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A Class of Homogeneous Einstein Manifolds***

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Abstract A Riemannian manifold (M, g) is called Einstein manifold if its Ricci tensor satisfies $r = c \cdot g$ for some constant c. General existence results are hard to obtain, e.g., it is as yet unknown whether every compact manifold admits an Einstein metric. A natural approach is to impose additional homogeneous assumptions. M. Y. Wang and W. Ziller have got some results on compact homogeneous space G/H. They investigate standard homogeneous metrics, the metric induced by Killing form on G/H, and get some classification results. In this paper some more general homogeneous metrics on some homogeneous space G/H are studies, and a necessary and sufficient condition for this metric to be Einstein is given. The authors also give some examples of Einstein manifolds with non-standard homogeneous metrics.

Keywords Einstein manifold, Homogeneous space, General homogeneous metric
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1 Introduction

A Riemannian manifold (M, g) is called Einstein manifold if its Ricci tensor satisfies $r = c \cdot g$ for some constant c. Einstein manifolds are not only interesting in themselves but are also related to many important topics of Riemannian geometry. For example: Riemannian submersions, homogeneous Riemannian geometry, Riemannian functionals and their critical points, Yang-Mills theory, holonomy groups, etc.

General existence results are hard to obtain, e.g., it is as yet unknown whether every compact manifold admits an Einstein metric. For $c > 0$, most known examples of Einstein manifolds are compact homogeneous space, so a natural approach is to impose additional homogeneous assumptions. M. Y. Wang and W. Ziller have got some results on compact homogeneous space G/H (cf. [5]). They investigate standard homogeneous metrics, the metric induced by Killing form on G/H, and get some classification results. In this paper we study some more general homogeneous metrics on some homogeneous space G/H and get a necessary and sufficient condition for this metric to be Einstein. We also give some examples of Einstein manifolds with non-standard homogeneous metrics.

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2 Geometric Structure

Suppose G is a compact connected simple Lie group with real simple Lie algebra $\mathfrak{g}$. Let θ, τ be two involutions of G, and $\theta\tau = \tau\theta$. Let $K = \{X \in G,\ \theta X = X\}$ and $K' = \{X \in G,\ \tau X = X\}$. Let $K^+ = K \bigcap K'$. It is clear that G/K^+ is a reductive homogeneous space. θ, τ induce two involutions of $\mathfrak{g}$, we still denote θ, τ, and we still have $\theta\tau = \tau\theta$. Let $\mathfrak{k}$, $\mathfrak{k}^+$ denote the Lie algebras of K, K^+ respectively. It is clear that $\mathfrak{k}^+ = \{X \in \mathfrak{k},\ \theta X = X\}$. We denote $\mathfrak{p} = \{X \in \mathfrak{g},\ \theta X = -X\}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. $\theta\tau = \tau\theta$ means that $\tau\mathfrak{k} \subset \mathfrak{k}$, $\tau\mathfrak{p} \subset \mathfrak{p}$. So we can write that $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$, $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$, where $\mathfrak{k}^- = \{X \in \mathfrak{k},\ \tau X = -X\}$, $\mathfrak{p}^\pm = \{X \in \mathfrak{p},\ \tau X = \pm X\}$. With an easy observation, we get

Proposition 2.1 $[\mathfrak{k}^+, \mathfrak{k}^+] \subset \mathfrak{k}^+$, $[\mathfrak{k}^+, \mathfrak{k}^-] \subset \mathfrak{k}^-$, $[\mathfrak{k}^+, \mathfrak{p}^+] \subset \mathfrak{p}^+$, $[\mathfrak{k}^+, \mathfrak{p}^-] \subset \mathfrak{p}^-$, $[\mathfrak{k}^-, \mathfrak{k}^-] \subset \mathfrak{k}^+$, $[\mathfrak{k}^-, \mathfrak{p}^+] \subset \mathfrak{p}^-$, $[\mathfrak{k}^-, \mathfrak{p}^-] \subset \mathfrak{p}^+$, $[\mathfrak{p}^+, \mathfrak{p}^+] \subset \mathfrak{k}^+$, $[\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}^-$, $[\mathfrak{p}^-, \mathfrak{p}^-] \subset \mathfrak{k}^+$.

Remark 2.1 From the above proposition we see that $\mathfrak{k}^+ + \mathfrak{k}^-$, $\mathfrak{k}^+ + \mathfrak{p}^+$, $\mathfrak{k}^+ + \mathfrak{p}^-$ are all subalgebras of $\mathfrak{g}$.

Suppose $\langle\ ,\ \rangle$ is an $\mathrm{ad}\,(K^+)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}/\mathfrak{k}^+$, which satisfies $\langle\ ,\ \rangle|_{\mathfrak{k}^-} = aB(\ ,\ )$, $\langle\ ,\ \rangle|_{\mathfrak{p}^+} = bB(\ ,\ )$, $\langle\ ,\ \rangle|_{\mathfrak{p}^-} = cB(\ ,\ )$, $\langle X, Y\rangle = 0$ for others, where $B(\ ,\ )$ denotes the Killing form of $\mathfrak{g}$, and a, b, c are any negative real constants. Let g be the G-invariant metric corresponding to $\langle\ ,\ \rangle$. Notice that if $a = b = c$ then g will be standard homogeneous metric.

Next, we investigate the Riemannian connection of $M = G/K^+$. For each $X \in \mathfrak{g}$ we denote by $\mathrm{Exp}(tX)$ the one-parameter subgroup of G generated by X. The action of $\mathrm{Exp}(tX)$ on M is defined by

$$\varphi_t(y) = \mathrm{Exp}(tX)y.$$

From now on we will identify $X \in \mathfrak{g}$ with the vector field on M generated by φ_t. In doing so, we identify $\mathfrak{g}$ with the set of those Killing vector field of (M, g) which generate one-parameter subgroups of G.

Remark 2.2 There is one subtle point in this identification. Let $[\ ,\ ]$ be the Lie bracket of vector fields in M and $[\ ,\ ]_{\mathfrak{g}}$ the Lie algebra bracket of $\mathfrak{g}$. Then using the identification given above, we have

$$[X, Y]_{\mathfrak{g}} = -[X, Y].$$

Moreover, the curvature tensor at $x \in \mathfrak{m}$ is identified with a tensor on the vector space $\mathfrak{m}$. Since K^+ acts by isometries, the resulting tensor on $\mathfrak{m}$ is in particular $\mathrm{Ad}_G(K^+)$-invariant. For any Killing vector field $X, Y \in \mathfrak{m}$, we only need to determine the value $\nabla_X Y$ at $o \in \mathfrak{m}$, denoted by $\Lambda_{\mathfrak{m}}(X)Y$.

Lemma 2.1 (Cf. [4]) *Let $M = K/H$ be a reductive homogeneous space with an* $\mathrm{ad}\,(H)$-*invariant decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ and an* $\mathrm{ad}\,(H)$*-invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{m}$. Let g be the K-invariant metric corresponding to $\langle\ ,\ \rangle$. Then the Riemannian connection for g is given by $\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)$, where $U(X, Y)$ is the symmetric* *bilinear mapping of* $\mathfrak{m} \times \mathfrak{m}$ *into* $\mathfrak{m}$ *defined by*

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle$$

for all $X, Y, Z \in \mathfrak{m}$.

Proposition 2.2 *The Riemannian connection of* $M = G/K^+$ *for* g *is given as follows:*

$$\begin{aligned}
\Lambda_{\mathfrak{m}}(X_1)X_2 &= 0, \quad \Lambda_{\mathfrak{m}}(X_1)Y_1 = \frac{b+c-a}{2c}[X_1,Y_1], \quad \Lambda_{\mathfrak{m}}(X_1)Z_1 = \frac{b+c-a}{2b}[X_1,Z_1],\\
\Lambda_{\mathfrak{m}}(Y_1)Y_2 &= 0, \quad \Lambda_{\mathfrak{m}}(Y_1)X_1 = \frac{a+c-b}{2c}[Y_1,X_1], \quad \Lambda_{\mathfrak{m}}(Y_1)Z_1 = \frac{a+c-b}{2a}[Y_1,Z_1],\\
\Lambda_{\mathfrak{m}}(Z_1)Z_2 &= 0, \quad \Lambda_{\mathfrak{m}}(Z_1)X_1 = \frac{a+b-c}{2b}[Z_1,X_1], \quad \Lambda_{\mathfrak{m}}(Z_1)Y_1 = \frac{a+b-c}{2a}[Z_1,Y_1],
\end{aligned}$$

where $X_i \in \mathfrak{k}^-, Y_i \in \mathfrak{p}^+, Z_i \in \mathfrak{p}^-, i = 1, 2$.

Proof Let $Y \in \mathfrak{m}$. Then

$$2\langle U(X_1,X_2), Y\rangle = \langle X_i, [Y,X_2]_{\mathfrak{m}}\rangle + \langle [Y,X_1]_{\mathfrak{m}}, X_2\rangle = 0.$$

We get

$$U(X_1,X_2) = 0,$$

so

$$\Lambda_{\mathfrak{m}}(X_1)X_2 = \frac{1}{2}[X_1,X_2]_{\mathfrak{m}} + U(X_1,X_2) = 0.$$

From

$$\begin{aligned}
2\langle U(X_1,Y_1), Y\rangle &= \langle X_1, [Y,Y_1]_{\mathfrak{m}}\rangle + \langle [Y,X_1]_{\mathfrak{m}}, Y_1\rangle\\
&= \langle X_1, [Y,Y_1]_{\mathfrak{k}^-}\rangle + \langle [Y,X_1]_{\mathfrak{p}^+}, Y_1\rangle\\
&= aB(X_1, [Y,Y_1]_{\mathfrak{k}^-}) + bB([Y,X_1]_{\mathfrak{p}^+}, Y_1)\\
&= -aB([X_1,Y_1], Y) + bB([X_1,Y_1], Y)\\
&= -aB([X_1,Y_1], Y_{\mathfrak{p}^-}) + bB([X_1,Y_1], Y_{\mathfrak{p}^-})\\
&= \Big\langle \frac{b-a}{c}[X_1,Y_1], Y\Big\rangle,
\end{aligned}$$

we get

$$U(X_1,Y_1) = \frac{b-a}{2c}[X_1,Y_1], \quad \Lambda_{\mathfrak{m}}(X_1)Y_1 = \frac{b+c-a}{2c}[X_1,Y_1].$$

By the same reason, we get the other identities.

Lemma 2.2 (Cf. [4]) *Let* $M = K/H$ *as in Lemma* 2.1. *Then the curvature tensor* R *of Riemannian connection correspinding to* $\Lambda_{\mathfrak{m}}$ *can be expressed at origin* o *as follows:*

$$R(X,Y)_o = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}\,([X,Y]_{\mathfrak{h}}).$$

Let $E_1, E_2, \cdots, E_m$ be the standard orthogonal basis of $\mathfrak{k}^-$ with respect to $\langle\ ,\ \rangle|_{\mathfrak{k}^-}$, $F_1, F_2, \cdots, F_n$ be the standard orthogonal basis of $\mathfrak{p}^+$ with respect to $\langle\ ,\ \rangle|_{\mathfrak{p}^+}$, and $G_1, G_2, \cdots, G_l$ be the standard orthogonal basis of $\mathfrak{p}^-$ with respect to $\langle\ ,\ \rangle|_{\mathfrak{p}^-}$.

Proposition 2.3 *The Ricci curvature of $M = G/K^+$ is given as follows:*

$$\begin{aligned}
r(E_\alpha, F_\beta) &= r(E_\alpha, G_\beta) = r(F_\alpha, G_\beta) = 0,\\
r(E_\alpha, E_\beta) &= \langle (C_{\mathfrak{k}} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle\\
&\quad + \frac{2c(a-c+b)-(c-b+a)(c-a+b)}{4ac}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle\\
&\quad + \frac{2b(a-b+c)-(b-c+a)(b-a+c)}{4ab}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle,\\
r(F_\alpha, F_\beta) &= \frac{2c(a-c+b)-(c-b+a)(c-a+b)}{4bc}\langle (C_{\mathfrak{k}} - C_{\mathfrak{k}^+})\cdot F_\alpha, F_\beta\rangle\\
&\quad + \langle (C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+})\cdot F_\alpha, F_\beta\rangle\\
&\quad + \frac{2a(b+c-a)-(a-c+b)(a-b+c)}{4ab}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\cdot F_\alpha, F_\beta\rangle,\\
r(G_\alpha, G_\beta) &= \frac{2b(a+c-b)-(b-a+c)(b-c+a)}{4bc}\langle (C_{\mathfrak{k}} - C_{\mathfrak{k}^+})\cdot G_\alpha, G_\beta\rangle\\
&\quad + \frac{2a(b+c-a)-(a-b+c)(a-c+b)}{4ac}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+})\cdot G_\alpha, G_\beta\rangle\\
&\quad + \langle (C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\cdot G_\alpha, G_\beta\rangle,
\end{aligned}$$

where $C_{\mathfrak{k}}, C_{\mathfrak{k}^+}, C_{\mathfrak{k}^+ + \mathfrak{p}^+}, C_{\mathfrak{k}^+ + \mathfrak{p}^-}$ denote the Casimir operator of corresponding Lie algebra.

Proof We only compute $r(E_\alpha, E_\beta)$. Using Lemma 2.2 we get

$$\begin{aligned}
\langle R_o(E_\alpha, E_i)\cdot E_\beta, E_i\rangle &= \langle \Lambda(E_\alpha)(\Lambda(E_i)E_\beta), E_i\rangle - \langle \Lambda(E_i)(\Lambda(E_\alpha)E_\beta), E_i\rangle\\
&\quad - \langle \Lambda([E_\alpha, E_i]_{\mathfrak{m}})E_\beta, E_i\rangle - \langle [[E_\alpha, E_i]_{\mathfrak{k}^+}, E_\beta], E_i\rangle\\
&= -\langle [[E_\alpha, E_i]_{\mathfrak{k}^+}, E_\beta], E_i\rangle = -aB([[E_\alpha, E_i]_{\mathfrak{k}^+}, E_\beta], E_i)\\
&= aB(\mathrm{ad}^2(E_i)(E_\alpha), E_\beta) = \langle \mathrm{ad}^2(E_i)(E_\alpha), E_\beta\rangle,\\
\langle R_o(E_\alpha, F_j)\cdot E_\beta, F_j\rangle &= \langle \Lambda(E_\alpha)(\Lambda(F_j)E_\beta), F_j\rangle - \langle \Lambda(F_j)(\Lambda(E_\alpha)E_\beta), F_j\rangle\\
&\quad - \langle \Lambda([E_\alpha, F_j]_{\mathfrak{m}})E_\beta, F_j\rangle - \langle [[E_\alpha, F_j]_{\mathfrak{k}^+}, E_\beta], F_j\rangle\\
&= \langle \Lambda(E_\alpha)(\Lambda(F_j)E_\beta), F_j\rangle - \langle \Lambda([E_\alpha, F_j]_{\mathfrak{m}})E_\beta, F_j\rangle\\
&= bB(\Lambda(E_\alpha)(\Lambda(F_j)E_\beta), F_j) - bB(\Lambda([E_\alpha, F_j])E_\beta, F_j)\\
&= \frac{2c(a+b-c)-(c-a+b)(c-b+a)}{4ac}\langle \mathrm{ad}^2(F_j)E_\alpha, E_\beta\rangle.
\end{aligned}$$

By the same reason,

$$(R_o(E_\alpha, G_k)\cdot E_\beta, G_k) = \frac{2b(a+c-b)-(b-a+c)(b-c+a)}{4ab}\langle \mathrm{ad}^2(G_k)E_\alpha, E_\beta\rangle,$$

so

$$\begin{aligned}
r(E_\alpha, E_\beta) &= \sum_{i=1}^{m}\langle R_o(E_\alpha, E_i)\cdot E_\beta, E_i\rangle + \sum_{j=1}^{n}\langle R_o(E_\alpha, F_j)\cdot E_\beta, F_j\rangle + \sum_{k=1}^{l}\langle R_o(E_\alpha, G_k)\cdot E_\beta, G_k\rangle \\
&= \Big\langle \sum_{i=1}^{m}\mathrm{ad}^2(E_i)E_\alpha, E_\beta\Big\rangle + \Big\langle \sum_{j=1}^{n}\mathrm{ad}^2(F_j)E_\alpha, E_\beta\Big\rangle + \Big\langle \sum_{k=1}^{l}\mathrm{ad}^2(G_k)E_\alpha, E_\beta\Big\rangle \\
&= \langle (C_{\mathfrak{k}} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle \\
&\quad + \frac{2c(a-c+b)-(c-b+a)(c-a+b)}{4ac}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle \\
&\quad + \frac{2b(a-b+c)-(b-c+a)(b-a+c)}{4ab}\langle (C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle.
\end{aligned}$$

Assume that $\mathfrak{k}^- = \mathfrak{k}_1^- + \mathfrak{k}_2^- + \cdots + \mathfrak{k}_r^-$, where the $\{\mathfrak{k}_i^-\}$ are irreducible representations of $\mathfrak{k}^+$. Then $C_{\mathfrak{k}^+}|_{\mathfrak{k}_i^-} = a_i\mathrm{Id}$. We see that

$$\langle (C_{\mathfrak{k}} - C_{\mathfrak{k}^+})\cdot E_\alpha, E_\beta\rangle = 0 \quad \text{if } \alpha \neq \beta.$$

By the same reason we have

Corollary 2.1 *If $\alpha \neq \beta$, then*

$$r(E_\alpha, E_\beta) = r(F_\alpha, F_\beta) = r(G_\alpha, G_\beta) = 0.$$

Theorem 2.1 *The metric g defined as above is an Einstein metric if and only if that g satisfies*

$$\begin{aligned}
&\Big\{(C_{\mathfrak{k}} - C_{\mathfrak{k}^+}) + \frac{2c(a-c+b)-(c-b+a)(c-a+b)}{4ac}(C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+}) \\
&+ \frac{2b(a-b+c)-(b-c+a)(b-a+c)}{4ab}(C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\Big\}\Big|_{\mathfrak{k}^-} = A\,\mathrm{Id}_{\mathfrak{k}^-}, \\
&\Big\{(C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+}) + \frac{2c(a-c+b)-(c-b+a)(c-a+b)}{4bc}(C_{\mathfrak{k}} - C_{\mathfrak{k}^+}) \\
&+ \frac{2a(b+c-a)-(a-c+b)(a-b+c)}{4ab}(C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+})\Big\}\Big|_{\mathfrak{p}^+} = A\,\mathrm{Id}_{\mathfrak{p}^+}, \\
&\Big\{(C_{\mathfrak{k}^+ + \mathfrak{p}^-} - C_{\mathfrak{k}^+}) + \frac{2b(a+c-b)-(b-a+c)(b-c+a)}{4bc}(C_{\mathfrak{k}} - C_{\mathfrak{k}^+}) \\
&\frac{2a(b+c-a)-(a-b+c)(a-c+b)}{4ac}(C_{\mathfrak{k}^+ + \mathfrak{p}^+} - C_{\mathfrak{k}^+})\Big\}\Big|_{\mathfrak{p}^-} = A\,\mathrm{Id}_{\mathfrak{p}^-},
\end{aligned}$$

where A is a constant.

3 Some Examples on SU$(n+1)$

Although we give the necessary and sufficient condition, the complete classification is hard to get. In this section we will investigate some examples on SU$(n+1)$.

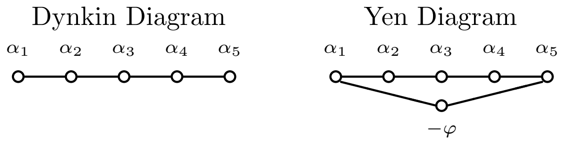
Let us first recall in brief some part of Yen's classification theory of real simple Lie algebra. In essence, his method is a better control of Gantmacher's canonical form of Cartan involutions.

Lemma 3.1 *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be Cartan decomposition of $\mathfrak{g}_0$ with respect to θ. Fix a maximally compact Cartan subalgebra $\mathfrak{h}_0^c$ of $\mathfrak{g}_0$ with decomposition $\mathfrak{h}_0^c = \mathfrak{k}_0^c + \mathfrak{a}_0^c$. Let $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$. Then $\mathfrak{h}_0^n = \mathfrak{k}_0^c + i\mathfrak{a}_0^c$ is a Cartan subalgebra of $\mathfrak{u}$ and there exists a $\gamma \in \mathrm{Ad}\,\mathfrak{u}$ such that $\gamma\theta\gamma^{-1} = \theta_0 e^{\mathrm{ad}\,H}$, where θ_0 is a diagram automorphism of $\mathfrak{u}$'s Dynkin diagram and $H \in \mathfrak{h}_0 = \{H_1 \in \mathfrak{h}_0^n, \theta_0(H_1) = H_1\}$.*

Let $\mathfrak{g}^c, \mathfrak{h}^c$ be the complexification of $\mathfrak{g}_0, \mathfrak{h}_0^c$ respectively. Let $\Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}^c$. Fix a system of positive roots, $\Delta^+(\mathfrak{g}^c, \mathfrak{h}^c)$, for $\Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ with the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_l\}$. Let $E_1, F_1, \cdots, E_l, F_l$ be a relative Weyl basis of $\mathfrak{g}^c$ such that $\alpha^\vee = [E_i, F_i]$. Let $\varphi = \sum\limits_{i=1}^{l} m_i\alpha_i$ be the highest weight relative to Π.

Definition 3.1 *Let Π, φ be as above. The Yen Diagram is the diagram of the system $\{\alpha_1, \cdots, \alpha_l, -\varphi\}$ defined in the same way as that of the Dynkin Diagram.*

Example 3.1 $\mathfrak{g}^c = A_5$, $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\varphi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$.



For any nonempty subset $I \subset \{1, \cdots, n\}$, we can define a unique automorphism $\theta(\Pi, I)$ of $\mathfrak{g}^c$, which satisfies:

(1) $\theta(\Pi, I)|_{\mathfrak{h}^c} = \mathrm{Id}$.

(2) $\theta(\Pi, I)(E_i) = \begin{cases} E_i, & i \notin I, \\ F_i, & i \in I. \end{cases}$

Lemma 3.2 *Let $\mathfrak{g}^c$ be a simple Lie algebra. Then for any $\theta(\Pi, I)$, there exists a system of simple roots $\Pi' = \{\alpha_1', \ldots, \alpha_l'\}$ with highest weight $\varphi' = m_1\alpha_1' + \cdots + m_l\alpha_l'$ and $i \in \{1, \cdots, l\}$ satisfying $m_i = 1$ or 2, such that $\theta(\Pi, I) = \theta(\Pi', \{i\})$.*

Set $\theta_i = \theta(\Pi, \{i\})$. Then Lemma 3.2 can be restated as: any $\theta(\Pi, I)$ is conjugate to some θ_i under $W(\mathfrak{g}^c, \mathfrak{h}^c)$. Set $\mathfrak{k}_i = \{x \in \mathfrak{g}^c, \theta_i(x) = x\}$, which is called the characteristic subalgebra of θ_i. Set $V_i^c = \{x \in \mathfrak{g}^c, \theta_i(x) = -x\}$.

Lemma 3.3 (Cf. [6]) (1) *If $m_i = 1$, then $\Pi_i = \Pi \backslash \{\alpha_i\}$ is a system of simple roots of $\Delta(\mathfrak{k}_i, \mathfrak{h}^c)$, whose Dynkin Diagram is the subdiagram of Π with α_i omitted. and $\mathrm{ad}_{V_i^c}\mathfrak{k}_i$ has two highest weights: φ and $-\alpha_i$.*

(2) *If $m_i = 2$, then $\Pi_i = \{\alpha_1, \cdots, \hat{\alpha}_i, \cdots, \alpha_l, -\varphi\}$ is a system of simple roots of $\Delta(\mathfrak{k}_i, \mathfrak{h}^c)$, whose Dynkin Diagram is the subdiagram of the Yen Diagram of Π with α_i omitted. and the highest weight of $\mathrm{ad}_{V_i^c}\mathfrak{k}_i$ is $-\alpha_i$.*

Remark 3.1 For the situation of $m_i = 1$, for $\mathrm{ad}_{V_i^c}\mathfrak{k}_i$, $V_i^c = V_1 + \overline{V}_1$, so the Casimir operator of $\mathfrak{k}_i$ on V^c is still $A\,\mathrm{Id}$, where A is a constant.

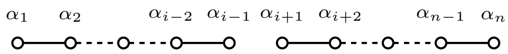
We also need to compute Casimir operator of a irreducible representation. For a simple Lie algebra $\mathfrak{g}^c$, let $V(\lambda)$ be the irreducible representation of g^c with highest weight λ, and let

$\delta = \frac{1}{2}\sum \alpha$, where α ranges from all positive roots of $\Delta(\mathfrak{g}^c, \mathfrak{h}^c)$.

Lemma 3.4 *Let $\mathfrak{g}^c, V(\lambda)$ as above, C be the Casimir operator relative to Killing form. Then*

$$C = (\langle \lambda + \delta, \lambda + \delta \rangle - \langle \delta, \delta \rangle)\mathrm{Id}\,.$$

Now we investigate some examples on $\mathrm{SU}(n+1)$. Set $\theta = \theta_i$, $\tau = \theta_j$ $(i < j)$, we give the Casimir operators which appear in Proposition 2.3. Using Lemma 3.3, we get the Dynkin Diagram of $\mathfrak{k}$ is



Notice that $\mathfrak{k}$ has a 1-dimensional center which is generated by λ_i, where λ_i satisfies $\frac{2\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$. We know

$$\lambda_i = \frac{1}{n+1}((n-i+1)\alpha_1 + \cdots + (i-1)(n-i+1)\alpha_{i-1} + i(n-i+1)\alpha_i + i(n-i)\alpha_{i+1} + \cdots + i\alpha_l),$$

so $\langle \lambda_i, \lambda_i \rangle = \frac{i(n-i+1)}{n+1}$, and the standard base of the center is $\lambda_i' = \sqrt{\frac{n+1}{i(n-i+1)}}\,\lambda_i$. We get $\mathrm{ad}^{\,2}(\lambda_i')|_{\mathfrak{p}} = \frac{n+1}{i(n-i+1)}\mathrm{Id}\,.$

From the Dynkin Diagram of $\mathfrak{k}$ we see that $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2 + R\lambda_i'$ where $\mathfrak{k}_1 = \mathfrak{su}(i)$ and $\mathfrak{k}_2 = \mathfrak{su}(l-i+1)$, and for $\mathrm{ad}_{\,\mathfrak{p}}\mathfrak{k}$, one has

$$\lambda = -\alpha_i, \quad \delta = \sum_{j=1}^{i-1} \frac{j(i-j)\alpha_j}{2} + \sum_{j=i+1}^{n} \frac{(j-i)(n+1-j)\alpha_j}{2},$$

so

$$C_{\mathfrak{k}}|_{\mathfrak{p}} = \Big(n+1+\frac{n+1}{i(n-i+1)}\Big)\mathrm{Id}\,.$$

Repeating this process, we get

$$\begin{aligned}
C_{\mathfrak{k}}|_{\mathfrak{k}^-} &= (2n-2i+2)\mathrm{Id}\,,\\
C_{\mathfrak{k}^+}|_{\mathfrak{k}^-} &= \Big((n-i+1)+\frac{n-i+1}{(j-i)(n-j+1)}\Big)\mathrm{Id}\,,\\
C_{\mathfrak{k}^++\mathfrak{p}^+}|_{\mathfrak{k}^-+\mathfrak{p}^-} &= \Big((n+1)+\frac{n+1}{j(n-j+1)}\Big)\mathrm{Id}\,,\\
C_{\mathfrak{k}^++\mathfrak{p}^+}|_{\mathfrak{p}^+} &= 2j\,\mathrm{Id}\,,\\
C_{\mathfrak{k}^+}|_{\mathfrak{p}^+} &= \Big(j+\frac{j}{i(j-i)}\Big)\mathrm{Id}\,,\\
C_{\mathfrak{k}^++\mathfrak{p}^-}|_{\mathfrak{k}^-+\mathfrak{p}^+} &= \Big((n+1)+\frac{n+1}{(j-i)(n+i-j+1)}\Big)\mathrm{Id}\,,\\
C_{\mathfrak{k}^++\mathfrak{p}^-}|_{\mathfrak{p}^-} &= 2(n-j+i+1)\mathrm{Id}\,,\\
C_{\mathfrak{k}^+}|_{\mathfrak{p}^-} &= \Big((i-j+i+1)+\frac{n-j+i+1}{i(n-j+1)}\Big)\mathrm{Id}\,.
\end{aligned}$$

Remark 3.2 For the situation of $\theta\tau = \theta_{\{i,j\}}$, there exists an $s \in W(\mathfrak{g})$ such that $s(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \hat{\alpha}_i, \cdots, \alpha_n, -\varphi)$ and $s \cdot \theta_{\{i,j\}} = e^{\mathrm{ad}\,H'}$, where H' satisfies: $\langle \alpha_j, H' \rangle =$

$\pi i,\ \langle \alpha_k, H'\rangle = 0\ (k \neq i, j),\ \langle -\varphi, H'\rangle = 0$. Now it returns to the situation which we have discussed.

Now using Theorem 2.1, for the examples we just investigated, we get the necessary and sufficient condition of Einstein metric. The equations that we get is still complicated. We only give the equations when $i + j = n + 1$.

Proposition 3.1 *Let $G = \mathrm{SU}(n+1)$, $\theta = \theta_i,\ \tau = \theta_j,\ i + j = n + 1$. Then g is an Einstein metric if and only if*

$$\begin{cases} \Big(4i - \dfrac{2}{i}\Big)a^2 - 4\Big(i + j - \dfrac{3}{2i} + \dfrac{1}{i-j}\Big)ac - \Big(4i - \dfrac{2}{i}\Big)b^2 + 4\Big(i + j - \dfrac{3}{2i} + \dfrac{1}{i-j}\Big)bc = 0, \\ \Big(2j + \dfrac{3}{j} + \dfrac{1}{i-j} - \dfrac{5}{2i}\Big)a^2 - \Big(4i + 4j + \dfrac{6}{j} - \dfrac{11}{i} + \dfrac{2}{i-j}\Big)ab + \Big(\dfrac{2}{j} + \dfrac{2}{i-j} + \dfrac{1}{i}\Big)ac \\ +\Big(2j - 4i + \dfrac{3}{j} - \dfrac{1}{2i} + \dfrac{1}{i-j}\Big)b^2 + \Big(4i + 4j - \dfrac{5}{i} + \dfrac{6}{i-j} + \dfrac{2}{j}\Big)bc - \Big(2j + \dfrac{5}{j} - \dfrac{3}{2i} + \dfrac{3}{i-j}\Big)c^2 = 0. \end{cases}$$

From the first equation, we get $a = b$ or $c = \dfrac{i - \frac{1}{2i}}{i + j - \frac{3}{2i} + \frac{1}{i-j}}(a + b)$. In case of $a = b$, the second equation turns into

$$8\Big(i - \frac{1}{i}\Big)a^2 - 4\Big(i + j - \frac{1}{i} + \frac{1}{j} + \frac{2}{i-j}\Big)ac + \Big(2j + \frac{5}{j} + \frac{3}{i-j} - \frac{3}{2i}\Big)c^2 = 0.$$

We see that if $i = 1$, then $a = b = \dfrac{2j + \frac{5}{j} + \frac{3}{1-j} - \frac{3}{2}}{4j + \frac{1}{j} + \frac{2}{1-j}}\, c$. If $i \neq 1$, we see that the second equation has two solutions when $j \gg i$. In the second case, the second equation turns into a bivariate quadratic homogeneous equation. We can find infinite many $\{i, j\}$ such that the equation has solutions which satisfy $a \neq b \neq c$, that is, the correspond metrics are non-standard homogenous.

Remark 3.3 In the case of $i + j = n + 1$, which we just discussed, it is easy to get that Einstein metric g is standard homogeneous, that is, $a = b = c$, if and only if $j = 2i$. In this situation, $\theta, \tau, \theta\tau$ are conjugate under Weyl group of $\mathfrak{g}$.

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