposition 3.1, we see that only the generators $b_{2,0}^2h_{1,n}$ and $b_{2,0}h_{2,1}b_{1,n-1}$ have May filtration $> 4p + 2$. From Lemma 4.1, we see that

$$\begin{aligned}
d_{2p-1}(b_{2,0}^2h_{1,n}) &= 2b_{2,0}(-b_{1,1}h_{1,1} + h_{1,2}b_{1,0})h_{1,n},\\
d_1(b_{2,0}h_{2,1}b_{1,n-1}) &= b_{2,0}h_{1,2}h_{1,1}b_{1,n-1}.
\end{aligned}$$

Thus

$$d_{2p-1}(b_{2,0}^2h_{1,n} - 2b_{2,0}h_{2,1}b_{1,n-1}) = 2b_{2,0}h_{1,2}(b_{1,0}h_{1,n} - h_{1,1}b_{1,n-1}) - 2b_{2,0}b_{1,1}h_{1,1}h_{1,n}.$$

The proposition follows.

Proposition 4.2 *For $n \geq 4$ and $3 \leq s < p-1$, no May differentials hit $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$ $(b_0h_n - h_1b_{n-1})$ in the MSS, and then $\tilde{\gamma}_s(b_0h_n - h_1b_{n-1})$ is a permanent cycle in the ASS.*

Proof It is easy to see that $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}(b_0h_n - h_1b_{n-1}) \in E_r^{s+3,*,M}$ has May filtration $M = 7s + p - 11$. From Proposition 3.2, we see that in $E_1^{s+2,*,M+r}$, only the generator $a_3^{s-3}h_{3,0}h_{2,1}h_{1,n}b_{2,0}$ has May filtration

$$M + r = 7s + 3p - 12 > 7s + p - 11.$$

Notice that

$$d_1(a_3^{s-3}h_{3,0}h_{2,1}h_{1,n}b_{2,0}) = -a_3^{s-3}h_{3,0}h_{1,2}h_{1,1}h_{1,n}b_{2,0} + \cdots \neq 0.$$

Thus the corresponding $E_2^{s+2,*,M+r} = 0$ and no May differentials hit $a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}(b_0h_n - h_1b_{n-1})$. The proposition follows.

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