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A Hybrid of Theorems of Goldbach and Piatetski-Shapiro***

Xianmeng MENG* Mingqiang WANG**

Abstract It is proved that for almost all sufficiently large even integers n, the prime variable equation $n = p_1 + p_2$, $p_1 \in P_\gamma$ is solvable, with $13/15 < \gamma \le 1$, where $P_\gamma = \{p \mid p = [m^{\frac{1}{\gamma}}],\ \text{for integer } m \text{ and prime } p\}$ is the set of the Piatetski-Shapiro primes.

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1 Introduction

The binary Goldbach problem seems out of reach at present. However we can get that almost all sufficiently large even integers n can be written as a sum of two primes,

$$n = p_1 + p_2. \tag{1.1}$$

Recently Li [1] obtained that for almost all sufficiently large even integers n, equation (1.1) has solutions with one or two prime variables restricted to the Piatetski-Shapiro primes.

The set of the Piatetski-Shapiro primes of type γ is a well-known thin set of prime numbers, which can be written as

$$P_\gamma = \{p \mid p = [m^{\frac{1}{\gamma}}],\ \text{for integer } m \text{ and prime } p\}.$$

The counting function $\pi_\gamma(x)$ of P_γ was studied by a number of authors. Piatetski-Shapiro [2] first proved that for $11/12 < \gamma \le 1$,

$$\pi_\gamma(x) = \sum_{\substack{p \le x \\ p = [n^{\frac{1}{\gamma}}]}} 1 = (1 + o(1)) \frac{x^\gamma}{\log x}.$$

The best results are given by [3] and [4], where it is proved that $\pi_\gamma(x)$ has an asymptotic formula for $2426/2817 < \gamma \le 1$, and $\pi_\gamma(x)$ has positive lower bound estimate for $205/243 < \gamma \le 1$.

The purpose of this paper is to establish the following result with one prime variable restricted to the Piatetski-Shapiro primes.

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*School of Computer Science and Technology, Shandong University, Jinan 250100, China; Department of Statistics and Mathematics, Shandong Finance Institute, Jinan 250014, China.
E-mail: mengxm@beelink.com
**School of Computer Science and Technology, Shandong University, Jinan 250100, China; Department of Mathematics, Qufu Normal University, Qufu 273165, Shandong, China. E-mail: mingqiangwang@sina.com
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Theorem 1.1 *If γ is fixed with $13/15 < \gamma \le 1$, then for almost all sufficiently large even integers n, the equation*

$$\begin{cases} n = p_1 + p_2, \\ p_1 \in P_\gamma \end{cases} \tag{1.2}$$

is solvable.

Li [1] proved that equation (1.2) is solvable for $8/9 < \gamma \le 1$. To get the better result, we shall apply the sieve method combined with the circle method. This method was invented by Jia [5], and was used later in many articles (see [6, 1], for example).

Notations Throughout this paper, both n and N are sufficiently large even integers, satisfying $N < n \le 2N$. ε is a sufficiently small positive constant. c, c_1 and c_2 are constants, which may have different values at different places. $m \sim M$ means $c_1 M < m \le c_2 M$. $N(d) = [-d^\gamma] - [-(d+1)^\gamma]$.

As usual, $\varphi(q)$, $\mu(q)$ and $\Lambda(n)$ stand for the functions of Euler, Mobius and von Mangoldt respectively.

2 Some Preliminary Lemmas

In the following, we assume that

$$H = N^{1-\gamma+\Delta+8\varepsilon}.$$

By the discussion in [7], the asymptotic formula that for $0 \le \Delta \le 1-\gamma$,

$$\sum_{N/10 < p \le N} N(p) e(\alpha p) = \gamma \sum_{N/10 < p \le N} p^{\gamma-1} e(\alpha p) + O(N^{\gamma-\Delta-5\varepsilon}),$$

depends on the fact that for $J \le H$ and $0 \le u \le 1$,

$$\min\Big(1, \frac{N^{1-\gamma}}{J}\Big) \sum_{h \sim J} \Big| \sum_{n \sim N} \Lambda(n) e(\alpha n + h(n+u)^\gamma) \Big| \ll N^{1-\Delta-6\varepsilon}.$$

Lemma 2.1 *Assume that $N^{1-\gamma+2\Delta+30\varepsilon} \ll M \ll N^{5\gamma-4-6\Delta-120\varepsilon}$ and that $a(m), b(k) = O(1)$. Then for $J \le H$ and $0 \le u \le 1$, we have*

$$\min\Big(1, \frac{N^{1-\gamma}}{J}\Big) \sum_{h \sim J} \Big| \sum_{m \sim M} \sum_{km \sim N} a(m) b(k) e(\alpha km + h(km+u)^\gamma) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

This is Proposition 2 of [7].

Lemma 2.2 *Assume that $M \ll N^{4\gamma-3-5\Delta-50\varepsilon}$, $a(m) = O(1)$ and*

$$6(1-\gamma) + \frac{19}{3}\Delta < 1.$$

Then for $J \le H$ and $0 \le u \le 1$, we have

$$\min\Big(1, \frac{N^{1-\gamma}}{J}\Big) \sum_{h \sim J} \Big| \sum_{m \sim M} a(m) \sum_{km \sim N} e(\alpha km + h(km+u)^\gamma) \Big| \ll N^{1-\Delta-10\varepsilon}.$$

This is Proposition 3 of [7].

Lemma 2.3 *Assume that* $|\alpha - a/q| < 1/q^2,\ (a,q) = 1$. *Then we have*

$$\sum_{p\sim x} e(\alpha p) \ll \Big(\frac{x}{\sqrt{q}} + \sqrt{xq} + x^{\frac{4}{5}}\Big)\log^5 x.$$

Refer to Section 25 in [8].

Lemma 2.4 *Assume that* $|\alpha - a/q| < 1/q^2, (a,q) = 1, x \geq 2$ *and* F_1, F_2 *be increasing sequences of positive integers. Then for any positive numbers* U *and* U' *satisfying* $1 \leq U \leq x$ *and* $U < U' \leq 2U$, *we have*

$$\sum_{\substack{U\leq u<U'\\ u\in F_1}} \sum_{\substack{1\leq uv\leq x\\ v\in F_2}} e(\alpha uv) \ll x\Big(\frac{1}{q} + \frac{q}{x}\log q + \frac{1}{U}\log q + \frac{U}{x}\Big)^{\frac{1}{2}}.$$

This is Lemma 5.7 of [9].

Lemma 2.5 *Assume that* $|\alpha - a/q| < 1/q^2, (a,q) = 1$. *Then we have*

$$\sum_{\substack{x<p_1p_2\leq 2x\\ x^{\frac{1}{3}}<p_1\leq x^{\frac{1}{2}}}} e(\alpha p_1 p_2) \ll x\log^{\frac{3}{2}} x\left(\frac{1}{q} + \frac{q}{x} + \frac{1}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}.$$

This follows from Lemma 2.4.

We define $w(u)$ as the continuous solution of the equations

$$\begin{aligned} w(u) &= \frac{1}{u}, && 1 \leq u \leq 2,\\ (uw(u))' &= w(u-1), && u > 2. \end{aligned}$$

$w(u)$ is called Buchstab's function. It plays an important role in finding asymptotic formulas in the sieve method. In particular,

$$w(u) = \begin{cases} \dfrac{1+\log(u-1)}{u}, & 2 \leq u \leq 3,\\ \dfrac{1+\log(u-1)}{u} + \dfrac{1}{u}\displaystyle\int_2^{u-1} \frac{\log(t-1)}{t}dt, & 3 \leq u \leq 4. \end{cases} \tag{2.1}$$

Lemma 2.6 *We get that*

(i) *for* $u \geq 2.47$, $w(u) \geq 0.5607$;
(ii) *for* $u \geq 3$, $w(u) \leq 0.5644$;
(iii) *for* $u \geq 1.7631$, $w(u) \leq 0.5672$;
(iv) *for* $u \geq 1$, $w(u) \geq 0.5$.

This is Lemma 8 of [6].

Lemma 2.7 *Assume that* $\mathcal{E} = \{n : x < n \leq 2x\}$, *and that* $z \leq x$. *Let* $P(z) = \prod_{p<z} p$. *Then for sufficiently large* x *and* z, *we have*

$$S(\mathcal{E}, z) = \sum_{\substack{x<n\leq 2x\\ (n,P(z))=1}} 1 = \Big(w\Big(\frac{\log x}{\log z}\Big) + O(\varepsilon)\Big)\frac{x}{\log z}.$$

This is Lemma 9 of [6].

3 Some Formulas in Sieve Method

In this section, we prove some lemmas which will be used next section.

Lemma 3.1 *Assume that* $M, K \ll N^{\frac{1}{3}}$, *and that* $a(m), b(k) = O(1)$. *Let*

$$I(n) = \sum_{\substack{n=n_1+n_2 \\ n/10<n_1,n_2\le n}} \frac{\gamma n_1^{\gamma-1}}{\log n_2}.$$

Then for $N < n \le 2N$, *except for* $O(N \log^{-2} N)$ *values, we have*

$$\sum_{\substack{m\sim M, k\sim K \\ (m,n)=(k,n)=1}} a(m)b(k)\Big(\sum_{\substack{n=mkl+p_2 \\ n/10<mkl,p_2\le n}} N(mkl) - \frac{1}{\varphi(mk)} I(n)\Big) = O\Big(\frac{N^\gamma}{\log^{20} N}\Big).$$

Proof Let

$$S(\alpha) = \sum_{n/10<p\le n} e(\alpha p).$$

We have

$$\begin{aligned}\sum_1 &= \sum_{\substack{m\sim M, k\sim K \\ (m,n)=(k,n)=1}} a(m)b(k) \sum_{\substack{n=mkl+p_2 \\ n/10<mkl,p_2\le n}} N(mkl) \\ &= \int_0^1 \sum_{\substack{n/10<mkl\le n \\ (m,n)=(k,n)=1 \\ m\sim M, k\sim K}} a(m)b(k)N(mkl)e(\alpha mkl)S(\alpha)e(-\alpha n)d\alpha.\end{aligned}$$

Define

$$\begin{aligned} g(\alpha) &= \sum_{\substack{n/10<mkl\le n \\ (m,n)=(k,n)=1 \\ m\sim M, k\sim K}} a(m)b(k)N(mkl)e(\alpha mkl), \\ f(\alpha) &= \gamma \sum_{\substack{n/10<mkl\le n \\ (m,n)=(k,n)=1 \\ m\sim M, k\sim K}} a(m)b(k)(mkl)^{\gamma-1}e(\alpha mkl).\end{aligned}$$

By the discussion in [7], the asymptotic formula

$$g(\alpha) = f(\alpha) + O(N^{\gamma-5\varepsilon}) \tag{3.1}$$

depends on the fact that for $J \le H_1 = N^{1-\gamma+8\varepsilon}$ and $0 \le u \le 1$,

$$\min\Big(1, \frac{N^{1-\gamma}}{J}\Big) \sum_{h\sim J} \Big| \sum_{m\sim M} \sum_{k\sim K} \sum_{mkl\sim N} a(m)b(k)e(\alpha mkl + h(mkl+u)^\gamma)\Big| \ll N^{1-6\varepsilon}. \tag{3.2}$$

If either M or K is larger than $N^{\frac{2}{15}}$, then by Lemma 2.1 with $\Delta = 0$, (3.2) holds. If $M, K \le N^{\frac{2}{15}}$, then $MK \ll N^{\frac{4}{15}} \ll N^{4\gamma-3-50\varepsilon}$. By Lemma 2.2 with $\Delta = 0$, (3.2) also holds. Hence (3.1) holds.

By (3.1) we have

$$g(\alpha)S(\alpha) - f(\alpha)S(\alpha) \ll N^{\gamma-5\varepsilon} \cdot |S(\alpha)|.$$

Thus

$$\sum_1 = \int_0^1 g(\alpha)S(\alpha)e(-n\alpha)d\alpha = \int_0^1 f(\alpha)S(\alpha)e(-n\alpha)d\alpha + \psi. \tag{3.3}$$

By Bessel's inequality, we have

$$\sum_{N<n\le 2N} |\psi|^2 \ll \int_0^1 |g(\alpha)S(\alpha) - f(\alpha)S(\alpha)|^2 d\alpha \ll N^{2\gamma-10\varepsilon} \int_0^1 |S(\alpha)|^2 d\alpha \ll N^{2\gamma+1-10\varepsilon}.$$

By the above, for $N < n \le 2N$, except for $O(N^{1-\varepsilon})$ values, we have

$$\psi \ll N^{\gamma-\varepsilon}. \tag{3.4}$$

In the following we investigate

$$\sum_2 = \int_0^1 f(\alpha)S(\alpha)e(-n\alpha)d\alpha.$$

Let $Q = N\log^{-80} N$. By Dirichlet's lemma, we divide $[-1/Q, 1-1/Q]$ into the major arcs E_1 and minor arcs E_2 as follows:

$$E_1 = \{\alpha : \alpha = a/q + \beta,\ q \le \log^{80} N,\ 0 \le a \le q,\ (a,q) = 1,\ |\beta| \le 1/(qQ)\},$$
$$E_2 = [-1/Q, 1-1/Q]\backslash E_1.$$

Then

$$\sum_2 = \Big(\int_{E_1} + \int_{E_2}\Big) f(\alpha)S(\alpha)e(-n\alpha)d\alpha. \tag{3.5}$$

For $\alpha \in E_2$, by Lemma 2.3, we have $S(\alpha) \ll N\log^{-35} N$. Hence by Bessel's inequality, we have

$$\sum_{N<n\le 2N} \Big|\int_{E_2} f(\alpha)S(\alpha)e(-n\alpha)d\alpha\Big|^2 \ll \int_{E_2} |f(\alpha)S(\alpha)|^2 d\alpha \ll |S(\alpha)|^2 \int_{E_2} |f(\alpha)|^2 d\alpha$$
$$\ll N^2 \log^{-70} N \cdot \int_0^1 |f(\alpha)|^2 d\alpha \ll N^{2\gamma+1} \log^{-60} N. \tag{3.6}$$

Here we have used the fact that for $d_r(n) = \sum_{n=n_1n_2\cdots n_r} 1$, and positive integer k, $\sum_{n\le x} d_r^k(n) \ll x(\log x)^{r^k-1}$. By (3.5), for $N < n \le 2N$, except for $O(N\log^{-20} N)$ values, we have

$$\int_{E_2} f(\alpha)S(\alpha)e(-n\alpha)d\alpha \ll N^\gamma \log^{-20} N.$$

If $\alpha \in E_1$, let $R = MK$, and $j(r) = \gamma \sum_{\substack{mk=r\\ m\sim M, k\sim K}} a(m)b(k)$. By (25) in [6], we have

$$f(\alpha) = \sum_{\substack{r\sim R\\ (r,n)=1, q|r}} j(r) \sum_{n/10<s\le n} s^{\gamma-1} e(\beta s) + O(N^{\gamma-\varepsilon}).$$

By Lemma 6.4 in [9], we have

$$S(\alpha)=\frac{\mu(q)}{\varphi(q)}\sum_{n/10<s\le n}\frac{e(\beta s)}{\log s}+O(N\exp(-c_2\sqrt{\log N}\,)). \tag{3.7}$$

Hence

$$\begin{aligned}\sum_3 &= \int_{E_1} f(\alpha)S(\alpha)e(-n\alpha)d\alpha\\ &=\sum_{q\le\log^{80}N}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} f\Big(\frac{a}{q}+\beta\Big)S\Big(\frac{a}{q}+\beta\Big)e(-\beta n)d\beta\\ &=\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\varphi(q)}\sum_{\substack{r\sim R\\(r,q)=1,q|r}}\frac{j(r)}{r}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\Big(\sum_{n/10<s\le n}s^{\gamma-1}e(\beta s)\Big)\\ &\quad\cdot\Big(\sum_{n/10<s\le n}\frac{e(\beta s)}{\log s}\Big)e(-\beta n)d\beta+O\Big(\frac{N^\gamma}{\log^{20}N}\Big),\end{aligned} \tag{3.8}$$

in which

$$C(q,m)=\sum_{\substack{a=1\\(a,q)=1}}^{q}e\Big(\frac{am}{q}\Big).$$

Since

$$\int_{-\frac{1}{qQ}}^{\frac{1}{2}}\Big(\sum_{n/10<s\le n}s^{\gamma-1}e(\beta s)\Big)\Big(\sum_{n/10<s\le n}\frac{e(\beta s)}{\log s}\Big)e(-\beta n)d\beta\ll\int_{-\frac{1}{qQ}}^{\frac{1}{2}}n^\gamma\frac{d\beta}{\beta^2}\ll\frac{qN^\gamma}{\log^{80}N}, \tag{3.9}$$

we obtain

$$\sum_3=\frac{1}{\gamma}I(n)\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\varphi(q)}\sum_{\substack{r\sim R\\(r,n)=1,q|r}}\frac{j(r)}{r}+O(N^\gamma\log^{-20}N).$$

Let

$$\Omega=\sum_{q\le\log^{80}N}\frac{\mu(q)C(q,-n)}{\varphi(q)}\sum_{\substack{r\sim R\\(r,n)=1,q|r}}\frac{j(r)}{r}.$$

By the discussion above and (4.11) in [1], we have

$$\Omega=\gamma\sum_{\substack{m\sim M,k\sim K\\(m,n)=(k,n)=1}}\frac{a(m)b(k)}{\varphi(mk)}+O\Big(\frac{1}{\log^{50}N}\Big).$$

Hence

$$\sum_3=I(n)\sum_{\substack{m\sim M,k\sim K\\(m,n)=(k,n)=1}}\frac{a(m)b(k)}{\varphi(mk)}+O\Big(\frac{N^\gamma}{\log^{20}N}\Big). \tag{3.10}$$

From (3.4), (3.7) and (3.10), the lemma follows.

Lemma 3.2 *Assume that $M, K \ll N^{\frac{1}{3}}$, and that $a(m), b(k) = O(1)$. Let*

$$J(n) = \sum_{n^{\frac{1}{3}} < p_1 \le n^{\frac{1}{2}}} \frac{1}{p_1} \sum_{\substack{n=n_1+n_2 \\ n/10<n_1,n_2\le n}} \frac{\gamma n_2^{\gamma-1}}{\log \frac{n_2}{p_1}}.$$

Then for $N < n \le 2N$, except for $O(N \log^{-2} N)$ values, we have

$$\sum_{\substack{m\sim M, k\sim K \\ (m,n)=(k,n)=1}} a(m)b(k)\Big(\sum_{[\,]} N(p_1p_2) - \frac{1}{\varphi(mk)} J(n)\Big) = O\Big(\frac{N^\gamma}{\log^{20} N}\Big),$$

where $\sum_{[\,]}$ denotes that the sum is taken over

$$n = mkl + p_1p_2, \quad n/10 < mkl, \quad p_1p_2 \le n, \quad n^{\frac{1}{3}} < p_1 \le n^{\frac{1}{2}}, \quad p_1 < p_2.$$

Proof This can be proved in almost the same way as Lemma 3.1. We only give the outline of the proof. Let

$$D(\alpha) = \sum_{\substack{n/10<p_1p_2\le n \\ n^{\frac{1}{3}}<p_1\le n^{\frac{1}{2}}, p_1<p_2}} N(p_1p_2)e(\alpha p_1p_2), \quad S_1(\alpha) = \gamma \sum_{\substack{n/10<p_1p_2\le n \\ n^{\frac{1}{3}}<p_1\le n^{\frac{1}{2}}, p_1<p_2}} (p_1p_2)^{\gamma-1} e(\alpha p_1p_2).$$

By Lemma 4 of [6], we have $D(\alpha) = S_1(\alpha) + O(N^{\gamma-5\varepsilon})$.

Let E_1, E_2 be defined as in Lemma 3.1. For $\alpha \in E_2$, by Lemma 2.5, we have $S_1(\alpha) \ll N^\gamma \log^{-35} N$.

For $\alpha \in E_1$, similarly to $g(\alpha)$ in Lemma 18 of [6], we have

$$S_1(\alpha) = \gamma \frac{\mu(q)}{\varphi(q)} \sum_{n^{\frac{1}{3}}<p_1\le n^{\frac{1}{2}}} \frac{1}{p_1} \sum_{n/10<s\le n} \frac{s^{\gamma-1}e(\beta s)}{\log \frac{s}{p_1}} + O(N^\gamma \exp(-c_2\sqrt{\log N}\,)).$$

Then we can prove the lemma in the same way used in Lemma 3.1.

Lemma 3.3 *Assume that $N^{\frac{2}{3}} \ll M \ll N^{\frac{13}{15}}$, $0 \le a(m) = O(1)$, and that $a(m) = 0$, if m has a prime factor $< N^\varepsilon$. Then for $N < n \le 2N$, except for $O(N \log^{-2} N)$ values, we have*

$$\begin{aligned}\sum_4 &= \sum_{\substack{n=mp_1+p_2 \\ n/10<mp_1,p_2\le n \\ m\sim M}} a(m)N(mp_1) \\ &= (1+O(\varepsilon))Z(\gamma)C(n)\frac{n^\gamma}{n\log n}\sum_{m\sim M} a(m) \sum_{n/m<p\le 2n/m} 1 + O\Big(\frac{N^\gamma}{\log^{10} N}\Big),\end{aligned}$$

in which

$$Z(\gamma) = \gamma \int_{\frac{1}{10}}^{\frac{9}{10}} u^{\gamma-1} du, \quad C(n) = \frac{n}{\varphi(n)} \prod_{p\nmid n} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

Proof In a way similar to the proof of Lemma 3.1, we have

$$\sum_4 = \int_0^1 \sum_{\substack{n/10<mp_1\le n \\ m\sim M}} a(m)N(mp_1)e(\alpha mp_1)S(\alpha)e(-\alpha n)d\alpha,$$

where $S(\alpha)$ is defined in the proof of Lemma 3.1. Then for $N < n \le 2N$, except for $O(N\log^{-2} N)$ values, we have

$$\sum_4 = \int_{E_1} h(\alpha)S(\alpha)e(-n\alpha)d\alpha + O\Big(\frac{N^\gamma}{\log^{20} N}\Big),$$

where E_1 is defined in the proof of Lemma 3.1 and

$$h(\alpha) = \gamma \sum_{\substack{n/10<mp_1\le n\\ m\sim M}} a(m)(mp_1)^{\gamma-1}e(\alpha m p_1).$$

By [6, p.22], we have

$$h(\alpha) = \gamma\frac{\mu(q)}{\varphi(q)}\sum_{m\sim M}\frac{a(m)}{m}\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log\frac{s}{p_1}} + O(N\exp(-c\sqrt{\log N})).$$

By the above and (3.8), we have

$$\begin{aligned}\sum_4 &= \sum_{q\le\log^{80} N}\sum_{\substack{a=1\\(a,q)=1}}^{q} e\Big(-\frac{an}{q}\Big)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} h\Big(\frac{a}{q}+\beta\Big)S\Big(\frac{a}{q}+\beta\Big)e(-\beta n)d\beta + O\Big(\frac{N^\gamma}{\log^{10} N}\Big)\\
&= \gamma\sum_{q\le\log^{80} N}\frac{\mu^2(q)C(q,-n)}{\varphi^2(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\sum_{m\sim M}\frac{a(m)}{m}\Big(\sum_{n/10<s\le n}\frac{s^{\gamma-1}e(\beta s)}{\log\frac{s}{m}}\Big)\\
&\quad\cdot\Big(\sum_{n/10<s\le n}\frac{e(\beta s)}{\log s}\Big)e(-\beta n)d\beta + O\Big(\frac{N^\gamma}{\log^{10} N}\Big)\\
&= \sum_{q\le\log^{80} N}\frac{\mu^2(q)C(q,-n)}{\varphi^2(q)}K(n) + O\Big(\frac{N^\gamma}{\log^{10} N}\Big),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}K(n) &= \gamma\sum_{m\sim M}\frac{a(m)}{m}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}}\frac{n_1^{\gamma-1}}{\log\frac{n_1}{m}\log n_2}\\
&= (1+O(\varepsilon))Z(\gamma)\frac{n^\gamma}{\log n}\sum_{m\sim M}\frac{a(m)}{m\log\frac{n}{m}}\\
&= (1+O(\varepsilon))Z(\gamma)\frac{n^{\gamma-1}}{\log n}\sum_{m\sim M}a(m)\sum_{n/m<p\le 2n/m}1.\end{aligned}\tag{3.12}$$

Moreover, by Lemma 11.6 in [9], we have

$$\sum_{q\le\log^{80} N}\frac{\mu^2(q)C(q,-n)}{\varphi^2(q)} = \sum_{q=1}^{\infty}\frac{\mu^2(q)C(q,-n)}{\varphi^2(q)} + O(\log^{-30} N) = C(n) + O(\log^{-30} N).$$

The lemma follows from the above, (3.11) and (3.12).

Lemma 3.4 *Assume that* $N^{\frac{2}{3}} \ll M \ll N^{\frac{13}{15}}$, $0\le a(m) = O(1)$, *and that* $a(m) = 0$, *if* m *has a prime factor* $< N^\varepsilon$. *Let*

$$\sum_5 = \sum_{\substack{n=mp+d\\ n/10<mp,d\le n\\ m\sim M}} a(m)N(d),$$

where $d = p_1p_2$ $(n^{\frac{1}{3}} < p \le n^{\frac{1}{2}}, p_1 < p_2)$. *Then for* $N < n \le 2N$, *except for* $O(N\log^{-2} N)$ *values, we have*

$$\sum_5 = (1+O(\varepsilon))Z(\gamma)C(n)\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{dt}{t(1-t)} \cdot \frac{n^\gamma}{n\log n}\sum_{m\sim M} a(m) \sum_{n/m<p\le 2n/m} 1 + O\Big(\frac{N^\gamma}{\log^{10} N}\Big).$$

Proof In almost the same way as in Lemma 3.3, and referring to Lemma 3.2, we obtain the lemma.

4 Sieve Method—Proof of Theorem 1.1

Let $T(n)$ denote the number of the solutions of the equation (1.2). Assume that

$$\begin{aligned} A &= \{a : a = n-p,\ N(a) = 1,\ n/10 < p \le n\},\\ B &= \{b : b = n-d,\ N(d) = 1,\ 0 < d \le 9n/10,\ d = p_1p_2\ (n^{\frac{1}{3}} < p \le n^{\frac{1}{2}},\ p_1 < p_2)\}, \end{aligned}$$

and that

$$P(z) = \prod_{p<z, p\nmid n} p, \quad S(A,z) = \sum_{\substack{a\in A\\ (a,P(z))=1}} 1, \quad S(B,w) = \sum_{\substack{b\in B\\ (b,P(w))=1}} 1.$$

Note once again that $p \in P_\gamma$ is equivalent to $N(p) = 1$. Applying Buchstab's identity, we get

$$T(n) \ge S(A, n^{\frac{1}{2}}) = S(A, n^{\frac{2}{15}}) - \sum_{n^{\frac{2}{15}}<p\le n^{\frac{1}{3}}} S(A_p, p) - \sum_{n^{\frac{1}{3}}<p\le n^{\frac{1}{2}}} S(A_p,p) = S_1 - S_2 - S_3. \quad (4.1)$$

Next

$$\begin{aligned} S_3 &= \sum_{n^{\frac{1}{3}}<p\le n^{\frac{1}{2}}} S(A_p, p)\\ &= \#\{d : d = n - p_4,\ N(d) = 1,\ n/10 < p_4 \le n,\ d = p_1p_2\ (n^{\frac{1}{3}} < p \le n^{\frac{1}{2}},\ p_1 < p_2)\}\\ &= \#\{p_4 : p_4 = n-d,\ N(d) = 1,\ 0 < d \le 9n/10,\ d = p_1p_2\ (n^{\frac{1}{3}} < p \le n^{\frac{1}{2}},\ p_1 < p_2)\}\\ &= S(B, n^{\frac{1}{2}}). \end{aligned}$$

Using Buchstab's identity again, we get

$$S_3 = S(B, n^{\frac{1}{2}}) = S(B, n^{\frac{2}{15}}) - \sum_{n^{\frac{2}{15}}<p\le n^{\frac{1}{3}}} S(B_p,p) - \sum_{n^{\frac{1}{3}}<p\le n^{\frac{1}{2}}} S(B_p,p) = T_1 - T_2 - T_3.$$

Lemma 4.1 *For* $N < n \le 2N$, *except for* $O(N\log^{-2} N)$ *values, we have*

$$S_1 = S(A, n^{\frac{2}{15}}) \ge 4.203525Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}.$$

Proof Take

$$X = I(n) = \sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\le n}} \frac{\gamma n_1^{\gamma-1}}{\log n_2},$$

$\omega(d) = \dfrac{d}{\varphi(d)}$, if $(d,n)=1$; $\omega(d)=0$, if $(d,n)>1$ and $r_d = \#A_d - \dfrac{X}{\varphi(d)}$.

By Theorem 7.11 and (7.40) in [9], we have

$$W(z) = \prod_{p<z}\Big(1-\frac{w(p)}{p}\Big) = C(n)\frac{e^{-\gamma_0}}{\log z}\Big(1+O\Big(\frac{1}{\log z}\Big)\Big),$$

where γ_0 is the Euler's constant.

Let $z = n^{\frac{2}{15}}, D = n^{\frac{2}{3}}$. By Iwaniec's bilinear sieve method (see [10, Theorem 1]), we obtain

$$S_1 \geq \frac{C(n)X}{\log z}\Big(f\Big(\frac{\log D}{\log z}\Big) - O(\varepsilon)\Big) - \sum_{\substack{m<n^{\frac{1}{3}}, k<n^{\frac{1}{3}}\\ (m,n)=(k,n)=1}} a(m)b(k)r(mk),$$

where $f(u)$ is a standard function in sieve method. In particular,

$$f(u) = \begin{cases} \dfrac{2}{u}\log(u-1), & 2\leq u\leq 4,\\ \dfrac{2}{u}\Big(\log(u-1) + \displaystyle\int_3^{u-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(s-1)}{s}ds\Big), & 4\leq u\leq 6.\end{cases}$$

By Lemma 3.1, for $N<n\leq 2N$, except for $O(N\log^{-2}N)$ values, we have

$$\sum_{\substack{m<n^{\frac{1}{3}}, k<n^{\frac{1}{3}}\\ (m,n)=(k,n)=1}} a(m)b(k)r(mk) = O\Big(\frac{N^\gamma}{\log^{10}N}\Big).$$

On the other hand,

$$X = \frac{(1+O(\varepsilon))\gamma}{\log n}\sum_{\substack{n=n_1+n_2\\ n/10<n_1,n_2\leq n}} n_1^{\gamma-1} = (1+O(\varepsilon))Z(\gamma)\frac{n^\gamma}{\log n}.$$

Combining the above we get the lemma.

Lemma 4.2 *For $N<n\leq 2N$, except for $O(N\log^{-2}N)$ values, we have*

$$S_2 = \sum_{n^{\frac{2}{15}}<p\leq n^{\frac{1}{3}}} S(A_p,p) < 2.5426Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}.$$

Proof By Lemmas 2.6, 2.7 and 3.3, it follows that

$$\begin{aligned} S_2 &= \sum_{\substack{n=rp+p_1\\ n/10<rp,p_1\leq n\\ n^{\frac{2}{15}}<p\leq n^{\frac{1}{3}}, (r,P(p))=1}} N(rp)\\ &= (1+O(\varepsilon))Z(\gamma)C(n)\frac{n^\gamma}{\log n}\sum_{n^{\frac{2}{15}}<p\leq n^{\frac{1}{3}}}\ \sum_{\substack{n/p<r\leq 2n/p\\ (r,P(p))=1}} 1 + O\Big(\frac{N^\gamma}{\log^8 N}\Big)\end{aligned}$$

$$= Z(\gamma)C(n)\frac{n^\gamma}{\log n}\sum_{n^{\frac{2}{15}}<p\le n^{\frac{1}{3}}}\frac{1}{p\log p}w\Big(\frac{\log\frac{n}{p}}{\log p}\Big)+O\Big(\varepsilon\frac{N^\gamma}{\log^2 N}\Big)$$

$$= Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}\int_{\frac{2}{15}}^{\frac{1}{3}}\frac{1}{u^2}w\Big(\frac{1-u}{u}\Big)du+O\Big(\varepsilon\frac{N^\gamma}{\log^2 N}\Big)$$

$$= Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}\int_{2}^{13/2}w(t)dt+O\Big(\varepsilon\frac{N^\gamma}{\log^2 N}\Big)$$

$$< 2.5426Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}.$$

Lemma 4.3 *For $N<n\le 2N$, except for $O(N\log^{-2}N)$ values, we have*

$$T_1=S(B,n^{\frac{2}{15}})<2.92389Z(\gamma)C(n)\frac{n^\gamma}{\log^2 n}.$$

Proof Take $Y=J(n)$, where $J(n)$ is defined in Lemma 3.2, and

$$r_d=\#B_d-\frac{Y}{\varphi(d)}.$$

Take $z=n^{\frac{2}{15}}$, $D=n^{\frac{2}{3}}$. By Iwaniec's bilinear sieve method, we have

$$T_1\le\frac{C(n)Y}{\log n\cdot\log z}\Big(F\Big(\frac{\log D}{\log z}\Big)+O(\varepsilon)\Big)+\sum_{\substack{m<n^{\frac{1}{3}},k<n^{\frac{1}{3}}\\(m,n)=(k,n)=1}}a(m)b(k)r(mk), \tag{4.2}$$

where $F(u)$ is a standard function. In particular

$$F(u)=\begin{cases}\dfrac{2}{u}, & 2\le u\le 3,\\[2mm] \dfrac{2}{u}\Big(1+\displaystyle\int_2^{u-1}\frac{\log(t-1)}{t}dt\Big), & 3\le u\le 5.\end{cases}$$

Applying Lemma 3.2, for $N<n\le 2N$, except for $O(N\log^{-2}N)$ values, we have

$$\sum_{\substack{m<n^{\frac{1}{3}},k<n^{\frac{1}{3}}\\(m,n)=(k,n)=1}}a(m)b(k)r(mk)=O\Big(\frac{N^\gamma}{\log^{10}N}\Big). \tag{4.3}$$

On the other hand,

$$\begin{aligned}Y&=(1+O(\varepsilon))\gamma\sum_{\substack{n=n_1+n_2\\n/10<n_1,n_2\le n}}n_2^{\gamma-1}\sum_{n^{\frac{1}{3}}<p\le n^{\frac{1}{2}}}\frac{1}{p\log\frac{n_2}{p}}\\&=(1+O(\varepsilon))Z(\gamma)\frac{n^\gamma}{\log n}\int_{\frac{1}{3}}^{\frac{1}{2}}\frac{dt}{t(1-t)}=\ln 2\cdot Z(\gamma)\frac{n^\gamma}{\log n}.\end{aligned}$$

Hence the lemma follows from the above, (4.2) and (4.3).

Lemma 4.4 *For $N < n \le 2N$, except for $O(N\log^{-2} N)$ values, we have*

$$T_2 = \sum_{n^{\frac{2}{15}} < p \le n^{\frac{1}{3}}} S(B_p, p) > 1.72914 Z(\gamma) C(n) \frac{n^\gamma}{\log^2 n}.$$

Proof We have

$$T_2 = \sum_{\substack{n = rp + d \\ n/10 < rp, d \le n \\ n^{\frac{2}{15}} < p \le n^{\frac{1}{3}}, (r, P(p)) = 1}} N(d),$$

where $d = p_1 p_2$ ($n^{\frac{1}{3}} < p_1 \le n^{\frac{1}{2}}$, $p_1 < p_2$). By Lemmas 2.6, 2.7, 3.4, and the deduction in Lemma 4.2, for $N < n \le 2N$, except for $O(N\log^{-2} N)$ values, we have

$$T_2 = \frac{C(n)Y}{\log n} \int_2^{\frac{13}{2}} w(u)du + O\Big(\frac{\varepsilon N^\gamma}{\log^2 N}\Big) > 1.72914 Z(\gamma) C(n) \frac{n^\gamma}{\log^2 n}.$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 By (4.1), Lemmas 4.1–4.4, for $N < n \le 2N$, except for $O(N\log^{-2} N)$ values, we have

$$T(n) > 0.466 Z(\gamma) C(n) \frac{n^\gamma}{\log^2 n}.$$

Thus Theorem 1.1 follows.

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