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Geometric Optics for One-Dimensional Schrödinger-Poisson System

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Abstract This paper considers a family of Schrödinger-Poisson system in one dimension, whose initial data oscillates so that a caustic appears. By using the Lagrangian integrals, the authors obtain a uniform description of the solution outside the caustic, and near the caustic.

Keywords Schrödinger-Poisson, Geometric optics, Lagrangian integral
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1 Introduction

In this paper we consider a family of one-dimensional Schrödinger-Poisson system

$$(\mathrm{SP}^\varepsilon)\quad \begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \dfrac{1}{2}\varepsilon^2\partial_x^2\psi^\varepsilon = \varepsilon^\alpha V^\varepsilon\psi^\varepsilon, \quad x\in\mathbb{R},\ t\geq 0, \\ \psi^\varepsilon|_{t=0} = e^{-ix^2/(2\varepsilon)}f(x), \end{cases} \tag{1.1}$$

where ψ^ε denotes the wave function in the quantum mechanics, $\varepsilon\in(0,1]$ is a parameter going to zero, $\alpha\geq 1$, and the potential V^ε is assumed to be given by Poisson's equation

$$-\partial_x^2 V^\varepsilon = |\psi^\varepsilon|^2 - b, \tag{1.2}$$

where V^ε vanishes at $x=0$ and $\partial_x V^\varepsilon$ vanishes as $x\to-\infty$, and the function $b(x)$ denotes the doping profile in the semiconductor application, and, in general, it denotes a fixed background charge; one may see [1] or [2] for more physical explanations. Finally, f is a smooth function (for example, $f\in\mathcal{S}(\mathbb{R})$).

This system has been widely studied. For the well-known case $\alpha=0$ and subcritical initial data (no caustic will be formed), the limit from Schrödinger-Poisson to Euler-Poisson equation is justified by H. Liu and E. Tadmor for general dimension [3, 4]. The limit from the Schrödinger-Poisson to the Vlasov-Poisson equation with general data in one dimension is proved by P. Zhang, Y. X. Zheng and N. J. Mauser [5]. Recently, P. Zhang justified the limit from Schrödinger-Poisson to Euler-Poisson equation with general data in any dimension before the formation of vortices (see [6]). The purpose of this paper is to describe the asymptotic behavior of the solution ψ^ε as $\varepsilon\to 0$. We proved that according to the value of α, we can

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speak, as far as geometric optics is concerned, of linear or nonlinear propagation. In the linear geometric optics case, we obtain the asymptotic expansion like the linear equation. In the nonlinear geometric case, we obtain the asymptotic expansion with the nonlinear shift in the phase.

With the initial phase $-\frac{x^2}{2}$, rays of geometric optics (which are the projection on the (t,x) space of the bicharacteristics) focus at the point $(t,x)=(1,0)$ so that a caustic appears (cf. [7, 8]). WKB expansions (i.e., the method of geometric optics) make it possible to describe the solution before the caustic. But, when a caustic is formed, the description of geometric optics has to be modified. In order to describe precisely the behavior of the solutions ψ^ε near the caustic, and perhaps beyond, as in [7], we use the Lagrangian integral representation. In [8], this notion is introduced for the linear equations, to get a uniform representation of the solution outside the caustic, and near the caustic. In the case of nonlinear equations, this approach was used in [9–11]. In our case, we describe the solution ψ^ε thanks to Lagrangian integrals:

$$\psi^\varepsilon(t,x)=\frac{1}{\varepsilon^{1/2}}\int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon}a_\varepsilon(t,\xi)\frac{d\xi}{2\pi}, \tag{1.3}$$

where $a_\varepsilon(t,\xi)$ is uniquely determined by ψ^ε, since we have the formula

$$a_\varepsilon(t,\xi)=\frac{1}{\varepsilon^{1/2}}e^{i(t-1)\xi^2/(2\varepsilon)}\widehat{\psi^\varepsilon}\Big(t,\frac{\xi}{\varepsilon}\Big). \tag{1.4}$$

To guess the limit transport equation of a_ε, we approximate ψ^ε with a function

$$e^{i\phi(t,x)/\varepsilon}\psi_0(t,x).$$

WKB expansions suggest that the profile ψ_0 is determined by the transport equation:

$$\partial_t\psi_0+\partial_x\phi\partial_x\psi_0+\frac{1}{2}\psi_0\triangle\phi=\begin{cases}0, & \text{if } \alpha>1,\\ -iV_0\psi_0, & \text{if } \alpha=1,\end{cases} \tag{1.5}$$

where V_0 is given by

$$-\partial_x^2V_0=|\psi_0|^2-b.$$

Therefore, for $\alpha>1$, we expect that the limit transport equation for the symbol a_ε is simply $\partial_t a=0$. When $\alpha=1$, the transport equation is a little more complicated (This question is addressed in Section 4):

$$i\partial_t a(t,\xi)=V(t,(t-1)\xi)a(t,\xi), \tag{1.6}$$

where the potential $V(t,x)$ satisfies

$$-\partial_x^2V(t,x)=\frac{1}{|1-t|}\Big|f\Big(\frac{x}{1-t}\Big)\Big|^2-b(x). \tag{1.7}$$

Our approach will consist in working with another symbol, which will satisfy a simple transport equation (the first one). Introduce $\tilde a_\varepsilon$ defined by

$$a_\varepsilon(t,\xi)=e^{ig(t,\xi)}\tilde a_\varepsilon(t,\xi),$$

where g solves

$$\begin{cases}\partial_t g(t,\xi) = -V(t,(t-1)\xi), \\ g|_{t=0} = 0.\end{cases} \tag{1.8}$$

Then (1.6) simply becomes $\partial_t \tilde{a} = 0$. With the convention $g = 0$ if $\alpha > 1$, the following writing will be rather general:

$$\psi^\varepsilon(t,x) = \frac{1}{\varepsilon^{1/2}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)} \tilde{a}_\varepsilon(t,\xi) \frac{d\xi}{2\pi}. \tag{1.9}$$

Thus we continued the writing of the linear to the nonlinear case.

Before the presentation of the main result, let us first make the following assumptions on the function $b(x)$:

$$b(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \qquad \int |\psi^\varepsilon(t,x)|^2 dx = \int b(x) dx. \tag{1.10}$$

Under the above assumptions, it is proved that (SP^ε) has a unique global smooth solution (cf. [5, 6]).

Now we state the main result of this paper.

Theorem 1.1 *Let $\alpha \geq 1$, $f \in \mathcal{S}(\mathbb{R})$, $b(x)$ satisfy* (1.10) . *Then we have the following asymptotic in L^2 :*

- *If $\alpha > 1$, then for $t < 1$, we have*

$$\psi^\varepsilon(t,x) \overset{\varepsilon \to 0}{\sim} \frac{1}{(1-t)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t))},$$

and for $t > 1$,

$$\psi^\varepsilon(t,x) \overset{\varepsilon \to 0}{\sim} \frac{1}{(t-1)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t))-i\pi/2},$$

- *If $\alpha = 1$, then for $t < 1$, we have*

$$\psi^\varepsilon(t,x) \overset{\varepsilon \to 0}{\sim} \frac{1}{(1-t)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t))+ig(t,x/(t-1))},$$

and for $t > 1$,

$$\psi^\varepsilon(t,x) \overset{\varepsilon \to 0}{\sim} \frac{1}{(t-1)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t))+ig(t,x/(t-1))-i\pi/2}.$$

Remark 1.1 For $t > 1$, the approximation we get differs from the one for $t < 1$ by a phase shift of $-\pi/2$, which is present in the linear case, for equation (1.1), and is explained in [8].

Notations Used Throughout This Paper Let $\phi \in \mathcal{S}(\mathbb{R})$, its Fourier transform $\mathcal{F}\phi$ or $\widehat{\phi}$ is defined by

$$\mathcal{F}\phi(\xi) = \int e^{-ix\xi} \phi(x) dx.$$

Denoting $d\bar{\xi} = d\xi/(2\pi)$, the inverse Fourier transform $\mathcal{F}^{-1}\phi$ or $\phi^\vee$ is defined by

$$\mathcal{F}^{-1}\phi(x) = \int e^{ix\xi} \phi(\xi) d\bar{\xi}.$$

2 Preliminaries

First we recall some lemmas that will be used in the proof of Theorem 1.1.

Lemma 2.1 (Cf. [7]) *Let* $a_0(\xi) = (\frac{2\pi}{i})^{1/2} f(-\xi)$, $f \in L^2(\mathbb{R})$. *Then*

$$a_\varepsilon(0,\xi) \overset{\varepsilon \to 0}{\to} a_0(\xi), \quad \text{in } L^2(\mathbb{R}).$$

Lemma 2.2 (Cf. [7]) *Let* $\sigma(t,\xi)$ *be locally bounded in time with values in* $L^2(\mathbb{R})$. *Denote*

$$H^\varepsilon(t,x) := \frac{1}{\sqrt{\varepsilon}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon} \sigma(t,\xi) d\bar{\xi},$$

and Λ^ε *the first term by stationary phase formula*

$$\Lambda^\varepsilon(t,x) = \frac{1}{|2\pi(1-t)|^{1/2}} e^{i(\pi/4)\mathrm{sgn}(1-t) - ix^2/(2\varepsilon(1-t))} \sigma\Big(t, \frac{x}{t-1}\Big).$$

Then there exists a continuous function h, *with* $h(0) = 0$, *such that*

$$\|H^\varepsilon(t,x) - \Lambda^\varepsilon(t,x)\|_{L^2_x} = h\Big(\frac{\varepsilon}{1-t}\Big).$$

Lemma 2.3 (Cf. [5, 6]) *Let* $s \geq 2$ *be an integer,* $b(x) \in L^1(\mathbb{R}) \cap H^{s-2}(\mathbb{R})$. *Then* (SP$^\varepsilon$) *has a unique global smooth solution* $\psi^\varepsilon \in L^\infty([0,T], H^s(\mathbb{R}))$ *for any* $T < \infty$. *Moreover, the following hold:*

$$\|\psi^\varepsilon(t,x)\|_{L^2} = \|f\|_{L^2}, \quad \frac{d}{dt}\Big\{\frac{\varepsilon^2}{2}\|\partial_x \psi^\varepsilon(t,x)\|_{L^2}^2 + \varepsilon^\alpha \|\partial_x V^\varepsilon\|_{L^2}^2\Big\} = 0.$$

To prove Theorem 1.1, we also need the following properties. If

$$v^\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon} b_\varepsilon(t,\xi) d\bar{\xi},$$

then

$$\varepsilon \partial_x v^\varepsilon(t,x) = \frac{i}{\sqrt{\varepsilon}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon} \xi b_\varepsilon(t,\xi) d\bar{\xi} \tag{2.1}$$

and

$$J^\varepsilon(t) v^\varepsilon(t,x) = \frac{i}{\sqrt{\varepsilon}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon} \partial_\xi b_\varepsilon(t,\xi) d\bar{\xi}, \tag{2.2}$$

where we denote

$$J^\varepsilon(t) := \frac{x}{\varepsilon} + i(t-1)\partial_x.$$

Finally, for the potential $V^\varepsilon(t,x)$, noticing that $\|\psi^\varepsilon(t,x)\|_{L^2} = \|f(x)\|_{L^2}$ and $b \in L^1(\mathbb{R})$, we have

$$|\partial_x V^\varepsilon(t,x)| = \Big|\int_{-\infty}^x (|\psi^\varepsilon|^2 - b) dx\Big| \leq \|f\|_{L^2}^2 + \|b\|_{L^1} \leq C. \tag{2.3}$$

3 Linear Case ($\alpha > 1$)

For $\alpha > 1$, since the transport equation of geometric optics is linear, we expect that the limit transport equation of a_ε is $\partial_t a = 0$. So we compare the exact solution ψ^ε with the approximate solution $\psi^\varepsilon_{\text{app}}$ defined by

$$\psi^\varepsilon_{\text{app}} = \frac{1}{\sqrt{\varepsilon}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon} a_0(\xi) d\bar{\xi}, \tag{3.1}$$

where $a_0(\xi)$ as in Lemma 2.1. Obviously, $\psi^\varepsilon_{\text{app}}$ solves

$$i\varepsilon\partial_t \psi^\varepsilon_{\text{app}} + \frac{1}{2}\varepsilon^2 \partial_x^2 \psi^\varepsilon_{\text{app}} = 0.$$

By Lemma 2.1, we have

$$a_\varepsilon(0,\xi) \stackrel{\varepsilon\to 0}{\longrightarrow} a_0(\xi), \quad \text{in } L^2(\mathbb{R}).$$

In particular, it follows that $\psi^\varepsilon(0,x) \stackrel{\varepsilon\to 0}{\longrightarrow} \psi^\varepsilon_{\text{app}}(0,x)$ in $L^2(\mathbb{R})$. Denote $w^\varepsilon = \psi^\varepsilon - \psi^\varepsilon_{\text{app}}$. Then it solves

$$\begin{cases} i\varepsilon\partial_t w^\varepsilon + \dfrac{1}{2}\varepsilon^2\partial_x^2 w^\varepsilon = \varepsilon^\alpha V^\varepsilon \psi^\varepsilon, \\ w^\varepsilon|_{t=0} = (\psi^\varepsilon - \psi^\varepsilon_{\text{app}})(0,x) = o(1). \end{cases} \tag{3.2}$$

First, by multiplying both sides of (3.2) by $\overline{w}^\varepsilon$, integrating with respect to x over $\mathbb{R}$, and taking the imaginary part, we obtain

$$\frac{i}{2}\partial_t \int |w^\varepsilon(t,x)|^2 dx = \varepsilon^\alpha \text{Im}\langle V^\varepsilon \psi^\varepsilon, w^\varepsilon\rangle. \tag{3.3}$$

Notice that

$$\langle V^\varepsilon\psi^\varepsilon, w^\varepsilon\rangle = \langle V^\varepsilon(\psi^\varepsilon - \psi^\varepsilon_{\text{app}}), w^\varepsilon\rangle + \langle V^\varepsilon \psi^\varepsilon_{\text{app}}, w^\varepsilon\rangle,$$

which together with (3.3) shows that

$$\partial_t \|w^\varepsilon(t)\|_{L^2} \le \varepsilon^{\alpha-1}\|V^\varepsilon\psi^\varepsilon_{\text{app}}\|_{L^2} = \varepsilon^{\alpha-1}\|(V^\varepsilon(t,x) - V^\varepsilon(t,0))\psi^\varepsilon_{\text{app}}\|_{L^2}. \tag{3.4}$$

Then by (2.1)–(2.3), we get

$$\begin{aligned} \partial_t\|w^\varepsilon(t)\|_{L^2} &\le \varepsilon^{\alpha-1}\|\partial_x V^\varepsilon\|_{L^\infty}\|x\psi^\varepsilon_{\text{app}}\|_{L^2} \\ &\le \varepsilon^{\alpha-1}\|\partial_x V^\varepsilon\|_{L^\infty}(\varepsilon\|J^\varepsilon(t)\psi^\varepsilon_{\text{app}}\|_{L^2} + |1-t|\|\varepsilon\partial_x\psi^\varepsilon_{\text{app}}\|_{L^2}) \\ &\le C\varepsilon^{\alpha-1}(\varepsilon + |1-t|). \end{aligned} \tag{3.5}$$

Integrating (3.5) with respect to t over $[0,t]$, we have

$$\|w^\varepsilon(t)\|_{L^2} \le \|w^\varepsilon(0)\|_{L^2} + C\varepsilon^{\alpha-1}(1 + \varepsilon t + |1-t|^2). \tag{3.6}$$

Since $\alpha > 1$, this implies that $\|w^\varepsilon(t)\|_{L^2} = o(1)$. On the other hand, by Lemma 2.2, we get for $t < 1$,

$$\psi^\varepsilon_{\text{app}}(t,x) \stackrel{\varepsilon\to 0}{\sim} \frac{1}{(1-t)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t))},$$

and for $t > 1$,

$$\psi^\varepsilon_{\text{app}}(t,x) \stackrel{\varepsilon\to 0}{\sim} \frac{1}{(t-1)^{1/2}} f\Big(\frac{x}{1-t}\Big) e^{-ix^2/(2\varepsilon(1-t)) - i\pi/2}.$$

This completes the proof of the first part of Theorem 1.1.

4 Nonlinear Case ($\alpha = 1$)

For $\alpha = 1$, since the transport equation of geometric optics is nonlinear, there is no reason why $\partial_t a_\varepsilon$ should go to zero. A formal calculation will show that we do have to expect some modifications.

In (1.3), let us forget the dependence in ε of the symbol a_ε. Then by stationary phase formula

$$\psi^\varepsilon(t,x) \overset{\varepsilon\to 0}{\sim} \frac{1}{|2\pi(1-t)|^{1/2}} e^{i(\pi/4)\mathrm{sgn}(1-t)+ix^2/(2\varepsilon(t-1))} a\Big(t, \frac{x}{t-1}\Big). \tag{4.1}$$

Thus, formally

$$-\partial_x^2 V^\varepsilon(t,x) = |\psi^\varepsilon|^2 - b \overset{\varepsilon\to 0}{\sim} \frac{1}{|2\pi(1-t)|}\Big|a\Big(t, \frac{x}{t-1}\Big)\Big|^2 - b.$$

Let the function $V(t,x)$ satisfy

$$-\partial_x^2 V(t,x) = \frac{1}{|2\pi(1-t)|}\Big|a\Big(t, \frac{x}{t-1}\Big)\Big|^2 - b. \tag{4.2}$$

Thus we have

$$V^\varepsilon\psi^\varepsilon \overset{\varepsilon\to 0}{\sim} \frac{1}{|2\pi(1-t)|^{1/2}} e^{i(\pi/4)\mathrm{sgn}(1-t)+ix^2/(2\varepsilon(t-1))} V(t,x) a\Big(t, \frac{x}{t-1}\Big).$$

If we calculate the Fourier transform at ξ/x of the right-hand side and apply the stationary phase formula again, we have

$$\mathcal{F}(V^\varepsilon\psi^\varepsilon)(t,\xi/\varepsilon) \overset{\varepsilon\to 0}{\sim} \varepsilon^{1/2} e^{-i(t-1)\xi^2/(2\varepsilon)} V(t,(t-1)\xi)a(t,\xi).$$

Notice that equation (1.1) can be written, in terms of a_ε, as

$$i\partial_t a_\varepsilon(t,\xi) = \varepsilon^{-1/2} e^{i(t-1)\xi^2/(2\varepsilon)} \mathcal{F}(V^\varepsilon\psi^\varepsilon)(t,\xi/\varepsilon). \tag{4.3}$$

This formal calculation eventually leads to the limit transport equation

$$i\partial_t a(t,\xi) = V(t,(t-1)\xi)a(t,\xi) \tag{4.4}$$

with initial data $a|_{t=0} = a_0(\xi)$. Multiplying (4.4) by $\bar{a}$, one notices that the modulus of a is constant. If we write $a = a_0 e^{ig(t,\xi)}$, the equation for g is

$$\begin{cases} \partial_t g(t,\xi) = -V(t,(t-1)\xi), \\ g|_{t=0} = 0. \end{cases} \tag{4.5}$$

Since $|a| = |a_0|$, $V(t,x)$ satisfies

$$-\partial_x^2 V(t,x) = \frac{1}{|1-t|}\Big|f\Big(\frac{x}{1-t}\Big)\Big|^2 - b(x). \tag{4.6}$$

If we wish to get as a limit transport equation the relation $\partial_t \tilde{a} = 0$, it seems natural to define a modified symbol $\tilde{a}_\varepsilon$ as

$$\psi^\varepsilon(t,x) = \frac{1}{\varepsilon^{1/2}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)} \tilde{a}_\varepsilon(t,\xi) d\bar{\xi}. \tag{4.7}$$

On the other hand, in the case of $\alpha = 1$, it is also natural to take as an approximate solution

$$\psi^\varepsilon_{\rm app}(t,x) = \frac{1}{\varepsilon^{1/2}} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)} a_0(\xi) d\bar{\xi}. \tag{4.8}$$

We can easily verify that the approximate solution $\psi^\varepsilon_{\rm app}(t,x)$ satisfies

$$i\varepsilon\partial_t \psi^\varepsilon_{\rm app} + \frac{1}{2}\varepsilon^2 \partial_x^2 \psi^\varepsilon_{\rm app} = \varepsilon^{1/2} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)} V(t,(t-1)\xi) a_0(\xi) d\bar{\xi}. \tag{4.9}$$

Let

$$\Delta^\varepsilon(t,x) = V^\varepsilon_{\rm app}\psi^\varepsilon_{\rm app} - \varepsilon^{-1/2} \int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)} V(t,(t-1)\xi) a_0(\xi) d\bar{\xi},$$

where the function $V^\varepsilon_{\rm app}(t,x)$ satisfies

$$-\partial_x^2 V^\varepsilon_{\rm app}(t,x) = |\psi^\varepsilon_{\rm app}|^2(t,x) - b(x) \tag{4.10}$$

and $V^\varepsilon_{\rm app}$ vanishes at $x = 0$, $\partial_x V^\varepsilon_{\rm app}$ vanishes as $x \to -\infty$. As in Section 3, let us denote $w^\varepsilon = \psi^\varepsilon - \psi^\varepsilon_{\rm app}$. Then it solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{1}{2}\varepsilon^2 \partial_x^2 w^\varepsilon = \varepsilon(V^\varepsilon\psi^\varepsilon - V^\varepsilon_{\rm app}\psi^\varepsilon_{\rm app}) - \varepsilon\Delta^\varepsilon(t,x). \tag{4.11}$$

In the following, in order to prove that $\|w^\varepsilon(t)\|_{L^2} = o(1)$, for $t \in \mathbb{R}^+$, we divide it into three cases:

Case 1 $0 \le t \le 1-\varepsilon$ (Before the caustic)

Proceeding as in the linear case, we have

$$\partial_t \|w^\varepsilon(t)\|_{L^2} \le \|(V^\varepsilon - V^\varepsilon_{\rm app})\psi^\varepsilon_{\rm app}(t)\|_{L^2} + \|\Delta^\varepsilon(t,x)\|_{L^2}. \tag{4.12}$$

First we estimate $\|(V^\varepsilon - V^\varepsilon_{\rm app})\psi^\varepsilon_{\rm app}(t)\|_{L^2}$. By (1.2) and (4.10), we have

$$\begin{aligned} -\partial_x(V^\varepsilon - V^\varepsilon_{\rm app})(t,x) &= \int_{-\infty}^x |\psi^\varepsilon|^2 - |\psi^\varepsilon_{\rm app}|^2 dx \\ &= \int_{-\infty}^x (\psi^\varepsilon - \psi^\varepsilon_{\rm app})\overline{\psi^\varepsilon} + \psi^\varepsilon_{\rm app}\overline{(\psi^\varepsilon - \psi^\varepsilon_{\rm app})} dx. \end{aligned}$$

Thus we get

$$\|\partial_x(V^\varepsilon - V^\varepsilon_{\rm app})\|_{L^\infty} \le \|\psi^\varepsilon - \psi^\varepsilon_{\rm app}\|_{L^2}(\|\psi^\varepsilon\|_{L^2} + \|\psi^\varepsilon_{\rm app}\|_{L^2}) \le C\|w^\varepsilon(t)\|_{L^2}. \tag{4.13}$$

As in (3.5), we obtain

$$\begin{aligned} \|(V^\varepsilon - V^\varepsilon_{\rm app})\psi^\varepsilon_{\rm app}(t)\|_{L^2} &\le \|\partial_x(V^\varepsilon - V^\varepsilon_{\rm app})\|_{L^\infty}\|x\psi^\varepsilon_{\rm app}\|_{L^2} \\ &\le C(\varepsilon + |1-t|)\|w^\varepsilon(t)\|_{L^2}. \end{aligned} \tag{4.14}$$

Now we turn to estimate $\|\Delta^\varepsilon(t,x)\|_{L^2}$. Let

$$\psi^\varepsilon_1(t,x) = \frac{1}{(1-t)^{1/2}} e^{ix^2/(2\varepsilon(t-1))+ig(t,x/(t-1))} f\Big(\frac{x}{1-t}\Big).$$

Then we rewrite $\Delta^\varepsilon(t,x)$ as

$$\begin{aligned}\Delta^\varepsilon(t,x) &= (V^\varepsilon_{\rm app}\psi^\varepsilon_{\rm app} - V\psi^\varepsilon_1)\\ &\quad + \Big(V\psi^\varepsilon_1 - \varepsilon^{-1/2}\int e^{-i(t-1)\xi^2/(2\varepsilon)+ix\xi/\varepsilon+ig(t,\xi)}V(t,(t-1)\xi)a_0(\xi)d\bar{\xi}\Big)\\ &:= \Delta^\varepsilon_1(t,x) + \Delta^\varepsilon_2(t,x). \end{aligned} \tag{4.15}$$

Denoting

$$\sigma(t,\xi) = e^{ig(t,\xi)}V(t,(t-1)\xi)a_0(\xi),$$

and by

$$-\partial_x V(t,x) = \int_{-\infty}^x \frac{1}{|1-t|}\Big|f\Big(\frac{x}{1-t}\Big)\Big|^2 - b(x)dx,$$

we have

$$\|\sigma(t,\xi)\|_{L^2} \le |1-t|\|\partial_\xi V(t,(t-1)\xi)\|_{L^\infty}\|\xi a_0(\xi)\|_{L^2} \le C|1-t|. \tag{4.16}$$

Thus by Lemma 2.2, there exists a continuous function h_1, with $h_1(0)=0$, such that

$$\|\Delta^\varepsilon_2(t,x)\|_{L^2} = h_1\Big(\frac{\varepsilon}{1-t}\Big). \tag{4.17}$$

In order to estimate $\|\Delta^\varepsilon_1(t,x)\|_{L^2}$, we write

$$\Delta^\varepsilon_1(t,x) = (V^\varepsilon_{\rm app} - V)\psi^\varepsilon_{\rm app} + V(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1) := \mathrm{I} + \mathrm{II}.$$

Now, as in (3.5), we have

$$\begin{aligned}\|\mathrm{II}\|_{L^2} &\le \|\partial_x V\|_{L^\infty}\|x(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1)\|_{L^2}\\ &\le \|\partial_x V\|_{L^\infty}(\varepsilon\|J^\varepsilon(t)(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1)\|_{L^2} + |1-t|\|\varepsilon\partial_x(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1)\|_{L^2}).\end{aligned}$$

By taking $\sigma(t,\xi) = \partial_\xi(e^{ig(t,\xi)}a_0(\xi))$ (resp. $\sigma(t,\xi) = \xi e^{ig(t,\xi)}a_0(\xi)$) in Lemma 2.2, we conclude that there exist $h_2, h_3 \in C(\mathbb{R})$, with $h_2(0) = h_3(0) = 0$, such that

$$\|J^\varepsilon(t)(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1)\|_{L^2} = h_2\Big(\frac{\varepsilon}{1-t}\Big), \quad \|\varepsilon\partial_x(\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1)\|_{L^2} = h_3\Big(\frac{\varepsilon}{1-t}\Big).$$

Hence we have

$$\|\mathrm{II}\|_{L^2} \le C\Big(\varepsilon h_2\Big(\frac{\varepsilon}{1-t}\Big) + |1-t|h_3\Big(\frac{\varepsilon}{1-t}\Big)\Big). \tag{4.18}$$

Notice that

$$\begin{aligned}-\partial_x(V^\varepsilon_{\rm app} - V) &= \int_{-\infty}^x |\psi^\varepsilon_{\rm app}|^2 - \frac{1}{|1-t|}\Big|f\Big(\frac{x}{1-t}\Big)\Big|^2 dx\\ &= \int_{-\infty}^x |\psi^\varepsilon_{\rm app}|^2 - |\psi^\varepsilon_1|^2 dx\\ &\le \|\psi^\varepsilon_{\rm app} - \psi^\varepsilon_1\|_{L^2}(\|\psi^\varepsilon_{\rm app}\|_{L^2} + \|\psi^\varepsilon_1\|_{L^2}).\end{aligned}$$

Then, as in (3.5), by Lemma 2.2 (with $\sigma(t,\xi) = a_0(\xi)$), there exists $h_4 \in C(\mathbb{R})$, with $h_4(0) = 0$, such that

$$\|\mathrm{I}\|_{L^2} \le C(\varepsilon + |1-t|) h_4\Big(\frac{\varepsilon}{1-t}\Big). \tag{4.19}$$

By (4.18) and (4.19), we obtain

$$\|\Delta_1^\varepsilon(t,x)\|_{L^2} \le C(\varepsilon + |1-t|)\Big(h_2\Big(\frac{\varepsilon}{1-t}\Big) + h_3\Big(\frac{\varepsilon}{1-t}\Big) + h_4\Big(\frac{\varepsilon}{1-t}\Big)\Big). \tag{4.20}$$

Hence, by (4.17), (4.20), there exists $h \in C(\mathbb{R})$, with $h(0) = 0$, such that

$$\|\Delta^\varepsilon(t,x)\|_{L^2} \le C(1 + \varepsilon + |1-t|) h\Big(\frac{\varepsilon}{1-t}\Big). \tag{4.21}$$

Finally, by combining (4.12), (4.14) and (4.21), we have

$$\partial_t \|w^\varepsilon(t)\|_{L^2} \le C(\varepsilon + |1-t|)\|w^\varepsilon(t)\|_{L^2} + C(1+\varepsilon+|1-t|) h\Big(\frac{\varepsilon}{1-t}\Big). \tag{4.22}$$

By the Gronwall inequality, we get

$$\begin{aligned}\|w^\varepsilon(t)\|_{L^2} \le \Big(&\|w^\varepsilon(0)\|_{L^2} + C\int_0^t (1+\varepsilon+|1-s|) h\Big(\frac{\varepsilon}{1-s}\Big) ds\Big) \\ &\cdot \exp\Big(C\int_0^t (\varepsilon + |1-s|) ds\Big).\end{aligned} \tag{4.23}$$

Notice that

$$\int_0^t (1+\varepsilon+|1-s|) h\Big(\frac{\varepsilon}{1-s}\Big) ds \overset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{for } t \le 1-\varepsilon.$$

Thus, we conclude that $\|w^\varepsilon(t)\|_{L^2} = o(1)$, for $0 \le t \le 1-\varepsilon$.

Case 2 $1-\varepsilon \le t \le 1+\varepsilon$ (Caustic crossing)

Let us denote $a_\varepsilon(t,\xi) = \tilde{a}_\varepsilon(t,\xi) e^{ig(t,\xi)}$. By (4.3), as in (3.5), we have

$$\|\partial_t a_\varepsilon(t,\xi)\|_{L^2} \le \|V^\varepsilon \psi^\varepsilon\|_{L^2} \le C(\varepsilon + |1-t|).$$

So, by (2.1) and Lemma 2.3, we find

$$\begin{aligned}\|\partial_t \tilde{a}_\varepsilon(t,\xi)\|_{L^2} &\le \|\partial_t g(t,\xi) a_\varepsilon(t,\xi)\|_{L^2} + \|\partial_t a_\varepsilon(t,\xi)\|_{L^2} \\ &= \|V^\varepsilon(t,(t-1)\xi) a_\varepsilon(t,\xi)\|_{L^2} + \|\partial_t a_\varepsilon(t,\xi)\|_{L^2} \\ &\le C|1-t| \|\xi a_\varepsilon(t,\xi)\|_{L^2} + \|\partial_t a_\varepsilon(t,\xi)\|_{L^2} \\ &\le C|1-t| \|\varepsilon \partial_x \psi^\varepsilon(t,x)\|_{L^2} + \|\partial_t a_\varepsilon(t,\xi)\|_{L^2} \\ &\le C(\varepsilon + |1-t|).\end{aligned} \tag{4.24}$$

Thus, by Case 1, for $1-\varepsilon \le t \le 1+\varepsilon$, we have

$$\begin{aligned}\|\tilde{a}_\varepsilon(t,\xi) - a_0(\xi)\|_{L^2} &\le \|\tilde{a}_\varepsilon(t,\xi) - a_0(\xi)\|_{L^2} + \|\tilde{a}_\varepsilon(t,\xi) - \tilde{a}_\varepsilon(1-\varepsilon,\xi)\|_{L^2} \\ &\le \|w^\varepsilon(1-\varepsilon)\|_{L^2} + \int_{1-\varepsilon}^{1+\varepsilon} \|\partial_t \tilde{a}_\varepsilon(t,\xi)\|_{L^2} dt \\ &\le o(1) + C\int_{1-\varepsilon}^{1+\varepsilon} (\varepsilon + |1-t|) dt = o(1).\end{aligned} \tag{4.25}$$

That is,

$$\|w^\varepsilon(t)\|_{L^2} = o(1), \quad 1-\varepsilon \le t \le 1+\varepsilon. \tag{4.26}$$

This completes the proof of Case 2.

Case 3 $t \ge 1+\varepsilon$ (Beyond the caustic)

We can resume the computations performed at the first step: the approximate solution $\psi^\varepsilon_{\rm app}$ remains the same, and the only difference is the initial data for the remainder. But from Case 2, this data is small, so we can repeat the computations of Case 1. We finally can prove that

$$w^\varepsilon(t,\xi) \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \quad \text{in } L^\infty_{\rm loc}(\mathbb{R}, L^2). \tag{4.27}$$

Then as in the proof of the linear case, this implies the second part of Theorem 1.1.

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