Chin. Ann. Math.
27B(4), 2006, 401–410
DOI: 10.1007/s11401-005-0101-4

Chinese Annals of Mathematics, Series B

Periodic Solutions of Schrödinger Flow from S^3 to S^2 **

Hao YIN*

Abstract This paper deals with the periodic solutions of Schrödinger flow from S^3 to S^2. It is shown that the periodic solution is related to the variation of some functional and there exist S^1-invariant critical points to this functional. The proof makes use of the Principle of Symmetric Criticality of Palais.

Keywords Schrödinger flow, Periodic solution, Variational method
2000 MR Subject Classification 35Q55, 58E20

1 Introduction

If (M, g) is a Riemannian manifold and (N, h, J) is a Kähler manifold with a complex structure J, $u : M \times [0, +\infty) \to N$ is said to satisfy the Schrödinger flow equation if

$$\frac{\partial}{\partial t} u = J\tau(u), \tag{1.1}$$

where $\tau(u)$ is the tension field of u as defined in the theory of harmonic maps. It can be regarded as the gradient of the energy functional

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dV_M = \frac{1}{2} \int_M g^{\alpha\beta} h_{jk} \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} dV_M.$$

In local coordinates, the tension field $\tau(u)$ can be written as

$$\tau(u)^i = \triangle_M u^i - g^{\alpha\beta} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta},$$

where Γ^i_{jk} is the Christoffel symbol of the Riemannian connection of N.

The Schrödinger flow can be considered as a generalization of the Schrödinger equation. The target changes from $\mathbb{C}$ to a Kähler manifold. This change causes nonlinearity and makes the study of Schrödinger flow very difficult. Ding and Wang have proved that the flow has a local solution provided the initial data is sufficiently smooth (see [1]). See also [2] for a survey on the subject.

In suitable sense, the Schrödinger flow is the Hamilton flow of the energy functional. From this point of view, it is natural to study the periodic solutions of the flow. The purpose of this article is to show the existence of periodic solutions in some special cases.

Manuscript received March 23, 2005.
*Department of Mathematics, East China Normal University, Shanghai 200062, China.
E-mail: hyin@math.ecnu.edu.cn
**Project supported by the National Natural Science Foundation of China (No.10321001).

Assume M to be some general compact Riemannian manifold and N a Kähler manifold admitting a holomorphic Killing field V. Then the one parameter transformation group S_t generated by V is both holomorphic and isometric. We ask the following question: Is there an initial map $u : M \to N$ such that the Schrödinger flow started from u will be $S_t \circ u$? In particular, if S_t is periodic, we get a periodic solution to the Schrödinger flow. It will be shown in the Reduction Lemma in Section 2 that such an initial map u is the solution of an elliptic system, and if (N, h) is a Kähler-Einstein metric with positive scalar curvature, u is the critical point of some functional very similar to the energy functional. Our previous paper (see [3]) deals with the variation problem when M is a surface. We used the blow up analysis of Sacks and Uhlenbeck and a delicate estimate to show an existence result for any period λ. As in the study of harmonic maps, when $\dim M = 3$, the direct variation method is not likely to be useful. In the case of $M = S^3$ and $N = S^2$, we consider a family of S^1 action $T_{k,l}$ on M where k and l are coprime positive integers. We use the Principle of Symmetric Criticality as formulated by Palais in [4] to reduce the dimension by 1 and the problem then becomes a variation problem of a perturbed weighted energy on the space of mappings from the orbit space Q to S^2. If $k = l = 1$, this Q is just S^2 and the weighted energy is (up to a constant) the energy. So the result in [3] implies

Theorem 1.1 *For any $\lambda > 0$, the Schrödinger flow from S^3 to S^2 has infinitely many inequivalent periodic solutions with period $\frac{2\pi}{\lambda}$. Moreover, these solutions are $T_{1,1}$-invariant.*

If $k \neq l$, we use the rotational invariance of Q to further reduce the problem to a one-dimensional problem. Therefore, we get compactness, however, at the cost of some singularities in the expression of weighted energy functional. Finally, we use the method of Lagrange multiplier to show the existence of critical points for some Lagrange multiplier λ. We have

Theorem 1.2 *For any coprime positve integer $k \neq l$, there exists at least one $T_{k,l}$-invariant map f from S^3 to S^2 such that Schrödinger flow starting from f is periodic.*

In Section 2, we recall the Reduction Lemma in [3] to show how the periodic solutions of the Schrödinger flow are related to an elliptic variation problem. In Section 3, we state the Principle of Symmetric Criticality and use it to reduce the dimension by 1. In Section 4, we see that Theorem 1.1 follows from [3]. In the last section, we prove Theorem 1.2.

2 Reduction Lemma

For the purpose of this section, it is enough to assume that M is any Riemannian manifold and N is a Kähler manifold that admits a holomorphic Killing field V. We recall the Reduction Lemma in [3].

Lemma 2.1 (Reduction Lemma) *If V is not identically zero and S_t is the one parameter transformation group generated by V, then for $f : M \to N$, $u(t) = S_t \circ f$ is a solution of* (1.1) *if and only if*

$$\tau(f) = -JV(f). \tag{2.1}$$

Since V is holomorphic Killing field, S_t is both holomorphic and isometric. The lemma follows from the fact that $\tau(S_t \circ f) = dS_t[\tau(f)]$ and $dS_t \circ J = J \circ dS_t$. For the details of the proof we refer to [3].

Remark 2.1 If N is a closed Kähler-Einstein manifold with positive scalar curvature, it is known (see [5]) that JV is the gradient vector field of a first eigenfunction F. In this case, if we denote ∇F as the gradient vector field of F, equation (2.1) becomes

$$\tau(f) = -\nabla F(f), \tag{2.2}$$

which is the Euler-Lagrange equation for the functional

$$J(f) = E(f) - \int_M F(f) dV_g.$$

In the case $N = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, everything is clear. We can choose F to be λx_3 for some $\lambda > 0$. Now

$$\nabla F(z) = \lambda P(z) e_3,$$

where $e_3 = (0, 0, 1)$ and $P(z) : \mathbb{R}^3 \to T_z S^2$ is the orthogonal projection. Since $V = -J\nabla F$, it is not difficult to see the period of S_t generated by V is $\frac{2\pi}{\lambda}$. Recall that in the study of harmonic maps when $N = S^2$ we have

$$\tau(f) = \triangle_g f + |\nabla f|_g^2 f.$$

Therefore the equation that we want to solve, i.e. the Euler-Lagrange equation of $J(f)$, is

$$\triangle_g f + |\nabla f|_g^2 f = -\lambda P(f) e_3. \tag{2.3}$$

3 Reduction to Two-Dimensional Problem

Before we start further reduction, let us recall the Principle of Symmetric Criticality of Palais (see [4]): Critical symmetric point is symmetric critical point. Precisely, suppose that X is some smooth manifold (possibly of infinite dimensions), f is some smooth function and a group G acts on X such that f is invariant under the action. Let Σ be the set of fixed points of G. The principle claims that if $p \in \Sigma$ is a critical point of $f|_\Sigma$ then it is a critical point of f on X. The principle is intuitively plausible. However, as pointed out in [4], this is only true under certain natural conditions. One of the conditions is that G is compact, which is our case. So the Principle of Symmetric Criticality ensures that the following reductions will lead us to the critical points of the original variational problem. We are going to use the principle twice in slightly different settings, but we will not mention it again.

We consider an S^1 action on S^3. (Wang [6] did the same reduction and his result will be useful.) Let $S^3 = \{(w, z) \in \mathbb{C} \times \mathbb{C} \mid |z|^2 + |w|^2 = 1\}$ be the unit 3-sphere. Given two coprime integers $k \geq l \geq 1$. Define isometric action of S^1 on S^3

$$T_{k,l} : S^1 \to \mathrm{Iso}(S^3)$$

by

$$T_{k,l}^\theta(w, z) = (e^{ik\theta} w, e^{il\theta} z). \tag{3.1}$$

A continuous map $f : S^3 \to S^2$ is S^1-invariant under the action $T_{k,l}$ if for any $\theta \in S^1$ and $(z, w) \in S^3$,

$$f(T_{k,l}^\theta(w, z)) = f(w, z).$$

To study the S^1-invariant maps, we need to know the space of orbits. Take coordinates on S^3, (η, α, β) which means

$$(\sin\eta\cos\alpha, \sin\eta\sin\alpha, \cos\eta\cos\beta, \cos\eta\sin\beta),$$

where $\eta \in [0, \frac{\pi}{2}]$ and $\alpha, \beta \in [0, 2\pi)$. Consider the equator of S^3 given by $\sin\alpha = 0$, i.e. $(*, 0, *, *)$ points. When we say S^2 in the following, we mean this equator. Obviously, there are two different types of orbits,

(1) Two special orbits.

The one given by $\eta = 0$ lies in S^2 as equator.

The one given by $\eta = \frac{\pi}{2}$ intersects S^2 at its poles.

(2) Generic orbits.

Any other orbit intersects each hemisphere of S^2 k times.

To see this, for a point $(\eta_0, \alpha_0, \beta_0)$ with $\eta_0 \in (0, \frac{\pi}{2})$. The orbit is given by

$$(\eta_0, \alpha_0 + k\theta, \beta_0 + l\theta),$$

where θ runs from 0 to 2π. The number of intersections with S^2 is the number of solutions

$$\sin(\alpha_0 + k\theta) = 0$$

for $\theta \in [0, 2\pi)$. There are $2k$ such θ's. We are interested in the intersection with upper hemisphere (determined by $\cos\alpha = 1$ and $\sin\alpha = 0$)

$$\alpha_0 + k\theta = 2t\pi.$$

There are k choices of integer t such that

$$\theta_t = \frac{2t\pi - \alpha_0}{k} \in [0, 2\pi)$$

which corresponds to

$$\begin{aligned} &\alpha_t = 0, \\ &\beta_t = \beta_0 + l\frac{2t\pi - \alpha_0}{k} = \beta_0 - \frac{l\alpha_0}{k} + \frac{l}{k}2t\pi. \end{aligned}$$

If we parameterize the upper hemisphere of S^2 by

$$(\sin\eta, 0, \cos\eta\cos\beta, \cos\eta\sin\beta).$$

The above calculation suggests that the space of generic orbits is the finite quotient of the upper hemisphere by the relation

$$(\eta, \beta) \sim \left(\eta, \beta + \frac{2\pi}{k}\right)$$

(Since k and l are coprime, $+\frac{2l\pi}{k}$ is the same as $+\frac{2\pi}{k}$).

The entire orbit space is the two-point compactification of this quotient with a singular north pole and a point representing the equator.

Next, we study the metric structure of the orbit space. Precisely, we find a metric (defined on the smooth part of the orbit space) such that the natural projection map is a Riemannian submersion.

Given a point $p = (\eta_0, 0, \beta_0)$ with $\eta_0 \neq 0, \frac{\pi}{2}$, there are three natural directions in T_pS^3. First the fibre direction

$$\begin{aligned} e_1 &= \frac{d}{d\theta}\Big|_{\theta=0}(\sin\eta\cos k\theta, \sin\eta\sin k\theta, \cos\eta\cos(\beta+l\theta), \cos\eta\sin(\beta+l\theta)) \\ &= (0, k\sin\eta, -l\cos\eta\sin\beta, l\cos\eta\cos\beta), \end{aligned}$$

then

$$\begin{aligned} e_2 &= \frac{\partial}{\partial\eta} = (\cos\eta, 0, -\sin\eta\cos\beta, -\sin\eta\sin\beta), \\ e_3' &= \frac{\partial}{\partial\beta} = (0, 0, -\cos\eta\sin\beta, \cos\eta\cos\beta). \end{aligned}$$

Now, $e_1 \perp e_2$, $e_2 \perp e_3'$. Set

$$\begin{aligned} e_3 &= e_3' - \frac{(e_3', e_1)e_1}{(e_1, e_1)} = \frac{\partial}{\partial\beta} - \frac{(l\cos^2\eta)e_1}{k^2\sin^2\eta + l^2\cos^2\eta} \\ &= \Big(0, \frac{-kl\cos^2\eta\sin\eta}{k^2\sin^2\eta + l^2\cos^2\eta}, \frac{-k^2\cos\eta\sin^2\eta\sin\beta}{k^2\sin^2\eta + l^2\cos^2\eta}, \frac{k^2\cos\eta\sin^2\eta\cos\beta}{k^2\sin^2\eta + l^2\cos^2\eta}\Big). \end{aligned}$$

Therefore, e_1, e_2, e_3 are perpendicular to each other.

Let ρ be the projection map from S^3 to the space of orbits (parametrized by η, β as above). We have

$$\rho_*(e_1) = 0, \quad \rho_*(e_2) = \partial_\eta, \quad \rho_*(e_3) = \partial_\beta.$$

(This ∂_β is a tangent vector of the orbit space. The last one is true because e_3 and e_3' differ by a multiple of fibre direction.) Let h be the metric tensor of the orbit space. To make ρ a Riemannian submersion at this point, we require

$$\begin{aligned} h(\partial_\eta, \partial_\eta) &= (e_2, e_2) = 1, \\ h(\partial_\beta, \partial_\beta) &= (e_3, e_3) = \frac{k^2\sin^2\eta\cos^2\eta}{k^2\sin^2\eta + l^2\cos^2\eta}, \\ h(\partial_\beta, \partial_\eta) &= (e_3, e_2) = 0, \end{aligned}$$

i.e.,

$$h = d\eta^2 + \frac{k^2\sin^2\eta\cos^2\eta}{k^2\sin^2\eta + l^2\cos^2\eta}d\beta^2.$$

(This is $\frac{1}{4}$ of (3.1) in the paper of Wang, where his η is twice ours.) This construction shows ρ is Riemannian submersion at the equator. Since each orbit passes through the equator and the action is by isometry, ρ is Riemannian submersion at every point.

Denote the orbit space by Q. By the above discussion, every S^1-invariant map f can be identified with some continuous map u_f from Q to S^2,

$$u_f(x) = f(\rho^{-1}(x)).$$

Set

$$W(\eta) = (k^2\sin^2\eta + l^2\cos^2\eta)^{\frac{1}{2}}$$

for simplicity. We want to express $E(f) = \frac{1}{2}\int_{S^3} |\nabla f|^2 \, dV$ in terms of integration of u_f. First, take an orthonormal basis e_1, e_2, e_3 at any point p in S^3 with e_1 the fibre direction. Then

$$|\nabla f|^2 = \sum_{i=1}^{3} |e_i f|^2.$$

Since f is S^1-invariant, we have $e_1 f = 0$. The projection ρ is Riemannian submersion, then $\rho_* e_2, \rho_* e_3$ is an orthonormal basis at $\rho(p)$. So

$$|\nabla f|^2 = |\nabla u_f|_h^2 .$$

The length of the orbit passing through $(\eta, \beta) \in Q$ is given by integrating $\left|\frac{\partial}{\partial\theta}\right|$ from 0 to 2π, that is, $2\pi W$.

$$E(f) = \frac{1}{2}\int_{S^3} |\nabla f|^2 \, dV \tag{3.2}$$

$$= \pi \int_Q |\nabla u_f|_h^2 \, W dV_Q. \tag{3.3}$$

The other integral that is important to us is $\int_{S^3} f^{(3)} dV$. By the same reason, it is reduced to

$$I(f) = 2\pi \int_Q u_f^{(3)} W dV_Q. \tag{3.4}$$

Remark 3.1 The metric h is defined for $\eta \in (0, \frac{\pi}{2})$ and $\beta \in [0, \frac{2\pi}{k})$. There are two singularities on Q corresponding to $\eta = 0, \frac{\pi}{2}$. They do not affect the integration.

4 The Case of $k = l = 1$

When $k = l = 1$, we first observe that W is a constant. It is easy to see that Q is topologically a sphere. Now the metric on Q becomes

$$h = d\eta^2 + \sin^2\eta \cos^2\eta d\beta^2 = \frac{1}{4}(d(2\eta)^2 + \sin^2(2\eta) d\beta^2),$$

where $\beta \in [0, 2\pi)$ and $\eta \in (0, \frac{\pi}{2})$. Therefore, Q is a round sphere of radius $\frac{1}{2}$. Moreover, up to a constant (3.2) is just the energy of maps from S^2 to S^2. So the reduced variational problem is the same as the original problem for $M = S^2$ and $N = S^2$. We have discussed this in [3]. Theorem 1.1 of that paper implies Theorem 1.1.

5 The Case of $k \neq l$

This case is more complicated because W is not constant. If we take a minimizing sequence $\{f_i\}$ for the functional $E(f) - \lambda I(f)$. It is not difficult to see $E(f)$ is bounded and since $l \le W \le k$, we know $W^{1,2}$-norms of f_i are bounded. However, this is the critical case for compactness needed for convergence. One might think naturally whether the famous Sacks-Uhlenbeck [7] method could be applied here. This will involve the proof of an a priori estimate and a removal of singularity theorem for a new equation. So far, the authors do not know whether this approach is possible. The method used in this paper is different. Notice that

Q has a rotation-invariant symmetry, which enables us to further reduce the problem to 1 dimensional. Hence, we get the compactness we need. Of cause, that causes some technical problems, i.e., the new expression for functional $E(f)$ will be singular near the ends of the interval.

We will focus on the following S^1-equivariant maps from Q to S^2,

$$u : (\eta, \beta) \mapsto (f(\eta), k\beta) = (\sin f \cos k\beta, \sin f \sin k\beta, \cos f), \tag{5.1}$$

where $f : [0, \frac{\pi}{2}] \to [0, \pi]$ and $f(0), f(\frac{\pi}{2}) \in \pi\mathbb{Z}$. Calculation shows

$$|\nabla u|^2 = (f')^2 + \frac{k^2 \sin^2 \eta + l^2 \cos^2 \eta}{\sin^2 \eta \cos^2 \eta} \sin^2 f,$$
$$dV = \frac{k \sin \eta \cos \eta}{W} d\eta d\beta.$$

(Recall that $\beta \in [0, \frac{2\pi}{k})$.) Then, the functionals are

$$E(u) = \int_Q |\nabla u|^2 \, W dV = 2\pi \int_0^{\frac{\pi}{2}} \left[(f')^2 + \frac{k^2 \sin^2 \eta + l^2 \cos^2 \eta}{\sin^2 \eta \cos^2 \eta} \sin^2 f \right] \sin \eta \cos \eta d\eta, \tag{5.2}$$

$$I(u) = 2\pi \int_0^{\frac{\pi}{2}} \cos f \sin \eta \cos \eta d\eta. \tag{5.3}$$

The Euler-Lagrange equation is

$$f'' + \frac{f' \cos 2\eta}{\sin \eta \cos \eta} - \frac{k^2 \sin^2 \eta + l^2 \cos^2 \eta}{\sin^2 \eta \cos^2 \eta} \sin f \cos f = \lambda \sin f. \tag{5.4}$$

Now we minimize $E(f)$ under the condition $I(f) = c$. If f is a critical point, then there should exist some λ such that f satisfies equation (5.4). Consider $H^1_{\mathrm{loc}}(0, \frac{\pi}{2})$ space, i.e. $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ is weakly differentiable,

$$\int_\epsilon^{\frac{\pi}{2} - \epsilon} (f')^2 + f^2 dt < +\infty$$

for any $\epsilon > 0$ and a subset

$$\mathcal{X} = \left\{ f \in H^1_{\mathrm{loc}}\left(0, \frac{\pi}{2}\right) \,\middle|\, E(f) < \infty, I(f) = c \right\} \tag{5.5}$$

for some $c \in (-\pi, +\pi)$, ($\pi = 2\pi \int_0^{\frac{\pi}{2}} \sin t \cos t dt$). Notice that we require no condition on the boundary value of f.

Take a minimizing sequence f_i in $\mathcal{X}$ of $E(f)$. We prove a series of properties of f_i.

Lemma 5.1 $|f_i(x) - f_i(y)|$ *is uniformly bounded for any* x, y, i.

Proof It is well known that the energy of a map u_f is greater than the area covered by the image. Since $l \leq W \leq k$, we know that $E(u_f)$ is comparable to the energy of u_f. The lemma follows from the fact that $E(u_{f_i})$ is uniformly bounded and the above two facts.

Remark 5.1 Because of this lemma, we can assume f_i are uniformly bounded.

Lemma 5.2 $f_i(0), f_i(\frac{\pi}{2})$ *must be* $k\pi$ *for* $k \in \mathbb{Z}$.

Proof Prove for $f_i(0)$ only. Since $E(f_i) < +\infty$, we know from (5.2) that for small $\epsilon > 0$,

$$\int_0^\epsilon \frac{\sin^2 f}{\sin\eta} d\eta < +\infty.$$

This implies that there exists a sequence of $\eta_i \to 0$ such that $\sin f(\eta_i) \to 0$. Due to the previous remark, we can assume $f(\eta_i)$ goes to some $k\pi$. It suffices to show that f_i can not vibrate too much when η approaches zero. The reason is the same as in the proof of the previous lemma.

We can see from (5.2) that for any $\epsilon > 0$, the $H^1(\epsilon, \frac{\pi}{2} - \epsilon)$ norms of f_i are bounded. Therefore, we can take a subsequence by diagonal method, still denoted by $\{f_i\}$, such that

$$f_i \to f \quad \text{weakly in } H^1\Big(\epsilon, \frac{\pi}{2} - \epsilon\Big), \quad \forall\, \epsilon > 0.$$

Lemma 5.3

$$\begin{aligned} &E(f) \le \liminf_{i\to\infty} E(f_i) < +\infty, \\ &I(f) = c. \end{aligned}$$

Proof Weak convergence in H^1 implies uniform convergence. So the second assertion is true. For the first one, the part of integrand involving f' is convex and therefore integration of this part is weakly lower-semi-continuous. For the other term, uniform convergence on $[\epsilon, \frac{\pi}{2} - \epsilon]$ implies

$$\begin{aligned} &\lim_{i\to\infty} \int_\epsilon^{\frac{\pi}{2}-\epsilon} \frac{k^2\sin^2\eta + l^2\cos^2\eta}{\sin^2\eta\cos^2\eta} \sin^2 f_i \sin\eta\cos\eta d\eta \\ = &\int_\epsilon^{\frac{\pi}{2}-\epsilon} \frac{k^2\sin^2\eta + l^2\cos^2\eta}{\sin^2\eta\cos^2\eta} \sin^2 f \sin\eta\cos\eta d\eta. \end{aligned}$$

Let ϵ go to zero and the fact that the integrand is nonnegative gives the required inequality. (Notice that the integrand may not be bounded, compare with $I(f)$.)

A corollary of this lemma is that f is a minimizer of $E(f)$ in $\mathcal{X}$.

Next, we will prove that f satisfies (5.4). The singular integrand in (5.2) causes some trouble. It will be overcome by the following consideration. Fix $\epsilon > 0$, let $l(t) : [\epsilon, \frac{\pi}{2} - \epsilon] \to R$ be the linear function such that $l(\epsilon) = f(\epsilon)$ and $l(\frac{\pi}{2} - \epsilon) = f(\frac{\pi}{2} - \epsilon)$.

Set

$$\begin{aligned} &\widetilde{E} : H_0^1\Big[\epsilon, \frac{\pi}{2} - \epsilon\Big] \to \mathbb{R}, \\ &\tilde{f} \mapsto E\Big(l + \tilde{f}, \epsilon, \frac{\pi}{2} - \epsilon\Big) \end{aligned}$$

and similar for $\tilde{I}$, where $E(l + \tilde{f}, \epsilon, \frac{\pi}{2} - \epsilon)$ means integration on the interval $[\epsilon, \frac{\pi}{2} - \epsilon]$. It is easy to see that $f - l$ minimizes $\widetilde{E}$ under the condition $\tilde{I}(f - l) = I(f, \epsilon, \frac{\pi}{2} - \epsilon)$. Now $\widetilde{E}$ and $\tilde{I}$ are C^1 functionals. So $f - l$ satisfies an Euler-Lagrange equation, which in terms of f is exactly (5.4). Notice that for different ϵ the Lagrange multipliers have to be the same unless $\sin f \equiv 0$. This is impossible due to our choice of c in the condition $I(f) = c$. This condition also shows that f is a nontrivial map.

We need to show $\lambda \neq 0$. This follows from a result of Wang [6]. Wang showed that there is no $T_{k,l}$-invariant harmonic maps from S^3 to S^2. If $\lambda = 0$, then the map from S^3 to S^2 determined by f is such a harmonic map. This is impossible, so $\lambda \neq 0$.

f determines a u_f from Q to S^2 and this u_f determines a u from S^3 to S^2. Since $f(0), f(\frac{\pi}{2}) \in \pi\mathbb{Z}$, we know u_f, hence u, is continuous. By the reduction process, it is not difficult to see that u is smooth outside two special orbits. We will prove that u is smooth on the entire S^3.

First, we will prove that u is a $W^{1,2}(S^3, S^2)$ weak solution to the equation (2.3). Then, the regularity follows from general PDE regularity theorems.

u is weakly differentiable because u is smooth away from two special orbits. For any p in the special orbits, take a coordinates around p. u is smooth along almost all line segment parallel to the coordinate axis. According to Theorem 2.1.4 in [8], u is weakly differentiable and to show $u \in W^{1,2}(S^3, S^2)$, it suffices to show integrability of the square of the classic derivatives of u. For a fixed coordinate system, the square of any partial derivative of u is dominated by $\frac{1}{2}|\nabla u|^2$. Moreover, we know that the energy of u is finite. So, $u \in W^{1,2}(S^3, S^2)$.

To show u is a weak solution to equation (2.3), it suffices to show for any $\phi \in C_c^\infty(S^3, \mathbb{R}^3)$,

$$\int_{S^3} -\nabla u \cdot \nabla\phi + |\nabla u|^2 u \cdot \phi + \lambda(P(u)e_3) \cdot \phi dV = 0. \tag{5.6}$$

For some $\epsilon > 0$, set

$$T_1(\epsilon) = \{(\eta, \alpha, \beta) \in S^3 \mid \eta < \epsilon\},$$
$$T_2(\epsilon) = \left\{(\eta, \alpha, \beta) \in S^3 \,\middle|\, \eta > \frac{\pi}{2} - \epsilon\right\}.$$

Since u satisfies (2.3) on $S^3 - T_1 - T_2$, we have

$$\int_{S^3 - T_1 - T_2} \triangle u \cdot \phi + |\nabla u| u \cdot \phi + \lambda(P(u)e_3) \cdot \phi dV = 0. \tag{5.7}$$

Integration by parts for the first term shows that we need only to prove there exists a sequence of $\{\epsilon_i\}$ going to zero, such that

$$\lim_{i \to +\infty} \int_{\partial T_1 + \partial T_2} \nabla u \cdot \vec{n} dV \tag{5.8}$$

can be arbitrarily small. It is obvious the area of ∂T_1 is proportional to $\sin 2\epsilon_i$. By the definition of T_1, the outward normal vector of T_1 is $\frac{\partial}{\partial\eta}$. Therefore

$$\nabla u \cdot \vec{n} = \frac{\partial u}{\partial \eta} = f'(\cos f \cos k\beta, \cos f \sin k\beta, -\sin f).$$

So

$$|\nabla u \cdot \vec{n}| = |f'|.$$

Since the energy is finite,

$$\int_0^\epsilon (f')^2 \sin\eta d\eta < +\infty.$$

So for any $\delta > 0$, there is a sequence $\{\epsilon_i\}$ such that

$$(f')^2 \sin\eta|_{\eta = \epsilon_i} < \frac{\delta}{\epsilon_i},$$

i.e.,

$$|f'(\epsilon_i)| \leq \sqrt{\frac{\delta}{\epsilon_i \sin \epsilon_i}}, \quad \lim_{i\to\infty} (f')(\epsilon_i) \sin 2\epsilon_i = 2\sqrt{\delta}.$$

Hence (5.8) can be arbitrarily small. The same argument works for T_2. Because the left-hand side of (5.6) is absolutely integrable, no matter how to choose ϵ_i, the limit of (5.7) is always the left side of (5.6). This shows the left side of (5.6) vanishes.

Now, u is a continuous weak solution of (2.3). Due to a theorem of Hildebrandt, S. [9], u is smooth. That completes the proof of Theorem 1.2.

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