Chin. Ann. Math.
27B(4), 2006, 393–400
DOI: 10.1007/s11401-005-0187-8

Chinese Annals of Mathematics, Series B

Uniqueness Theorem of Meromorphic Mappings with Moving Targets***

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Abstract Some uniqueness theorems of meromorphic mappings with moving targets are given under the inclusion relations between the zeros sets of meromorphic mappings.

Keywords Uniqueness theorem, Meromorphic mapping, Moving hyperplane
2000 MR Subject Classification 32H30, 32A22

1 Introduction

In 1926, R. Nevanlinna [1] showed that, for two distinct nonconstant meromorphic functions f and g on the complex plane C, they can not have the same inverse images for five distinct values.

Over the last few decades, there have been several generalizations of Nevanlinna's result to the case of meromorphic mappings of C^n into the complex projective space $P^N(C)$.

Recently, motivated by the accomplishment of the second main theorem of value distribution theory for moving targets, the uniqueness problem of meromorphic mappings of C^n into $P^N(C)$ started to be discussed.

Firstly, we must introduce some notions.

For $z = (z_1, \ldots, z_n) \in C^n$ we set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. For $r > 0$, define

$$\begin{aligned}
&B(r) = \{z \in C^n \mid \|z\| < r\},\\
&S(r) = \{z \in C^n \mid \|z\| = r\},\\
&d^c = (4\pi\sqrt{-1})^{-1}(\partial - \bar{\partial}),\\
&v = (dd^c\|z\|^2)^{n-1},\\
&\sigma = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}.
\end{aligned}$$

Let $f : C^n \to P^N(C)$ be a meromorphic mapping. We can choose holomorphic functions $f_0, \cdots, f_N$ on C^n such that $I_f := \{z \in C^n \mid f_0(z) = \cdots = f_N(z) = 0\}$ is of dimension at most $n-2$, and $f = (f_0(z) : \cdots : f_N(z))$ is called a reduced representation of f. The characteristic

Manuscript received May 17, 2005.
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***Project supported by the National Natural Science Foundation of China (No.10571135, No.10511140543), the Doctoral Program Foundation of the Ministry of Education of China (No.20050240711) and the Foundation of Committee of Science and Technology of Shanghai (No.03JC14027).

function of f is defined by

$$T(r,f)=\int_{S(r)}\log\|f\|\sigma-\int_{S(1)}\log\|f\|\sigma,\quad r>1.$$

Note that $T(r,f)$ is independent of the choice of the reduced representation of f.

A moving hyperplane assigns, to every $z\in C^n$, a hyperplane given by

$$H(z)=\Big\{(x_0:\cdots:x_N)\in P^N(C)\Big|\sum_{i=0}^{N}b_i(z)x_i=0\Big\},$$

where $b_i,\ 0\le i\le N$, are entire functions whose common zeros set is of dimension at most $n-2$. A moving hyperplane H gives a meromorphic mapping $a=(b_0:\cdots:b_N):C^n\to P^N(C)$. We define $T_H(r)=T(r,a)$. We say that a is "small" with respect to the meromorphic mapping f if $T(r,a)=o(T(r,f))$ as $r\to+\infty$.

We say that moving hyperplanes $\{H_1,\cdots,H_q\}$ (or $\{a_1,\cdots,a_q\}$, where $a_j=(a_{j0}:\cdots:a_{jN})$) are in general position if $\{H_1(z),\cdots,H_q(z)\}$ are in general position for some (and hence for almost all) $z\in C^n$.

Let $\mathcal{M}$ be the field of all meromorphic functions on C^n. Denote by $\mathcal{R}(\{a_j\}_{j=1}^q)\subset\mathcal{M}$ the smallest subfield which contains C and all a_{jk}/a_{jl} with $a_{jl}\not\equiv0$, where $1\le j\le q,\ 0\le k,l\le N$.

Let $f:C^n\to P^N(C)$ be a meromorphic mapping, and $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings of C^n into $P^N(C)$ in general position such that

$$\dim\{z\in C^n\mid (f,a_i)(z)=(f,a_j)(z)=0\}\le n-2,\quad 1\le i<j\le q.$$

Assume that f is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$.

Consider the set $\mathcal{F}(f,\{a_j\}_{j=1}^q,d)$ of all nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$ meromorphic mappings $g:C^n\to P^N(C)$ satisfying the conditions:

(a) $\min(\nu_{(f,a_j)},d)=\min(\nu_{(g,a_j)},d),\quad 1\le j\le q,$

(b) $f(z)=g(z)$ on $\bigcup_{j=1}^{q}\{z\mid (f,a_j)(z)=0\}$.

In [2], Chen and Ru proved the following:

Theorem A *If* $q\ge 2N^2+4N$, *then* $\sharp\mathcal{F}(f,\{a_j\}_{j=1}^q,2)\le2$.

And Thai [3] showed that

Theorem B *If* $q=2N^2+4N$ *and* $N\ge2$, *then* $\sharp\mathcal{F}(f,\{a_j\}_{j=1}^q,1)=1$.

And under the assumption that these meromorphic mappings are nonconstant, Chen and Li [4] proved that

Theorem C *If* $q=4N^2+2N$ *and* $N\ge2$, *then* $\sharp\mathcal{F}(f,\{a_j\}_{j=1}^q,1)=1$.

In this paper, we will prove some generalized uniqueness theorems with truncated multiplicities for moving targets under the inclusion relations between the zeros sets of meromorphic mappings. This is the first time to discuss the uniqueness problem of meromorphic mappings for moving targets under this assumption.

Let $f(z)$ be a meromorphic mapping and $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings in general position. We use $\overline{E}(a_j,f)$ to denote the zero set of $(f(z),a_j(z))$, in which each zero is counted only once.

Let S be a subvariety in C^n with $\dim S \leq n-2$. Denote by $T[N+1,q]$ the set of all injective maps from $\{1,\cdots,N+1\}$ to $\{1,\cdots,q\}$. For every

$$z \in C^n \Big\backslash \Big(\bigcup_{\beta \in T[N+1,q]} \{z \mid a_{\beta(1)}(z) \wedge \cdots \wedge a_{\beta(N+1)}(z) = 0\} \bigcup S \Big),$$

we define

$$\rho_f(z) = \sharp\{j \mid z \in \overline{E}(a_j, f)\}.$$

Since a_j $(j=1,\cdots,q)$ are located in general position, $\rho_f(z) \leq N$. For every positive number r, define

$$\rho_f(r) = \sup\{\rho_f(z) \mid \|z\| \geq r\},$$

where the sup is taken over all

$$z \in C^n \Big\backslash \Big(\bigcup_{\beta \in T[N+1,q]} \{z \mid a_{\beta(1)}(z) \wedge \cdots \wedge a_{\beta(N+1)}(z) = 0\} \bigcup S \Big) \quad \text{with } \|z\| \geq r.$$

Then $\rho_f(r)$ is a decreasing function. Let

$$d_{f,S} = \lim_{r \to +\infty} \rho_f(r).$$

Then

$$1 \leq d_{f,S} \leq N.$$

There exists a number $r_0 > 1$ such that

$$\rho_f(r) = d_{f,S} \leq N, \quad \text{as } r \geq r_0.$$

Let

$$d_f = \inf\{d_{f,S} \mid \dim S \leq n-2\},$$

where the inf is taken over all subvarieties $S \subset C^n$ with $\dim S \leq n-2$.

If, for each $i \neq j$,

$$\dim \overline{E}(a_i, f) \cap \overline{E}(a_j, f) \leq n-2,$$

then $d_f = 1$.

Our main results are stated as follows:

Theorem 1.1 *Let $f(z)$ and $g(z)$ be two meromorphic mappings, and let $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings of C^n into $P^N(C)$ in general position such that $(f,a_j) \not\equiv 0$ and $(g,a_j) \not\equiv 0$ $(1 \leq j \leq q)$. Assume that*

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g), \quad 1 \leq j \leq q.$$

And $f = g$ on $\bigcup_{j=1}^{q} \{z \mid (f,a_j)(z) = 0\}$. If $q = 2d_fN(2N+1)+1$, and

$$\liminf_{r \to +\infty} \sum_{j=1}^{2d_fN(2N+1)+1} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{2d_fN(2N+1)+1} N^1_{(g,a_j)}(r) > \frac{d_fN(2N+1)}{d_fN(2N+1)+1},$$

then

$$f(z) \equiv g(z).$$

Furthermore, if, for each $i \neq j$,

$$\dim \overline{E}(a_i, f) \cap \overline{E}(a_j, f) \leq n-2, \quad q = 4N^2 + 2N + 1$$

and

$$\liminf_{r \to +\infty} \sum_{j=1}^{4N^2+2N+1} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{4N^2+2N+1} N^1_{(g,a_j)}(r) > \frac{N(2N+1)}{N(2N+1)+1},$$

then

$$f(z) \equiv g(z).$$

Theorem 1.2 *Let* $f(z)$ *and* $g(z)$ *be two meromorphic mappings, and let* $\{a_j\}_{j=1}^q$ *be "small" (with respect to* f*) meromorphic mappings of* C^n *into* $P^N(C)$ *in general position such that* $(f, a_j) \not\equiv 0$ *and* $(g, a_j) \not\equiv 0$ $(1 \leq j \leq q)$*. And* f, g *are linearly nondegenerate over* $\mathcal{R}(\{a_j\}_{j=1}^q)$*. Assume that*

$$\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g), \quad 1 \leq j \leq q.$$

And $f = g$ *on* $\bigcup_{j=1}^{q} \{z \mid (f, a_j)(z) = 0\}$*. If* $q = 2d_f N(N+2) + 1$*, and*

$$\liminf_{r \to +\infty} \sum_{j=1}^{2d_f N(N+2)+1} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{2d_f N(N+2)+1} N^1_{(g,a_j)}(r) > \frac{d_f N(N+2)}{d_f N(N+2)+1},$$

then

$$f(z) \equiv g(z).$$

Furthermore, if, for each $i \neq j$,

$$\dim \overline{E}(a_i, f) \cap \overline{E}(a_j, f) \leq n-2, \quad q = 2N^2 + 4N + 1$$

and

$$\liminf_{r \to +\infty} \sum_{j=1}^{2N^2+4N+1} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{2N^2+4N+1} N^1_{(g,a_j)}(r) > \frac{N(N+2)}{N(N+2)+1},$$

then

$$f(z) \equiv g(z).$$

2 Preliminaries and Some Lemmas

We first introduce some preliminaries in Nevanlinna theory.

Let $F(z)$ be a nonzero entire function on C^n. For $a \in C^n$, set

$$F(z) = \sum_{m=0}^{\infty} P_m(z-a),$$

where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m. The number $\nu_F(a) := \min\{m \mid P_m \neq 0\}$ is said to be the zero-multiplicity of F at a and $|\nu_F| = \{z \in C^n \mid \nu_F(z) \neq 0\}$ is the support of ν_F.

We now define counting function. For a moving hyperplane H (or a), we define

$$\nu^M_{(f,a)}(z) = \min\{M, \nu_{(f,a)}(z)\}$$

for positive integer M or $M = \infty$. And

$$n(t) = \begin{cases} \displaystyle\int_{|\nu_{(f,a)}| \bigcap B(t)} \nu_{(f,a)}(z)v, & \text{if } n \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu_{(f,a)}(z), & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$. Define

$$N_{(f,a)}(r) = \int_1^r \frac{n(t)}{t^{2n-1}} dt, \quad 1 < r < +\infty.$$

Similarly, we define $N^M_{(f,a)}(r)$.

We define the proximity function of a by

$$m_{f,a}(r) = \int_{S(r)} \log \frac{\|f\|\|a\|}{|(f,a)|} \sigma - \int_{S(1)} \log \frac{\|f\|\|a\|}{|(f,a)|} \sigma, \quad r > 1.$$

Now we state the first and second main theorems of meromorphic mapping, that will be used in the proof of our theorems.

The first main theorem:

$$T(r,f) = m_{f,a}(r) + N_{(f,a)}(r) + T(r,a).$$

Some second main theorems are stated as follows:

Theorem D (See [5]) *Let $f : C^n \to P^N(C)$ be a meromorphic mapping. Let $\{a_j\}_{j=1}^q$ be meromorphic mappings of C^n into $P^N(C)$ in general position such that f is linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$. Then*

$$\| \frac{q}{N+2} T(r,f) \leq \sum_{j=1}^q N^N_{(f,a_j)}(r) + o(T(r,f)) + O\Big(\max_{1 \leq j \leq q} T(r,a_j)\Big),$$

where "$\|$" means the estimate holds for all large r outside a set of finite Lebesgue measure.

Theorem E (See [6]) *Let $f : C^n \to P^N(C)$ be a meromorphic mapping. Let $\{a_j\}_{j=1}^q$ ($q \geq 2N+1$) be meromorphic mappings of C^n into $P^N(C)$ in general position. Then*

$$\| \frac{q}{2N+1} T(r,f) \leq \sum_{j=1}^q N^N_{(f,a_j)}(r) + o(T(r,f)) + O\Big(\max_{1 \leq j \leq q} T(r,a_j)\Big).$$

Lemma 2.1 *Under the assumptions in Theorems* 1.1 *and* 1.2*, we obtain that $\{a_j\}_{j=1}^q$ are "small" with respect to g.*

Proof If f, g are linearly nondegenerate over $\mathcal{R}(\{a_j\}_{j=1}^q)$, then, by Theorem D, we have

$$\begin{aligned}\|\frac{q}{N+2}T(r,f) &\le \sum_{j=1}^{q} N_{(f,a_j)}^{N}(r) + o(T(r,f)) \le \sum_{j=1}^{q} N_{(f,a_j)}^{1}(r) + o(T(r,f)) \\ &\le N\sum_{j=1}^{q} N_{(g,a_j)}^{1}(r) + o(T(r,f)) \le qNT(r,g) + o(T(r,f)).\end{aligned}$$

If f, g are nonconstant, by Theorem E, we have

$$\begin{aligned}\|\frac{q}{2N+1}T(r,f) &\le \sum_{j=1}^{q} N_{(f,a_j)}^{N}(r) + o(T(r,f)) \le N\sum_{j=1}^{q} N_{(g,a_j)}^{1}(r) + o(T(r,f)) \\ &\le qNT(r,g) + o(T(r,f)).\end{aligned}$$

Since $\{a_j\}_{j=1}^q$ are "small" with respect to f, it is easy to see that $\{a_j\}_{j=1}^q$ are also "small" with respect to g.

3 Proof of Main Results

Proof of Theorem 1.1 For f, g, we set $T(r) = T(r,f) + T(r,g)$. Assume $f(z) \not\equiv g(z)$. Denote by $N_{\mu_{f\wedge g}}(r)$ the counting function associated with the divisor $\mu_{f\wedge g}$.

Let $z \in \bigcup_{j=1}^{q} \overline{E}(a_j, f)$. We verify that $f \wedge g$ vanishes at z. In fact, we can write

$$f(\zeta) = \alpha_{\mathbf{1}} + \sum_{i=1}^{n} (\zeta_i - z_i)\mathbf{h}_{\mathbf{1}}^{i}(\zeta), \quad g(\zeta) = \alpha_{\mathbf{2}} + \sum_{i=1}^{n} (\zeta_i - z_i)\mathbf{h}_{\mathbf{2}}^{i}(\zeta),$$

where α_i is a constant vector, and $\mathbf{h}_{\mathbf{1}}^{i}, \mathbf{h}_{\mathbf{2}}^{i}$ are holomorphic vector-valued functions defined around z.

Since $f = g$ on $\bigcup_{j=1}^{q} \overline{E}(a_j, f)$, we have $\alpha_{\mathbf{1}} \wedge \alpha_{\mathbf{2}} = 0$. Hence

$$f \wedge g = \sum_{i=1}^{n} (\alpha_{\mathbf{1}} \wedge \mathbf{h}_{\mathbf{2}}^{i} - \alpha_{\mathbf{2}} \wedge \mathbf{h}_{\mathbf{1}}^{i})(\zeta_i - z_i) + \sum_{i,j=1}^{n} \mathbf{h}_{\mathbf{1}}^{i} \wedge \mathbf{h}_{\mathbf{2}}^{j}(\zeta_i - z_i)(\zeta_j - z_j).$$

So z is a zero of $f \wedge g$.

By the definition of d_f, when $\|z\|$ is large enough, it is easy to see that

$$\sum_{j=1}^{q} N_{(f,a_j)}^{1}(r) \le d_f N_{\mu_{f\wedge g}}(r) + \sum_{\beta} N_{\mu_{a_{\beta(1)}\wedge\cdots\wedge a_{\beta(N+1)}}}(r),$$

where the sum is over all injective maps $\beta : \{1, \cdots, N+1\} \to \{1, \cdots, q\}$.

By the First Main Theorem of exterior product (cf. [7, p.327] and [8]),

$$\begin{aligned}&N_{\mu_{f\wedge g}}(r) \le T(r,f) + T(r,g) + O(1), \\ &N_{\mu_{a_{\beta(1)}\wedge\cdots\wedge a_{\beta(N+1)}}}(r) \le \sum_{j=1}^{q} T(r, a_j) + O(1).\end{aligned}$$

Hence,

$$\sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le d_f T(r,f) + d_f T(r,g) + o(T(r)).$$

By Theorem E, we have

$$\| \frac{q}{N(2N+1)} T(r,f) \le \sum_{j=1}^{q} N^1_{(f,a_j)}(r) + o(T(r,f)),$$

$$\| \frac{q}{N(2N+1)} T(r,g) \le \sum_{j=1}^{q} N^1_{(g,a_j)}(r) + o(T(r,g)).$$

Hence

$$\| \sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le \frac{d_f N(2N+1)}{q} \sum_{j=1}^{q} N^1_{(f,a_j)}(r) + \frac{d_f N(2N+1)}{q} \sum_{j=1}^{q} N^1_{(g,a_j)}(r) + o(T(r)).$$

We have

$$\| \frac{q - d_f N(2N+1)}{q} \sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le \frac{d_f N(2N+1)}{q} \sum_{j=1}^{q} N^1_{(g,a_j)}(r) + o(T(r)).$$

It follows that

$$\liminf_{r\to+\infty} \sum_{j=1}^{q} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{q} N^1_{(g,a_j)}(r) \le \frac{d_f N(2N+1)}{q - d_f N(2N+1)}.$$

For $q = 2d_f N(2N+1) + 1$, we get

$$\liminf_{r\to+\infty} \sum_{j=1}^{2d_f N(2N+1)+1} N^1_{(f,a_j)}(r) \Big/ \sum_{j=1}^{2d_f N(2N+1)+1} N^1_{(g,a_j)}(r) \le \frac{d_f N(2N+1)}{d_f N(2N+1)+1},$$

which contradicts our assumption, and hence $f(z) \equiv g(z)$.

If, for each $i \ne j$,

$$\dim \overline{E}(a_i,f) \cap \overline{E}(a_j,f) \le n-2,$$

then $d_f = 1$.

Proof of Theorem 1.2 Under the argument in the proof of Theorem 1.1, we get

$$\sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le d_f T(r,f) + d_f T(r,g) + o(T(r)).$$

Using Theorem D, we have

$$\| \sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le \frac{d_f N(N+2)}{q} \sum_{j=1}^{q} N^1_{(f,a_j)}(r) + \frac{d_f N(N+2)}{q} \sum_{j=1}^{q} N^1_{(g,a_j)}(r) + o(T(r)).$$

This means that

$$\| \frac{q - d_f N(N+2)}{q} \sum_{j=1}^{q} N^1_{(f,a_j)}(r) \le \frac{d_f N(N+2)}{q} \sum_{j=1}^{q} N^1_{(g,a_j)}(r) + o(T(r)).$$

It follows that

$$\liminf_{r\to+\infty}\sum_{j=1}^{q}N^1_{(f,a_j)}(r)\Big/\sum_{j=1}^{q}N^1_{(g,a_j)}(r)\le\frac{d_fN(N+2)}{q-d_fN(N+2)}.$$

For $q=2d_fN(N+2)+1$, we get

$$\liminf_{r\to+\infty}\sum_{j=1}^{2d_fN(N+2)+1}N^1_{(f,a_j)}(r)\Big/\sum_{j=1}^{2d_fN(N+2)+1}N^1_{(g,a_j)}(r)\le\frac{d_fN(N+2)}{d_fN(N+2)+1},$$

which contradicts our assumption, and hence $f(z)\equiv g(z)$.

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