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On Nonlinearly Elastic Membranes under Compression

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Abstract The classical equations of a nonlinearly elastic plane membrane made of Saint Venant-Kirchhoff material have been justified by Fox, Raoult and Simo (1993) and Pantz (2000). We show that, under compression, the associated minimization problem admits no solution. The proof is based on a result of non-existence of minimizers of non-convex functionals due to Dacorogna and Marcellini (1995). We generalize the application of their result from plane elasticity to three-dimensional plane membranes.

Keywords Nonlinear elasticity, Minimization, Quasiconvexity, Quasiconvex envelope, Rank-1-convexity
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0 Introduction

Let $\mathbb{R}^{3\times 2}$ be the space of real 3×2 matrices endowed with the usual Euclidean norm $|F| = (\mathrm{tr}(F^T F))^{1/2}$. The classical two-dimensional stored energy function for a nonlinearly elastic plane membrane is the following:

$$\forall\, \xi \in \mathbb{R}^{3\times 2}, \quad W(\xi) = \frac{E\nu}{2(1-\nu^2)}(\mathrm{tr}(\xi^t \xi - I))^2 + \frac{E}{2(1+\nu)}\,\mathrm{tr}(\xi^t\xi - I)^2, \tag{0.1}$$

where ξ stands for the two dimensional deformation gradient, $E > 0$ is the Young modulus and $0 \le \nu < \frac{1}{2}$ is Poisson's ratio. The above function expresses the exact difference between the metric tensor of the unknown surface and that of the reference configuration. We are interested in the associated minimization problem. Let $\omega \subset \mathbb{R}^2$ be a bounded open set. Let $\varphi : \omega \longrightarrow \mathbb{R}^3$ denote the deformation. Finally, let $f \in L^{12}(\omega;\mathbb{R}^3)$ and $\xi_0 \in \mathbb{R}^{3\times 2}$. Then the considered minimization problem is

$$\text{(P)} \qquad \inf\Big\{ I(\varphi) = \int_\omega W(\nabla\varphi)\,dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega;\mathbb{R}^3)\Big\},$$

where $\varphi_0 = \xi_0 x,\ x \in \overline{\omega}$.

The classical two-dimensional equations of a nonlinearly elastic membrane plate, as found in the mechanical literature (see [17], for instance), have been justified by Fox, Raoult and Simo [15] by means of the method of formal asymptotic expansions, introduced by Ciarlet and Destuynder [7, 8] (see also [6] for an extensive presentation of the different nonlinear plate theories) applied to the three-dimensional equations of the nonlinear elasticity for a Saint

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Venant-Kirchhoff material. A remarkable property of this nonlinear plate theory is the material frame indifference of its stored energy function (0.1) which it inherited from the original model.

Unfortunately, the associated energy functional $J(\varphi) = \int_\omega W(\nabla\varphi)dx$ is not weakly lower semicontinuous on $W^{1,\infty}(\omega;\mathbb{R}^3)$ which is a necessary condition for the lower semicontinuity with respect to the topology of $W^{1,4}(\omega;\mathbb{R}^3)$. Actually, Morrey [25, 26] has shown the equivalence between weakly lower semicontinuity and quasiconvexity, under some ad hoc growth and coercivity conditions, for a whole class of finite functionals depending on a gradient. See also [24, 1] for refined versions of this result. In the case of problem (P), Coutand [10] gave a counterexample to the weak lower semi-continuity of its functional when the membrane is subjected to plane forces. In fact, the above stored energy function (0.1) is neither polyconvex nor quasiconvex since it is not rank-one-convex, as it is shown in [16] (see also [29, 5] for the case of the Saint Venant-Kirchhoff stored energy function), which is a necesary condition for weak-$\star$ lower semicontinuity by Tartar [30].

Le Dret and Raoult [20, 23] have computed the quasiconvex envelope of the three-dimensional Saint Venant-Kirchhoff stored energy function. It turns out that the latter function is not quasiconvex and consequently not weakly lower semicontinuous in light of Morrey's result as well.

For the reasons stated above, it is natural to consider the relaxed minimization problem (see for example [13]), in which the stored energy function is substituted by its quasiconvex envelope. It is well known that the infimum of the associated relaxed problem

$$\text{(QP)} \qquad \inf\Big\{\bar{I}(\varphi) = \int_\omega QW(\nabla\varphi)\,dx : \ \varphi \in \varphi_0 + W_0^{1,4}(\omega;\mathbb{R}^3)\Big\}$$

coincides with that of the original problem (P) and that any minimizing sequence of problem (P) contains a subsequence which weakly converges in $W^{1,4}(\omega;\mathbb{R}^3)$ towards a minimizer of problem (QP) (see [13] for a complete survey of the subject).

The motivation for the computation of the quasiconvex envelope carried out by Le Dret and Raoult [23] was their justification of another nonlinear plane membrane model by means of Γ-convergence theory (see [19, 22]). Their approach gives a convergence result, as the thickness of the plate tends to zero, of a diagonal infimizing sequence of deformations of the original three-dimensional energy towards a minimizer of the two-dimensional membrane energy. The latter is proved to be equal to the quasiconvex envelope of the functional obtained by minimizing the Saint Venant-Kirchhoff Stored energy function with respect to the third column vector. Besides giving the first rigorous result in the derivation of nonlinear membrane theories, the existence of a minimizer to the energy they obtain is established through the convergence. An important feature of their membrane model is that it cannot sustain any compression. That is, it can be compressed with nil stored energy which is not, intuitively, that surprising since a membrane energy only measures the changes in the surface metric.

Coutand [9–12] has managed to show some existence results of minimizers for the classical model albeit in particular instances. He proves the existence of a solution to the equations by means of the inverse function theorem before showing it actually is a local or global minimizer depending on the case. In one of the considered cases, he proves that the found solution (critical point) does not minimize the energy in any reasonable space (see [10]). In fact, it turns out to be the only case where the membrane is under slight compression on its boundary whereas all the other results deal with membranes extended or clamped at their boundary. In a similar vein, Dacorogna and Marcellini [14] give necessary and sufficient conditions to the existence of minimizers to the two-dimensional Saint Venant-Kirchhoff stored energy functional. These conditions draw a direct link between the existence of minimizers and the prescribed condition on the boundary whether it is a compression or an extension in absence of body loads. Their application is based on a general non-existence theorem that they establish for non-convex functionals with a linear application as the prescribed deformation on the boundary and on the computation of the quasiconvex envelope of the Saint Venant-Kirchhoff stored energy function performed by Le Dret and Raoult [20, 23]. For issues dealing with the uniqueness of the solution, we send the reader to Knops and Stuart [18].

These existence results were the incentive behind this chapter. Whereas Coutand [9–12] uses an implicit method which does not bring out the link between the convexity properties of the functional and the prescribed conditions on the boundary, Dacorogna and Marcellini apply a direct method but to a less satisfying instance because of an algebraic restriction namely the lack of convenient expressions for the singular values of the deformation gradient. Here, we use the non-existence result obtained by Dacorogna and Marcellini to prove the non-existence of minimizers to the classical nonlinear plane membrane energy model under compression conditions on the boundary.

In the next section, we recall the notion of quasiconvexity and give the quasiconvex envelope of the nonlinear plate membrane stored energy function. We also recall the non-existence theory established by Dacorogna and Marcellini which yields the announced result. In Section 2, we give and prove an algebraic lemma, that allows the application of this theory. We terminate the chapter with a compilation of remarks on nonlinear membrane plates in the last section. Namely we derive an upper bound for the minimum of the elastic membrane energy with dead body loads and remark that membranes do not resist compression.

1 Preliminaries

We recall that a Borel measurable and locally integrable function $G : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is said to be quasiconvex if

$$G(x, u, A) \leq \frac{1}{\operatorname{meas} D} \int_D G(x, u, A + \nabla\varphi)\, dx$$

for every bounded domain $D \subset \mathbb{R}^n$, for every $A \in \mathbb{R}^{n\times N}$ and for every $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^N)$. The notion of quasiconvexity was first introduced by Morrey [25]. Since, several authors have worked on the relationship between quasiconvexity and lower semicontinuity. Here, we give the result as stated in [4] and we send the reader, for instance, to Morrey [26], Dacorogna [13], Acerbi and Fusco [1] and Marcellini [14] for proofs of this result and other variations on the data.

Theorem 1.1 *Let $1 \leq p \leq \infty$, let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $G : \Omega \times \mathbb{R}^N \times \mathbb{R}^{n\times N} \to$*

$\mathbb{R}$ be a Cathéodory integrand satisfying the following estimate

$$
\begin{aligned}
0 \le G(x,u,A) &\le a(x,|u|)(1+|A|^p) && \text{if } p<\infty,\\
0 \le G(x,u,A) &\le \alpha(x,|u|,|A|) && \text{if } p=\infty,
\end{aligned}
$$

where $a(x,s)$ and $\alpha(x,s,t)$ are summable in x and increasing in s and t. Then the following conditions are equivalent:

(1) *for a.e. $x \in \Omega$ and every $u \in \mathbb{R}^N$, the function $G(x,u,\cdot\,)$ is quasiconvex;*

(2) *the functional $F : u \in \mathbb{R}^N \to \int_\Omega G(x,u,\nabla u)\,dx$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(\Omega;\mathbb{R}^N)$ (sequentially weakly-$*$ lower semicontinuous on $W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ if $p=\infty$).*

Remark 1.1 Ball and Murat [2] introduced the notion of $W^{1,p}$-quasiconvexity in an attempt to weaken the notion of quasiconvexity of Morrey (which corresponds in their terminology to the $W^{1,\infty}$-quasiconvexity) and thus draw a direct link between the $W^{1,p}$-quasiconvexity of a stored energy function and the sequential weak lower semicontinuity with respect to the topology of $W^{1,p}$. In particular, they show that in certain cases $W^{1,p}$-quasiconvexity and $W^{1,\infty}$-quasiconvexity are equivalent. However, the above theorem clearly shows that for finite functionals quasiconvexity lies behind sequential weak lower semicontinuity in Sobolev spaces.

Next we give the definitions of the convex and quasiconvex envelopes respectively of a function $G : \mathbb{R}^{n\times N} \to \mathbb{R}^N$,

$$
\begin{aligned}
G^{**} &= \sup\{Z : \mathbb{R}^{n\times N} \to \mathbb{R}^N;\ Z \text{ convex},\ Z \le g\},\\
\mathbf{Q}G &= \sup\{Z : \mathbb{R}^{n\times N} \to \mathbb{R}^N;\ Z \text{ quasiconvex},\ Z \le g\}.
\end{aligned}
$$

Furthermore if G is locally bounded and Borel measurable then a characterization of the quasiconvex envelope, due to Dacorogna [13], is given by

$$
\mathbf{Q}G(\xi) = \inf_{\varphi \in W_0^{1,\infty}(D;\mathbb{R}^N)} I_\xi(\varphi), \tag{1.1}
$$

where

$$
I_\xi(\varphi) = \frac{1}{\operatorname{meas} D} \int_\omega G(\xi + \nabla\varphi)\,dx - \int_\omega f\cdot\varphi\,dx, \tag{1.2}
$$

for every $\xi \in \mathbb{R}^{n\times N}$, and $D \subset \mathbb{R}^n$ a bounded domain. In particular, the infinum in identity (1.1) is independent of the choice of D.

All of the above notions as well as their properties are fully exposed in [13] (see also [4]). So we refer to these books for details, proofs and references. In the next section, we will explain how we intend to use the above theory. The quasiconvex envelope of the stored energy function (0.1) is easily deduced from the expression of the quasiconvex envelope of the Saint-Venant Kirchhoff stored energy function carried out by Le Dret and Raoult [20, 23], or can be computed in the same fashion as they have also done. We give its expression in the next proposition, but first we recall the membrane stored energy function (0.1) expressed through the singular values s_1 and s_2 (ordered such that $s_1 \le s_2$), also known as the principal stretches, of the matrix ξ (i.e. the eigenvalues of $(\xi^t\xi)^{\frac{1}{2}}$)

$$W(\xi) = \frac{E\nu}{2(1-\nu^2)}(s_1^2+s_2^2-2)^2 + \frac{E}{2(1+\nu)}((s_1^2-1)^2+(s_1^2-1)^2). \tag{1.3}$$

Proposition 1.1 *The quasiconvex envelope of the nonlinear plane membrane is*

$$\mathbf{Q}W(\xi) = W^{**}(\xi) = \begin{cases} 0, & \text{if } \xi \in D_1, \\ \dfrac{E}{2}(s_2^2-1)^2, & \text{if } \xi \in D_2, \\ W(\xi), & \text{if } \xi \notin D_1 \cup D_2, \end{cases} \qquad \forall\, \xi \in \mathbb{R}^{3\times 2}, \tag{1.4}$$

where

$$\begin{aligned} &0 \le s_1(\xi) \le s_2(\xi), \\ &D_1 = \{\xi \in \mathbb{R}^{32} : s_2 \le 1 \text{ and } s_1^2+\nu s_2^2 < 1+\nu\}, \\ &D_2 = \{\xi \in \mathbb{R}^{32} : s_2 > 1 \text{ and } s_1^2+\nu s_2^2 < 1+\nu\}. \end{aligned} \tag{1.5}$$

Remark 1.2 Inspecting the Saint Venant-Kirchhoff stored energy function expressed in terms of the singular values $0 \le s_1 \le s_2 \le s_3$ of the deformation gradient F,

$$\overline{W}(F) = \frac{E\nu}{8(1+\nu)(1-2\nu)}(\mathrm{tr}(F^tF-I))^2 + \frac{E}{8(1+\nu)}\,\mathrm{tr}(F^tF-I)^2, \tag{1.6}$$

it is clear that, from a mathematical viewpoint, it is not very different from the energy (1.3) we are concerned with. In [22], Le Dret and Raoult prove a similar result by fixing one of the singular values at a time and considering the convex envelope with respect to the other thereby concluding by symmetry arguments. However, Dacorogna and Marcellini do not consider the right quasiconvex envelope of the two-dimensional stored energy function for an elastic body made of Saint Venant-Kirchhoff material in their paper [14]. Indeed, they seem to have recovered the result by merely taking the greatest singular value equal to zero in the three-dimensional case. Whereas, the right guess would have been to make the smallest one vanish.

We now present the non-existence theorem we shall use to achieve our goal. This result is due to Dacorogna and Marcellini [14]. First of all, we need to introduce a notion of strict convexity that is central to the proof.

Definition 1.1 *A convex function $H : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ is said to be strictly convex at $\xi_0 = (\xi_0^\alpha)_{1\le\alpha\le N} \in \mathbb{R}^{n\times N}$ in at least N directions if there exists $\lambda = (\lambda^\alpha)_{1\le\alpha\le N} \in \mathbb{R}^{n\times N}$ such that*

$$\lambda^\alpha \neq 0 \quad \text{and} \quad \langle \lambda^\alpha ; \xi^\alpha - \xi_0^\alpha \rangle_{\mathbb{R}^n} = 0, \quad \forall\, \alpha = 1, 2, \cdots, N,$$

whenever $\xi = (\xi^\alpha)_{1\le\alpha\le N}$ satisfies the condition

$$\frac{H(\xi)+H(\xi_0)}{2} = H\Big(\frac{\xi+\xi_0}{2}\Big). \tag{1.7}$$

Theorem 1.2 *Let $g : \mathbb{R}^{n\times N} \longrightarrow \mathbb{R}$ be a lower semicontinuous function and let $\xi_0 \in \mathbb{R}^{n\times N}$ be such that*

(i) $G^{**}(\xi_0) = \mathbf{Q}\,G(\xi_0) < g(\xi_0)$,

(ii) G^{**} *is strictly convex in at least N directions.*

Then (P) *has no solution.*

Remark 1.3 As pointed out by the authors, the weakness of this theorem is that it is expressed in terms of G^{**} and not of $\mathbf{Q}\,G$. Fortunately, as already observed above, in our case the two envelopes coincide and the same goes for the three-dimensional Saint Venant-Kirchhoff stored energy function (see [20, 23]).

2 Main Result

As already mentioned in the Introduction, the purpose of this paper is to draw a link between weak lower semicontinuity of the minimization functional and the boundary condition in the case without exterior energy.

Theorem 2.1 *Problem* (P) *has solutions if and only if*

$$s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu.$$

Remark 2.1 The above result is still valid if we replace W_m in problem (P) by the following stored energy function

$$W_0(\xi) = W(\xi) + \frac{1}{4}\Big[\frac{\nu}{1-\nu}\,\mathrm{tr}(\xi^t\xi - I) - 2\Big]_+^2, \quad \forall\, \xi \in \mathbb{R}^{3\times 2},$$

which was obtained by Pantz in [27, 28] for a membrane plate made of Saint Venant-Kirchhoff material without imposing the orientation-preserving condition on the set of admissible three-dimensional deformations. Note that this function can also be obtained by minimizing the functional $\overline{W}$ along the third column vector as in [22].

In practice, the above result means that a plane membrane *behaves badly* under compression. This behaviour was already observed by Le Dret and Raoult [19, 22] in their asymptotic analysis. As a matter of fact, their two-dimensional limit model is a relaxed problem. In another context, Coutand [10] showed that the solution he found to the local boundary-value problem of the membrane under slight compression via the implicit function theorem is not even a local minimizer. This property is further investigated in the next section.

We can now announce the algebraic lemma that allows us to generalize the result obtained by Dacorogna and Marcellini, who applied their Theorem 1.2 to the two-dimensional Saint Venant-Kirchhoff material, to the nonlinear plane membrane made of Saint Venant-Kirchhoff material. We shall give the proof later after proving Theorem 2.1.

Lemma 2.1 *Let $\zeta_0 \in \mathbb{R}^{3\times 2}$ such that $s_1(\zeta_0) < s_2(\zeta_0)$. Then there exists $\lambda = (\lambda^\alpha)_{1\leq\alpha\leq N} \in \mathbb{R}^{3\times 2}$ such that*

$$\lambda^\alpha \neq 0 \quad \textit{and} \quad \langle \lambda^\alpha, \zeta^\alpha - \zeta_0^\alpha\rangle_{\mathbb{R}^3} = 0, \quad \forall\, \alpha = 1, 2,$$

whenever $\zeta \in E = \big\{\zeta \in \mathbb{R}^{3\times 2} : \frac{s_2(\xi)+s_2(\xi_0)}{2} = s_2\big(\frac{\xi+\xi_0}{2}\big)\big\}$.

Remark 2.2 Note that the statement above does not say that function s_2 is strictly convex in at least two directions at ζ_0 since function s_2 is not even convex.

Proof of Theorem 2.1 Here we follow Dacorogna and Marcellini [14]. First we assume that $s_1^2(\xi_0) + \nu s_2^2(\xi_0) \geq 1 + \nu$. By definition of quasiconvexity we write

$$|\omega|\mathbf{Q}W(\xi_0) \leq \int_\omega \mathbf{Q}W(\xi_0 + \nabla\varphi(x))\,dx$$

for all $\varphi \in W^{1,\infty}(\omega;\mathbb{R}^3)$ and by a result in Ball and Murat [2], the above still holds for all $\varphi \in W^{1,4}(\omega;\mathbb{R}^3)$ by the lower semicontinuity of W. Since by defintion $\mathbf{Q}W \leq W$ on $\mathbb{R}^{3\times 2}$ and by assumption $\mathbf{Q}W(\xi_0) = W(\xi_0)$ (see Proposition 1.1), we infer

$$|\omega|W(\xi_0) \leq \int_\omega W(\xi_0 + \nabla\varphi(x))\,dx$$

for all $\varphi \in W^{1,4}(\omega;\mathbb{R}^3)$ which means that the function defined by $\varphi_0(x) = \xi_0 x$ on $\bar{\omega}$ is a solution of problem (P).

Next, we assume that $s_1^2(\xi_0) + \nu s_2^2(\xi_0) < 1 + \nu$. Then ξ_0 is either in D_1 or in D_2 (see (1.4) and (1.5)). We study these two cases separately.

Step 1 Assume that $\xi_0 \in D_2$. The non-existence of solutions in this case will follow from Theorem 1.2. Therefore, we have to prove that W^{**} is strictly convex at ξ_0 in at least two directions. Consider $\xi_0 \in D_2$ (this is possible since D_2 is open) satisfying condition (1.7), i.e.,

$$\frac{W^{**}(\xi) + W^{**}(\xi_0)}{2} = W^{**}\Big(\frac{\xi + \xi_0}{2}\Big). \tag{2.1}$$

Consider now the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \frac{E}{2}(x^2 - 1)^2.$$

h is strictly convex as long as $x > 1$. And we can write

$$W^{**}(\xi) = h(s_2(\xi)). \tag{2.2}$$

Since $s_2(\xi) > 1$, we deduce from (2.1), (2.2) and the strict convexity of h that

$$\frac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\Big(\frac{\xi + \xi_0}{2}\Big).$$

In other words, we have shown that

$$\xi \in E = \Big\{\zeta \in \mathbb{R}^{32} : \frac{s_2(\xi) + s_2(\xi_0)}{2} = s_2\Big(\frac{\xi + \xi_0}{2}\Big)\Big\}.$$

Now $\xi_0 \in D_2$ entails that $s_1(\xi_0) \leq 1 < s_2(\xi_0)$. Hence, we can apply Lemma 2.1 to infer that W^{**} is strictly convex at ξ_0 in at least two directions. Then we conclude by Theorem 1.2 that (P) has no solution if $\xi_0 \in D_2$.

Step 2 Assume that $\xi_0 \in D_1$. According to Proposition 1.1, we have

$$\mathbf{Q}W(\xi_0) = W^{**}(\xi_0) = 0.$$

Assume now that (P) has a solution $\varphi \in \varphi_0 + W_0^{1,\infty}(\omega;\mathbb{R}^3)$. Then, recalling that the infima of problems (P) and (QP) coincide (see Preliminaries), we necessarily have

$$W(\nabla\varphi) = 0 \quad \text{a.e. in } \omega.$$

From (1.3), we deduce that

$$s_1(\nabla\varphi) = s_2(\nabla\varphi) = 1 \quad \text{and} \quad \det \nabla\varphi^t\nabla\varphi = 1,$$

almost everywhere in ω. On the one hand, this implies that

$$\int_\omega \det \nabla\varphi^t\nabla\varphi\, dx = \operatorname{meas}\omega,$$

and on the other hand, the boundary data and the fact that $\varphi \in W^{1,\infty}(\omega;\mathbb{R}^3)$ yield

$$\int_\omega \det \nabla\varphi^t\nabla\varphi\, dx = \frac{1}{2}\int_{\partial\omega} \{(\operatorname{Cof}\nabla{\varphi_0}^t\nabla\varphi_0)n(x)\}\cdot\{(\nabla{\varphi_0}^t\nabla\varphi_0)x\}\, da. \tag{2.3}$$

For a proof of (2.3), we refer to Ciarlet [5, Theorem 2.7-1]. Next, observing that

$$(\operatorname{Cof}\nabla{\varphi_0}^t\nabla\varphi_0) = \det\nabla{\varphi_0}^t\nabla\varphi_0(\nabla{\varphi_0}^t\nabla\varphi_0)^{-1},$$

identity (2.3) entails

$$\operatorname{meas}\omega = \frac{1}{2}\,\det\nabla{\varphi_0}^t\nabla\varphi_0\int_{\partial\omega}\{(\nabla{\varphi_0}^t\nabla\varphi_0)^{-1}n(x)\}\cdot\{(\nabla{\varphi_0}^t\nabla\varphi_0 x)\}da.$$

Then, by the symmetry of $\nabla{\varphi_0}^t\nabla\varphi_0$, we have

$$\operatorname{meas}\omega = \frac{1}{2}\,\det\nabla{\varphi_0}^t\nabla\varphi_0\int_{\partial\omega} n(x)\cdot x\, da. \tag{2.4}$$

Besides, Stokes formula applied to the identity vector field gives

$$\int_{\partial\omega} n(x)\cdot x\, da = \int_\omega \operatorname{div} x\, dx = 2\operatorname{meas}\omega,$$

and injecting the above in (2.4) we get

$$\det\nabla{\varphi_0}^t\nabla\varphi_0 = \det\xi_0^t\xi_0 = s_1(\xi_0)s_2(\xi_0) = 1.$$

And as $0 \le s_1(\xi_0) \le s_2(\xi_0) \le 1$, since $\xi_0 \in D_1$, we necessarily have

$$s_1(\xi_0) = s_2(\xi_0) = 1.$$

However this contradicts the data since $s_1(\xi_0)^2 + \nu s_2(\xi_0)^2 < 1 + \nu$, when $\xi_0 \in D_1$. The proof is complete.

Proof of Lemma 2.1 We divide the proof in two steps.

Step 1 For all matrices $\zeta \in M_{32}$, let us note that $v_1(\zeta)$ and $v_2(\zeta)$ are the two positive eigenvalues of the symmetric matrix $\zeta^t\zeta$, $s_\alpha(\zeta) = v_\alpha^{\frac{1}{2}}(\zeta)$ for $\alpha = 1, 2$ are its singular values and

$\{e_1, e_2\}$ is an orthonormal basis of eigenvectors of $\zeta_0^t\zeta_0$ so that $\zeta_0^t\zeta_0\, e_\alpha = v_\alpha(\zeta_0)e_\alpha$. First of all, we remark that

$$\begin{aligned} v_2(\zeta_0) &= \sup_{a_1^2+a_2^2=1}\{a_1^2 v_1(\zeta_0) + a_2^2 v_2(\zeta_0)\} = \sup_{a_1^2+a_2^2=1}\langle a_1e_1 + a_2e_2, (\zeta_0^t\zeta_0)(a_1e_1+a_2e_2)\rangle \\ &= \sup_{|u|=1}\langle u, (\zeta_0^t\zeta_0)u\rangle = \sup_{|u|=1}|\zeta_0 u|^2. \end{aligned}$$

In the same manner as above, we prove that

$$\sup_{|u|=1}|\zeta u| = s_2(\zeta), \quad \forall\,\zeta\in M_{3\times 2}. \tag{2.5}$$

Let us show now that

$$\zeta e_2 = \lambda\zeta_0 e_2, \quad \text{where} \ \ \lambda\in\mathbb{R}. \tag{2.6}$$

Let $w\in\mathbb{R}^2$ be such that

$$|w| = 1 \ \text{ and } \ |(\zeta+\zeta_0)w| = \sup_{|u|=1}|(\zeta+\zeta_0)u|.$$

Then

$$s_2(\zeta+\zeta_0) = \sup_{|u|=1}|(\zeta+\zeta_0)u| \le |\zeta w| + |\zeta_0 w| \le \sup_{|u|=1}|\zeta u| + \sup_{|u|=1}|\zeta_0 u| = s_2(\zeta) + s_2(\zeta_0).$$

Since $\zeta\in E$, the above necessarily implies that

$$|\zeta w| + |\zeta_0 w| = \sup_{|u|=1}|\zeta u| + \sup_{|u|=1}|\zeta_0 u|.$$

Next, as

$$0 \ge |\zeta w| - \sup_{|u|=1}|\zeta u| = \sup_{|u|=1}|\zeta_0 u| - |\zeta_0 w| \le 0,$$

we draw

$$|\zeta w| = \sup_{|u|=1}|\zeta u| = s_2(\zeta) \ \text{ and } \ |\zeta_0 w| = \sup_{|u|=1}|\zeta_0 u| = s_2(\zeta_0).$$

Recalling that e_2 is the unique vector satisfying $|e_2| = 1$ and $\zeta_0 e_2 = s_2(\zeta_0)e_2$, we deduce that $w = e_2$. We can now write that

$$s_2(\zeta_0+\zeta) = |(\zeta+\zeta_0)e_2| \le |\zeta e_2| + |\zeta_0 e_2| = s_2(\zeta) + s_2(\zeta_0).$$

As $\zeta\in E$, we necessarily have

$$|(\zeta+\zeta_0)e_2| = |\zeta e_2| + |\zeta_0 e_2|,$$

and it follows that

$$\exists\,\lambda\in\mathbb{R}_+ \ \text{ such that } \ \zeta e_2 = \lambda\zeta_0 e_2.$$

We have so far shown that

$$E = \Big\{\zeta\in M_{32}:\ \sup_{|u|=1}|\zeta u| = |\zeta e_2| \text{ and } \exists\,\lambda\in\mathbb{R}_+:\ \zeta e_2 = \lambda\zeta_0 e_2\Big\}.$$

Step 2 First of all we remark that $\zeta_0 e_1$ and $\zeta_0 e_2$ are orthogonal since

$$\langle \zeta_0 e_1, \zeta_0 e_2 \rangle = \langle e_1, \zeta_0^t \zeta_0 e_2 \rangle = v_2(\zeta_0) \langle e_1, e_2 \rangle = 0.$$

Therefore we can set $f_\alpha = \dfrac{\zeta e_\alpha}{|\zeta e_\alpha|}$ and let $\{f_1, f_2, f_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Now recall that $|\zeta e_2| = s_2(\zeta)$ and $\zeta e_2 = \lambda \zeta_0 e_2$, so we can write

$$\zeta_0 = \begin{pmatrix} b_0 & 0 \\ 0 & s_2(\zeta_0) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} p & 0 \\ 0 & s_2(\zeta) \\ q & 0 \end{pmatrix},$$

where

$$b_0 = |\zeta_0 e_1|, \quad p = \langle \zeta e_1, f_1 \rangle \quad \text{and} \quad q = \langle \zeta e_1, f_3 \rangle.$$

Finally, $\zeta - \zeta_0 = \begin{pmatrix} p-b_0 & 0 \\ 0 & s_2(\zeta)-s_2(\zeta_0) \\ q & 0 \end{pmatrix}$ thereby choosing, for instance, $\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ completes the argument.

Remark 2.3 The result presented in this section was announced in [31]; see also [32].

3 Some Remarks on Nonlinear Plane Membranes

For completeness, in this section we compile some general remarks on nonlinear membrane models.

3.1 An upper bound for the minimization problem with non-vanishing external forces

In [2], Ball and Murat obtain a non-existence result similar to Theorem 1.2 albeit in the particular instance of a strictly positive work of external forces. Their method does not apply to the case of vanishing external loads, nor does the latter theorem apply in their instance. Moreover, neither method applies to the case of ad hoc body forces. Nevertheless, the method of Ball and Murat still gives an upper bound for the infimum of the total energy in the general case. More precisely, if we define $M_{\xi_0} = \xi_0 x + W_0^{1,4}(\omega; \mathbb{R}^3)$, we have

Proposition 3.1 *Let $f \in L^2(\omega; \mathbb{R}^3)$ and suppose $\partial\omega = 0$. Then*

$$\inf_{\bar{\varphi} \in M_{\xi_0}} J(\bar{\varphi}) \le \inf_{\bar{\varphi} \in M_{\xi_0}} I(\bar{\varphi}) - \int_\omega f \cdot \xi_0 x \, dx,$$

where

$$J(\bar{\varphi}) = I(\bar{\varphi}) - \int_\omega f \cdot \bar{\varphi} \, dx.$$

Proof Let $\varepsilon > 0$ and $\varphi = \xi_0 x + \psi \in M_{\xi_0}$ satisfy the following inequality

$$I(\varphi) \le \inf_{\bar{\varphi} \in M_{\xi_0}} I(\bar{\varphi}) + \varepsilon.$$

By the Vitali covering theorem, given $n \in \mathbb{N}$ there exists a finite or countable disjoint sequence $(a_p + \varepsilon_p \overline{\omega})_{p \in A_n} \subset \omega$ where $a_p \in \mathbb{R}^2$ and $0 < \varepsilon_p \leq \frac{1}{n}$ such that $\text{meas}\big(\omega \setminus \bigcup_{p \in A_n} (a_p + \varepsilon_p \overline{\omega})\big) = 0$. Since $\partial\omega = 0$, we also have that $\sum_{p \in A_n} \varepsilon_p^2 = 1$. Now define

$$\varphi_n(x) = \begin{cases} \xi_0 x + \varepsilon_p \psi\Big(\dfrac{x - a_p}{\varepsilon_p}\Big), & \text{if } \; x \in a_p + \varepsilon_p \overline{\omega}, \\ \xi_0 x, & \text{otherwise.} \end{cases}$$

Then $\varphi_n \in M_{\xi_0}$ verifies

$$J(\varphi_n) = \sum_{p \in A_n} \int_{a_p + \varepsilon_p \omega} W\Big(\xi_0 + \nabla\psi\Big(\frac{x - a_p}{\varepsilon_p}\Big)\Big) dx - \int_\omega f \cdot \varphi_n \, dx.$$

Next we perform in each of the integrals above the corresponding change of variable $y = \frac{x - a_p}{\varepsilon_p}$ to bring up

$$\begin{aligned} J(\varphi_n) &= \sum_{p \in A_n} \varepsilon_p^2 \int_\omega W(\xi_0 + \nabla\psi(y)) dy - \int_\omega f \cdot \xi_0 x \, dx - \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f \cdot \psi(y) \, dy, \\ &= I(\varphi) - \int_\omega f \cdot \xi_0 x \, dx - \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f(a_p + \varepsilon_p y) \cdot \psi(y) \, dy. \end{aligned}$$

Now recalling the properties of $(\varepsilon_p)_{p \in A_n}$, the following holds

$$\Big| \sum_{p \in A_n} \varepsilon_p^3 \int_{a_p + \varepsilon_p \omega} f(a_p + \varepsilon_p y) \cdot \psi(y) \, dy \Big| \leq \frac{1}{n} \sum_{p \in A_n} \varepsilon_p^2 |f|_{L^2(\omega;\mathbb{R}^3)} |\psi|_{L^2(\omega;\mathbb{R}^3)}$$

Hence, by Lebesgue's dominated convergence theorem we deduce that

$$\inf_{\bar{\varphi} \in M_{\xi_0}} J(\bar{\varphi}) \leq J(\varphi_n) \leq I(\varphi) - \int_\omega f \cdot \xi_0 x \, dx \leq \inf_{\bar{\varphi} \in M_{\xi_0}} I(\bar{\varphi}) - \int_\omega f \cdot \xi_0 x \, dx + \varepsilon$$

for all $\varepsilon > 0$, which yields the aforementioned bound.

3.2 Membranes do not resist compression on the boundary

This fact was observed by Le Dret and Raoult [22] who show that the stored energy function of the membrane model they obtain by Γ-convergence arguments from finite nonlinear elasticity vanishes for deformations whose singular values are less than one, which correspond to compressive states. Here we show that any sensible membrane stored energy function verifies the latter property. First, let us recall that for all matrix $\xi \in \mathbb{R}^{32}$, $0 \leq s_1(\xi) \leq s_2(\xi)$ are its singular values, that is, the eigenvalues of the matrix $(\xi^t \xi)^{\frac{1}{2}}$. We also recall that a function $F : \mathbb{R}^{n \times N} \longrightarrow \mathbb{R}$ is rank-one-convex if

$$F(\lambda A + (1 - \lambda) B) \leq \lambda F(A) + (1 - \lambda) F(B)$$

for all $\lambda \in [0, 1]$ and all $A, B \in \mathbb{R}^{n \times m}$ such that $\text{rank}(A - B) \leq 1$.

Proposition 3.2 *Let $W : \mathbb{R}^{3\times 2} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ be a rank-one-convex isotropic and frame-indifferent function, i.e.,*

$$W(\xi) = W(\rho\xi), \quad \forall\, \xi \in \mathbb{R}^{3\times 2},\ \forall\, \rho \in \mathrm{SO}(3).$$

Suppose furthermore that $W(\eta) = 0$ where $\eta = (\delta^i_j)_{i\le 3, j\le 2}$. Then

$$W(\xi) = 0, \quad \forall\, \xi \in \mathbb{R}^{3\times 2} \ \ \textit{such that} \ \ s_2(\xi) \in [0,1].$$

Proof We first show that

$$W(\xi) = 0, \quad \forall\, \xi \in \mathbb{R}^{3\times 2} \ \ \text{such that} \ \ s_1(\xi)s_2(\xi) = 1. \tag{3.1}$$

Indeed, as pointed out by Le Dret and Raoult [22], any matrix $\xi \in \mathbb{R}^{3\times 2}$ admits the following *polar factorization*

$$\xi = \rho\eta(\xi^t\xi)^{\frac{1}{2}}, \tag{3.2}$$

where $\rho \in \mathrm{SO}(3)$. Next, let $\zeta \in \mathrm{SO}(2)$ be such that

$$\xi^t\xi = \zeta^t \mathrm{diag}\,(s^2_\alpha(\xi))\zeta.$$

Then identity (3.2) gives

$$\xi\zeta^t = \rho\eta(\zeta(\xi^t\xi)\zeta^t)^{\frac{1}{2}} = \rho\eta\,\mathrm{diag}\,(s_\alpha(\xi)).$$

Hence isotropy and frame-indifference of the membrane stored energy function W imply

$$W(\xi) = W(\xi\zeta^t) = W(\rho\eta\,\mathrm{diag}\,(s_\alpha(\xi))) = W(\eta\,\mathrm{diag}\,(s_\alpha(\xi))),$$

so that if we suppose furthermore that $s_1(\xi)s_2(\xi) = 1$ then $\mathrm{diag}\,(s_\alpha(\xi)) \in \mathrm{SO}(2)$. This together with isotropy brings about

$$W(\xi) = W(\eta\,\mathrm{diag}\,(s_\alpha(\xi))) = W(\eta) = 0$$

by definition of the function W, and property (3.1) is proved.

Now we set out to prove the announced result. For simplicity, let $\xi \in \mathbb{R}^{3\times 2}$ be such that $s_2(\xi) \le 1$ and $\xi = \mathrm{diag}\,(s_\alpha(\xi))$. Let $r_\alpha \in [0,1]$ be such that $s_\alpha(\xi) = -r_\alpha + (1-r_\alpha)$ and note $\zeta^\alpha_\beta = ((-1)^\alpha e_1|(-1)^\beta e_2) \in \mathbb{R}^{3\times 2}$. Accordingly writing ξ in this fashion

$$\xi = r_1[r_2\zeta^1_1 + (1-r_2)\zeta^1_2] + (1-r_1)[r_2\zeta^2_1 + (1-r_2)\zeta^2_2]$$

yields

$$W(\xi) \le r_1 W(r_2\zeta^1_1 + (1-r_2)\zeta^1_2) + (1-r_1)W(r_2\zeta^2_1 + (1-r_2)\zeta^2_2),$$

using the rank-one-convexity of W. Again the rank-one-convexity of W raises

$$W(\xi) \le r_1r_2W(\zeta^1_1) + r_1(1-r_2)W(\zeta^1_2) + (1-r_1)r_2W(\zeta^2_1) + (1-r_1)(1-r_2)W(\zeta^2_2).$$

Lastly, since $s_1(\zeta^\alpha_\beta)s_2(\zeta^\alpha_\beta) = 1$, property (3.1) entails that $W(\zeta^\alpha_\beta) = 0$ and the result is fully justified.

Remark 3.1 (i) Note that in the particular case of a finite membrane energy W, a corollory of the above is the result obtained by Le Dret and Raoult [21, 22] who show that quasiconvex finite energy membranes behave likewise. Indeed, in the finite case quasiconvexity implies rank-one-convexity and accordingly $\mathbf{QR}W = \mathbf{Q}W$. Thus, to conclude it suffices to mention that 0 is quasiconvex. This argument is false in the general case since rank-one-convexity does not imply quasiconvexity; we send the reader to Ball and Murat [2] for a counterexample when the stored energy function is not continuous. What is more, the issue of weak lower semicontinuity related to vectorial convexity in the general case is far from being fully understood (see [3] for a partial answer).

(ii) As was already remarked in [22], the observed phenomenon is a consequence of the reference configuration being a natural state. In fact, if the reference configuration was an extended state the membrane would tend to shrink back to a natural position and compressive deformations would accordingly be expected.

In light of the above observation, we deduce that compressive states do not realize a finite minimum of the membrane energy if an external load is applied whether the stored energy function is rank-one-convex or even polyconvex since polyconvexity implies rank-one-convexity. This phenomenon agrees with Tartar [30] who showed that for such a functional to be $W^{1,\infty}$ sequentially weakly lower semicontinuous, it has to be rank-one-convex.

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