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Global Bifurcation of a Perturbed Double-Homoclinic Loop***

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Abstract This paper deals with a kind of fourth degree systems with perturbations. By using the method of multi-parameter perturbation theory and qualitative analysis, it is proved that the system can have six limit cycles.

Keywords Perturbation, Bifurcation, Cubic system, Limit cycle, Hamiltonian system, Homoclinic loop
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1 Introduction and Main Result

Regarding the second part of Hilbert's sixteenth problem, a lot of important results have been achieved on the study of the number and the distribution of limit cycles of polynomial planar vector fields. C. S. Coleman in his survey (cf. [1]) stated that "For $n > 2$, the maximal number of eyes is not known, nor is it known just which complex patterns of eyes within eyes, or eyes enclosing more than a single critical point can exist." Here so called "eye" means the limit cycle. In recent years, the problem of limit cycles and the application were studied for some polynomial systems by the bifurcation theory (cf. [2–5, 9–23]).

For the following Liénard system

$$\begin{cases}\dot{x} = y,\\ \dot{y} = x(1-x^2) - \varepsilon y(a_1 + a_2 x + a_3 x^2),\end{cases} \tag{1.1}$$

it was proved in paper [18] that there are at most three limit cycles if a_1, a_2 and a_3 are analytic functions of ε for $|\varepsilon|$ small. Recently, Han (cf. [5]) studied the global bifurcation of Liénard system

$$\begin{cases}\dot{x} = y,\\ \dot{y} = x(1-bx-x^2) - \varepsilon y(a_1 + a_2 x + a_3 x^2),\end{cases} \tag{1.2}$$

by using the method of analysis and the Poincaré map in a neighborhood of homoclinic and double-homoclinic loop. He found that this system can have four limit cycles.

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In this paper, we consider the following system

$$\begin{cases}\dot{x}=y(c-x),\\ \dot{y}=x(1-x^2)(c-x)+\varepsilon y f(x),\end{cases} \tag{1.3}$$

where $c>\sqrt{2}$, $f(x)=a_1+a_2x+a_3x^2+a_4x^3$. Our main result is as follows.

Theorem 1.1 *There exists a $\bar{c}>\sqrt{2}$, such that for $c>\bar{c}$, $0<|\varepsilon|\ll 1$ and suitable $a_i, i=1,\cdots,4$, the system* (1.3) *can have six limit cycles with the two different distributions* $1+(3,2)$ *and* $2+(2,2)$ (*see Figure* 4(1)–(2)).

2 Perturbation of Homoclinic Loops

When $c>\sqrt{2}$, $x<c$, by using the transformation $t\mapsto \dfrac{t}{c-x}$, it follows from (1.3) that

$$\begin{cases}\dot{x}=y,\\ \dot{y}=x(1-x^2)+\dfrac{\varepsilon y}{c-x}(a_1+a_2x+a_3x^2+a_4x^3).\end{cases} \tag*{$(2.1)_\varepsilon$}$$

Obviously, the unperturbed system $(2.1)_0$ is Hamiltonian, with three singular points $O(0,0)$, $A_i((-1)^{i-1},0)$, $i=1,2$, and $O(0,0)$ is saddle, $A_i((-1)^{i-1},0)$, $i=1,2$ are both centers. The Hamiltonian function is

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{4}x^4=h,\quad h\in\Big[-\frac{1}{4},+\infty\Big), \tag{2.2}$$

where the value $h=-\frac{1}{4}$ corresponds to the centers $A_i((-1)^{i-1},0)$, $i=1,2$, and $h=0$ to the double-homoclinic orbit Γ. Thus $\Gamma=L_1\bigcup L_2$ can be expressed as

$$\begin{aligned} L_1:&\quad y=\pm x\sqrt{1-\frac{x^2}{2}},\qquad 0<x<\sqrt{2},\\ L_2:&\quad y=\pm x\sqrt{1-\frac{x^2}{2}},\qquad -\sqrt{2}<x<0. \end{aligned} \tag{2.3}$$

For $\varepsilon>0$ small enough, the system $(2.1)_\varepsilon$ has separatrices L_k^s, L_k^u near $L_k, k=1,2$, such that $L_1^s\bigcup L_2^s$ and $L_1^u\bigcup L_2^u$ are the stable and unstable manifolds of saddle $O(0,0)$. Let $a=(a_1,a_2,a_3,a_4)$ with $a_4>0$. Then the directed distance from L_k^u to L_k^s is measured by

$$d_k(\varepsilon,a)=\varepsilon N_k M_k(a)+O(\varepsilon^2), \tag{2.4}$$

where N_k is a positive constant, and

$$M_k(a)=\oint_{L_k}\frac{f(x)}{c-x}y\,dx=\sum_{i=0}^{3}A_{k,i+1}a_{i+1},\quad k=1,2 \tag{2.5}$$

with

$$\begin{aligned} A_{1,i+1}&=\oint_{L_1}\frac{x^i}{c-x}y\,dx=2\int_0^{\sqrt{2}}\frac{x^{i+1}}{c-x}\sqrt{1-\frac{x^2}{2}}\,dx=A_{1,i+1}(c),\\ A_{2,i+1}&=\oint_{L_2}\frac{x^i}{c-x}y\,dx=(-1)^{i+1}A_{1,i+1}(-c)=A_{2,i+1}(c),\quad i=0,1,2,3. \end{aligned}$$

Making integral transformation $x = \sqrt{2}\sin t$, we have

$$A_{1,i+1} = 2(\sqrt{2}\,)^{i+1}\int_0^{\frac{\pi}{2}} \frac{\sin^{i+1} t\cos^2 t}{c_1 - \sin t}dt, \quad c_1 = \frac{c}{\sqrt{2}}.$$

Note that

$$\int_0^{\frac{\pi}{2}} \frac{1}{c_1 - \sin t}dt = \frac{1}{\sqrt{c_1^2-1}}\Big(\frac{\pi}{2}+n\Big), \quad \int_0^{\frac{\pi}{2}} \frac{1}{c_1 + \sin t}dt = \frac{1}{\sqrt{c_1^2-1}}\Big(\frac{\pi}{2}-n\Big),$$

where $n = \arctan[c_1^2-1]^{-\frac{1}{2}} \in (0, \frac{\pi}{2})$, we easily obtain

$$\begin{aligned}
A_{11} =&\ 2\sqrt{2}\Big[\frac{\pi}{4}+\frac{\pi}{2}(c_1^2-1)+c_1-c_1\sqrt{c_1^2-1}\Big(\frac{\pi}{2}+n\Big)\Big],\\
A_{12} =&\ 4\Big[\frac{2}{3}+\frac{\pi}{4}c_1+(c_1^2-1)+\frac{\pi}{2}c_1(c_1^2-1)-c_1^2\sqrt{c_1^2-1}\Big(\frac{\pi}{2}+n\Big)\Big],\\
A_{13} =&\ 4\sqrt{2}\Big[\frac{3\pi}{16}+\frac{2}{3}c_1+\frac{\pi}{4}(c_1^2-1)+c_1(c_1^2-1)+\frac{\pi}{2}c_1^2(c_1^2-1)-c_1^3\sqrt{c_1^2-1}\Big(\frac{\pi}{2}+n\Big)\Big],\\
A_{14} =&\ 8\Big[\frac{8}{15}+\frac{3\pi}{16}c_1+\frac{2}{3}(c_1^2-1)+\frac{\pi}{4}c_1(c_1^2-1)+c_1^2(c_1^2-1)\\
&+\frac{\pi}{2}c_1^3(c_1^2-1)-c_1^4\sqrt{c_1^2-1}\Big(\frac{\pi}{2}+n\Big)\Big],\\
A_{21} =&\ 2\sqrt{2}\Big[-\frac{\pi}{4}-\frac{\pi}{2}(c_1^2-1)+c_1-c_1\sqrt{c_1^2-1}\Big(\frac{\pi}{2}-n\Big)\Big],\\
A_{22} =&\ 4\Big[\frac{2}{3}-\frac{\pi}{4}c_1+(c_1^2-1)-\frac{\pi}{2}c_1(c_1^2-1)-c_1^2\sqrt{c_1^2-1}\Big(\frac{\pi}{2}-n\Big)\Big],\\
A_{23} =&\ 4\sqrt{2}\Big[-\frac{3\pi}{16}+\frac{2}{3}c_1-\frac{\pi}{4}(c_1^2-1)+c_1(c_1^2-1)-\frac{\pi}{2}c_1^2(c_1^2-1)-c_1^3\sqrt{c_1^2-1}\Big(\frac{\pi}{2}-n\Big)\Big],\\
A_{24} =&\ 8\Big[\frac{8}{15}-\frac{3\pi}{16}c_1+\frac{2}{3}(c_1^2-1)-\frac{\pi}{4}c_1(c_1^2-1)+c_1^2(c_1^2-1)\\
&-\frac{\pi}{2}c_1^3(c_1^2-1)-c_1^4\sqrt{c_1^2-1}\Big(\frac{\pi}{2}-n\Big)\Big].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
A_{21}A_{12}-A_{11}A_{22} &= \frac{4}{3}\sqrt{2}\,(2c_1^2\pi-\pi-4c_1\sqrt{c_1^2-1}\,n),\\
A_{23}A_{12}-A_{13}A_{22} &= -\frac{\sqrt{2}\pi}{3}\,[2+3c_1^2(-2+\sqrt{c_1^2-1}\,\pi)],\\
A_{24}A_{12}-A_{14}A_{22} &= -\frac{2}{15}c_1\,[c_1^2\pi(2+15\sqrt{c_1^2-1}\,\pi)-64c_1\sqrt{c_1^2-1}\,n-6\pi],\\
A_{24}A_{13}-A_{14}A_{23} &= \frac{8\sqrt{2}}{15}\,[\pi(1+4c_1^2-8c_1^4)+c_1^3\sqrt{c_1^2-1}\,n]
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\lim_{c_1\to+\infty} A_{12}(c_1)\\
=&\ 4\lim_{c_1\to+\infty}\Big[\frac{2}{3}+\frac{\pi}{4}c_1+(c_1^2-1)+\frac{\pi}{2}c_1(c_1^2-1)-c_1^2\sqrt{c_1^2-1}\Big(\frac{\pi}{2}+n\Big)\Big]\\
=&\ 4\lim_{c_1\to+\infty}\Big\{\frac{2}{3}+\frac{\pi}{4}c_1+(c_1^2-1)+c_1\sqrt{c_1^2-1}\Big[\frac{\pi}{2}(\sqrt{c_1^2-1}-c_1)-c_1\arctan\frac{1}{\sqrt{c_1^2-1}}\Big]\Big\}\\
=&\ 4\lim_{c_1\to+\infty}\Big[\frac{2}{3}+\frac{\pi}{4}c_1+(c_1^2-1)-\frac{\pi}{2}\frac{c_1\sqrt{c_1^2-1}}{\sqrt{c_1^2-1}+c_1}-c_1^2\Big]
\end{aligned}$$

$$= 4 \lim_{c_1 \to +\infty} \Big(-\frac{1}{3} + \frac{\pi}{4} \frac{c_1^2 - c_1\sqrt{c_1^2-1}}{\sqrt{c_1^2-1}+c_1} \Big) = -\frac{4}{3},$$

which implies $A_{12}(c_1) < 0$ for c_1 large sufficiently. For $c_1 = 1+0$, we have $A_{12}(1+0) = \frac{2}{3} + \frac{\pi}{4} > 0$. Therefore, there exists a $c_1^* > 1$ such that $A_{12}(c_1^*) = 0$.

Thus, we obtain the following result.

Lemma 2.1 *For $A_{12} \neq 0$ and ε small enough, there exists a function*

$$K_2(a_1, a_3, a_4, \varepsilon) = -\frac{A_{11}}{A_{12}} a_1 - \frac{A_{13}}{A_{12}} a_3 - \frac{A_{14}}{A_{12}} a_4 + O(\varepsilon) \tag{2.7}$$

such that for $\varepsilon > 0$, $d_1 \geq 0$ (< 0) if and only if $A_{12}a_2 \geq (<) A_{12}K_2(a_1, a_3, a_4, \varepsilon)$, and that the system (1.3) *or* $(2.1)_\varepsilon$ *has a homoclinic loop L_1^* near L_1 if and only if $a_2 = K_2(a_1, a_3, a_4, \varepsilon)$.*

Furthermore when $a_2 = K_2(a_1, a_3, a_4, \varepsilon)$, there exists a function

$$K_3(a_1, a_4, \varepsilon) = -\frac{A_{21}A_{12} - A_{11}A_{22}}{A_{23}A_{12} - A_{13}A_{22}} a_1 - \frac{A_{24}A_{12} - A_{14}A_{22}}{A_{23}A_{12} - A_{13}A_{22}} a_4 + O(\varepsilon) \tag{2.8}$$

such that $d_2 \geq 0$ (< 0) if and only if $A_{12}a_3 \leq (>) A_{12}K_3(a_1, a_4, \varepsilon)$, and that the system (1.3) *or* $(2.1)_\varepsilon$ *has a double-homoclinic loop $\Gamma^* = L_1^* \bigcup L_2^*$ near Γ if and only if $a_3 = K_3(a_1, a_4, \varepsilon)$.*

Proof (2.7) follows directly from (2.5). Using the implicit function theorem and substitute $a_2 = K_2(a_1, a_3, a_4, \varepsilon)$ into (2.5), we get

$$M_2(a, \varepsilon) = \frac{A_{21}A_{12} - A_{11}A_{22}}{A_{12}} a_1 + \frac{A_{23}A_{12} - A_{13}A_{22}}{A_{12}} a_3 + \frac{A_{24}A_{12} - A_{14}A_{22}}{A_{12}} a_4 + O(\varepsilon).$$

It follows from the figure of $g_1(c_1)$ that

$$A_{23}A_{12} - A_{13}A_{22} < 0 \quad \text{for } c > \sqrt{2}c_{1,2},$$

where $c_{1,2}$ will be given later. Thus, $d_2 \geq 0$ (< 0) if and only if $A_{12}a_3 \leq (>) A_{12}K_3(a_1, a_4, \varepsilon)$. This completes the proof.

Next, we study the stability of Γ^* under conditions $a_2 = K_2$ and $a_3 = K_3$.

The origin $O(0,0)$ is always a saddle of the system $(2.1)_\varepsilon$. The divergence of $(2.1)_\varepsilon$ at the origin has the value

$$\sigma_0(O_\varepsilon) = (P_x + Q_y)|_{O_\varepsilon} = -\frac{a_1}{c}\varepsilon + O(\varepsilon^2).$$

As we know, the sign of $\sigma_0(O_\varepsilon)$ determines the stability of Γ^*. More precisely, if $\sigma_0(O_\varepsilon) > 0$ (< 0), then the homoclinic loop L_1^*, L_2^* and the double-homoclinic loop Γ^* are unstable (stable) (cf. [6–8]). Hence, we have

Lemma 2.2 *Under the conditions $a_2 = K_2(a_1, a_3, a_4, \varepsilon)$ and $a_3 = K_3(a_1, a_4, \varepsilon)$, there exists a function*

$$K_1(a_4, \varepsilon) = O(\varepsilon), \tag{2.9}$$

such that the homoclinic loop L_1^*, L_2^* and the double-homoclinic loop Γ^* are unstable (*stable*) *for $a_1 < (>) K_1(a_4, \varepsilon)$.*

Lemma 2.3 (Cf. [6]) *Let $a_i = K_i,\ i = 1,2,3$. Then the following results hold:*

(i) *The integral $\sigma_{1k} = \oint_{L_k^*}(P_x + Q_y)dt = \sigma_{1k}(a_4,\varepsilon)$ converges finitely, and L_k^* is stable (unstable) if $\sigma_{1k} < 0\ (> 0)$;*

(ii) $\sigma_{1k} = \oint_{L_k}(P_x + Q_y)dt + O(\varepsilon)$;

(iii) *If $\sigma_{11} + \sigma_{12} > 0\ (< 0)$, then the double-homoclinic loop Γ^* is unstable (stable) outside, where $k = 1,2$.*

Suppose $a_i = K_i,\ i = 1,2,3$. Then we have

$$a_1 + a_2x + a_3x^2 + a_4x^3 = \Big[x^3 - \frac{A_{24}A_{12} - A_{14}A_{22}}{A_{23}A_{12} - A_{13}A_{22}}x^2 + \frac{A_{24}A_{13} - A_{14}A_{23}}{A_{23}A_{12} - A_{13}A_{22}}x\Big]a_4 + O(\varepsilon)$$

and

$$\sigma_{1k} = \oint_{L_k}\frac{a_1 + a_2x + a_3x^2 + a_4x^3}{(c-x)y}dx + O(\varepsilon) = \Delta_k a_4 + O(\varepsilon), \tag{2.10}$$

where

$$\Delta_k = J_{k3} - \frac{A_{24}A_{12} - A_{14}A_{22}}{A_{23}A_{12} - A_{13}A_{22}}J_{k2} + \frac{A_{24}A_{13} - A_{14}A_{23}}{A_{23}A_{12} - A_{13}A_{22}}J_{k1},$$

$$J_{ki} = \oint_{L_k}\frac{x^i}{(c-x)y}dx, \quad k = 1,2,\ i = 1,2,3.$$

Note that

$$J_{1i}(c) = 2(\sqrt{2})^{i-1}\int_0^{\frac{\pi}{2}}\frac{\sin^{i-1}t}{c_1 - \sin t}dt, \quad J_{2i}(c) = (-1)^{i-1}J_{1i}(-c) \quad \text{for } i = 1,2,3.$$

Thus we have

$$J_{11} = 2\int_0^{\frac{\pi}{2}}\frac{1}{c_1 - \sin t}dt = \frac{2}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} + n\Big),$$

$$J_{12} = 2\sqrt{2}\Big[c_1\int_0^{\frac{\pi}{2}}\frac{1}{c_1 - \sin t}dt - 1\Big] = 2\sqrt{2}\Big[-\frac{\pi}{2} + \frac{c_1}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} + n\Big)\Big],$$

$$J_{13} = 4\int_0^{\frac{\pi}{2}}\Big[-\sin t - c_1 + \frac{c_1^2}{c_1 - \sin t}\Big]dt = 4\Big[-1 - \frac{\pi}{2}c_1 + \frac{c_1^2}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} + n\Big)\Big],$$

$$J_{21} = \frac{2}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} - n\Big),$$

$$J_{22} = 2\sqrt{2}\Big[\frac{\pi}{2} + \frac{c_1}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} - n\Big)\Big],$$

$$J_{23} = 4\Big[-1 + \frac{\pi}{2}c_1 + \frac{c_1^2}{\sqrt{c_1^2 - 1}}\Big(\frac{\pi}{2} - n\Big)\Big].$$

Therefore, by (2.6), we have

$$\Delta_1 = \frac{4g_2(c_1)}{5g_1(c_1)\sqrt{c_1^2 - 1}} = \Delta_1(c_1),$$

$$\Delta_2 = \frac{-4g_3(c_1)}{5g_1(c_1)\sqrt{c_1^2 - 1}} = \Delta_2(c_1),$$

$$\Delta_1 + \Delta_2 = \frac{8g_4(c_1)}{5g_1(c_1)\sqrt{c_1^2 - 1}},$$

where

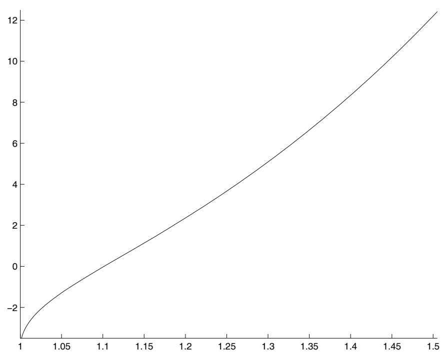
$$g_1(c_1) = 2 + 3c_1^2(\sqrt{c_1^2-1}\,\pi - 2),$$

$$g_2(c_1) = -8c_1\sqrt{c_1^2-1}\,\pi + 16c_1^3\sqrt{c_1^2-1}\,\pi - 2(5\sqrt{c_1^2-1} + \pi + 2n)$$
$$-c_1^4(15\pi + 32n) + c_1^2[32n + 15(2\sqrt{c_1^2-1} + \pi)],$$

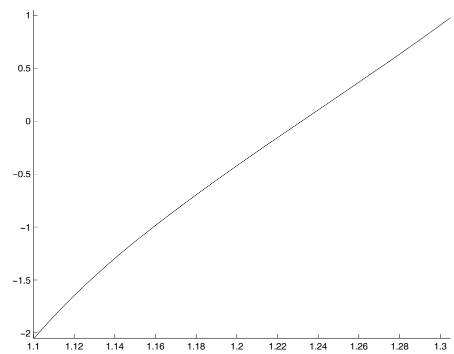
$$g_3(c_1) = -8c_1\sqrt{c_1^2-1}\,\pi + 16c_1^3\sqrt{c_1^2-1}\,\pi + 2(5\sqrt{c_1^2-1} + \pi - 2n)$$
$$+c_1^4(15\pi - 32n) + c_1^2[32n - 15(2\sqrt{c_1^2-1} + \pi)],$$

$$g_4(c_1) = -15c_1^4\pi + 15c_1^2(2\sqrt{c_1^2-1} + \pi) - 2(5\sqrt{c_1^2-1} + \pi).$$

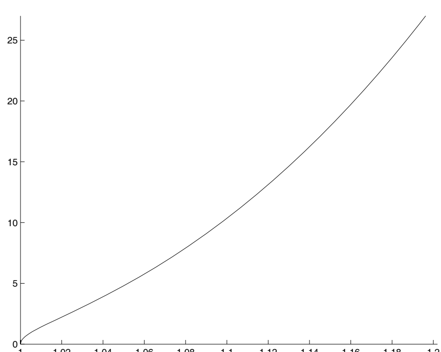
By [24], we know that g_1 and g_2 has unique zero $c_{1,2} = 1.1014386685799062\cdots$ and $c_{1,5} = 1.232010827\cdots$, respectively, and $g_3(c_1) > 0$, $g_4(c_1) < 0$ for $c_1 > 1$ (The figure of $g_i(c_1)$, $i = 1, \cdots, 4$, see Figure 1(1)–(4)).



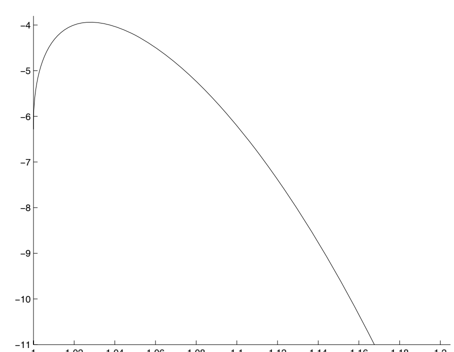
(1) The figure of $g_1(c_1)$



(2) The figure of $g_2(c_1)$



(3) The figure of $g_3(c_1)$



(4) The figure of $g_4(c_1)$

Figure 1

3 Stability Analysis

In this part, firstly, we determine the stability of the foci $P_i((-1)^{i-1}, 0),\ i = 1, 2$ under the conditions $a_i = K_i,\ i = 1, 2, 3$. It is direct that

$$\begin{aligned}\operatorname{div}(1.3)|_{P_i} &= \frac{\varepsilon}{c-(-1)^{i-1}}\{a_1 + a_2(-1)^{i-1} + a_3[(-1)^{i-1}]^2 + a_4[(-1)^{i-1}]^3\} + O(\varepsilon^2)\\ &= \varepsilon a_4\Big\{[(-1)^{i-1}]^3 - \frac{A_{24}A_{12}-A_{14}A_{22}}{A_{23}A_{12}-A_{13}A_{22}}[(-1)^{i-1}]^2 + \frac{A_{24}A_{13}-A_{14}A_{23}}{A_{23}A_{12}-A_{13}A_{22}}(-1)^{i-1}\Big\} + O(\varepsilon^2)\\ &\equiv \varepsilon a_4 f_i + O(\varepsilon^2), \quad i = 1, 2. \end{aligned} \tag{3.1}$$

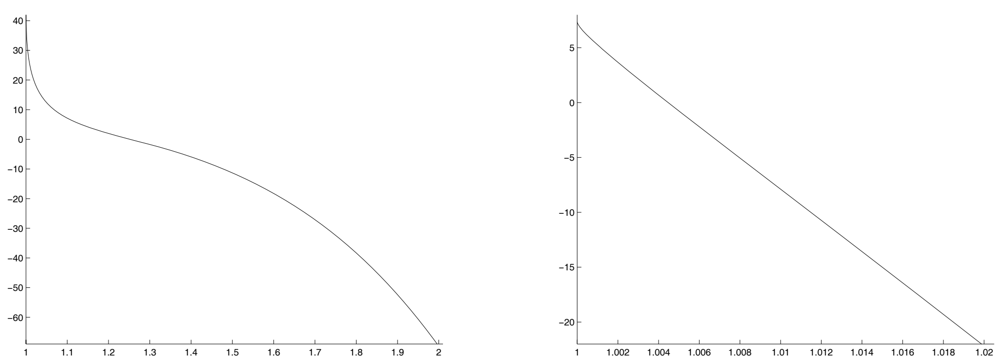
We easily obtain by (2.6)

$$f_1 = \frac{h_1(c_1)}{5\sqrt{2}\pi g_1(c_1)}, \quad f_2 = \frac{h_2(c_1)}{5\sqrt{2}\pi g_1(c_1)},$$

where, for $c_1 > 1$,

$$\begin{aligned} h_1(c_1) =\ & 2\sqrt{2}\,\pi + 12c_1\pi + 64\sqrt{2}\,c_1^4\pi + c_1^2[128\sqrt{c_1^2-1}\,n + \sqrt{2}\,\pi(-62 + 15\sqrt{c_1^2-1}\,\pi)]\\ & -2c_1^3[64\sqrt{2}\,n\sqrt{c_1^2-1} + (2 + 15\pi\sqrt{c_1^2-1})\,\pi],\\ h_2(c_1) =\ & -2\sqrt{2}\,\pi + 12c_1\pi - 64\sqrt{2}\,c_1^4\pi + c_1^2[128\sqrt{c_1^2-1}\,n + \sqrt{2}\,\pi(62 - 15\sqrt{c_1^2-1}\,\pi)]\\ & +2c_1^3[64\sqrt{2}\,n\sqrt{c_1^2-1} - (2 + 15\pi\sqrt{c_1^2-1}\,)\pi]. \end{aligned}$$

By [24], we can know that $h_1(c_1)$ has a unique zero $c_{1,6} = 1.2526178\cdots$, and $h_2(c_1)$ has a unique zero $c_{1,1} = 1.004468630935794\cdots$ (see Figure 2(1)–(2)).



(1) The figure of $h_1(c_1)$ (2) The figure of $h_2(c_1)$

Figure 2

In order to determine the existence and the number of the large limit cycles surrounding all three singular points, we should prove that, $b_0 \in (\sqrt{2}, c)$, the positive orbit γ_B^+ of $(2.1)_\varepsilon$ starting at $B(b_0, 0)$ is bounded. In other words, we need to prove that the first intersection point $\gamma_B^+ \bigcap \{x > 0, y = 0\} = B^*(b^*, 0)$ satisfies $b^* < b_0$. For this purpose, we can take B as the intersection point of the closed curve $H(B) = h$, where $h \to h^* + 0,\ h^* = H(c, 0)$. Then we have

$$\begin{aligned} H(B^*) - H(B) &= \varepsilon a_4 \oint_{\Gamma_h} \frac{y}{c-x}\Big(x^3 - \frac{A_{24}A_{12}-A_{14}A_{22}}{A_{23}A_{12}-A_{13}A_{22}}x^2 + \frac{A_{24}A_{13}-A_{14}A_{23}}{A_{23}A_{12}-A_{13}A_{22}}x\Big)dx + O(\varepsilon^2)\\ &= -\varepsilon a_4 D^*(h, c_1) + O(\varepsilon^2), \end{aligned} \tag{3.2}$$

where

$$D^*(h,c_1)=\oint_{\Gamma_h}x^2ydx+N\oint_{\Gamma_h}xydx+M\oint_{\Gamma_h}ydx-cM\oint_{\Gamma_h}\frac{y}{c-x}dx,$$

and from (2.6),

$$M=c^2-\frac{A_{24}A_{12}-A_{14}A_{22}}{A_{23}A_{12}-A_{13}A_{22}}c+\frac{A_{24}A_{13}-A_{14}A_{23}}{A_{23}A_{12}-A_{13}A_{22}}=-\frac{8}{5[2+3c_1^2(\sqrt{c_1^2-1}\,\pi-2)]},$$

$$N=c-\frac{A_{24}A_{13}-A_{14}A_{23}}{A_{23}A_{12}-A_{13}A_{22}}.$$

The inequality $b^*<b_0$ will be satisfied for $\varepsilon>0$ small and $D^*(h,c)>0$.

Since

$$H(x,y)=\frac{y^2}{2}-\frac{x^2}{2}+\frac{x^4}{4}=\frac{x_0^4}{4}-\frac{x_0^2}{2},$$

where $H(x_0,0)=h$, $x_0>0$, with $\sqrt{2}<x_0<c$, and $\sqrt{c^2-2}<\sqrt{x_0^2+x^2-2}<\sqrt{2(c^2-1)}$ as $h\to h^*$ i.e., $x_0\to c$, we have

$$\oint_{\Gamma_h}\frac{y}{c-x}dx=2\sqrt{2}c\int_{-x_0}^{x_0}\frac{\sqrt{\frac{x_0^4}{4}-\frac{x_0^2}{2}+\frac{x^2}{2}-\frac{x^4}{4}}}{c^2-x^2}dx=2\sqrt{2}c\int_0^{x_0}\frac{\sqrt{x_0^2-x^2}\sqrt{x_0^2+x^2-2}}{c^2-x^2}dx.$$

Noting that

$$0<\sqrt{x_0^2-2}\le\sqrt{x_0^2+x^2-2}\le\sqrt{2x_0^2-2}$$

and

$$\begin{aligned}2c\int_0^{x_0}\frac{\sqrt{x_0^2-x^2}}{c^2-x^2}dx&=\,c\pi+\frac{x_0^2-c^2}{x_0}\Big(\int_0^{\frac{\pi}{2}}\frac{1}{c/x_0-\sin\theta}d\theta+\int_0^{\frac{\pi}{2}}\frac{1}{c/x_0+\sin\theta}d\theta\Big)\\&=\,c\pi-\pi\sqrt{c^2-x_0^2}\to c\pi,\quad\text{as } x_0\to c,\end{aligned}$$

we have, as $x_0\to c$,

$$\begin{aligned}&\frac{\sqrt{2}\pi c^4}{8}\sqrt{c^2-2}\le\oint_{\Gamma_h}x^2ydx\le\frac{\pi c^4}{4}\sqrt{c^2-1},\quad\oint_{\Gamma_h}xydx=0,\\&\frac{\sqrt{2}\pi c^2}{2}\sqrt{c^2-2}\le\oint_{\Gamma_h}ydx\le\pi c^2\sqrt{c^2-1},\\&\sqrt{2}\pi c\sqrt{c^2-2}\le\oint_{\Gamma_h}\frac{y}{c-x}dx\le2\pi c\sqrt{c^2-1}.\end{aligned}$$

Therefore, denoting $D^*(c_1)=\lim\limits_{h\to h^*}D^*(h,c_1)$, we have

$$\begin{aligned}&\frac{c^2}{8}\sqrt{c^2-2}+\frac{M}{2}\sqrt{c^2-2}-\sqrt{2}M\sqrt{c^2-1}\le D^*(c_1)\\&\le\frac{\sqrt{2}c^2}{8}\sqrt{c^2-1}+\frac{\sqrt{2}M}{2}\sqrt{c^2-1}-M\sqrt{c^2-2}\quad\text{for } M>0\end{aligned}$$

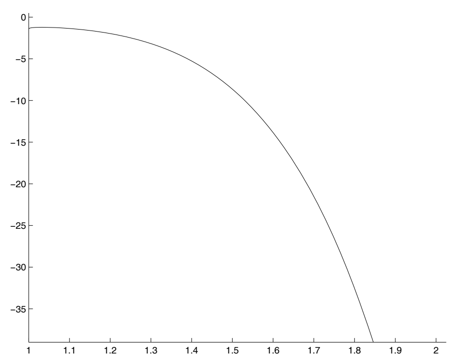
and

$$\begin{aligned}&\frac{c^2}{8}\sqrt{c^2-2}+\frac{\sqrt{2}M}{2}\sqrt{c^2-1}-M\sqrt{c^2-2}\le D^*(c_1)\\&\le\frac{\sqrt{2}c^2}{8}\sqrt{c^2-1}+\frac{M}{2}\sqrt{c^2-2}-\sqrt{2}M\sqrt{c^2-1}\quad\text{for } M<0.\end{aligned}$$

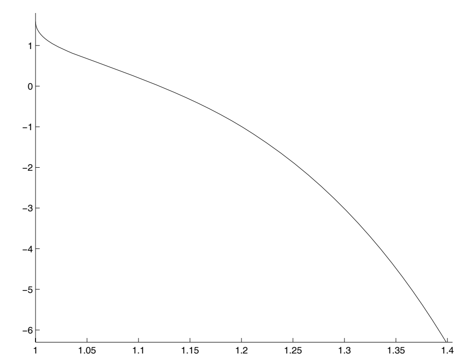
Denote

$$\begin{aligned}
r_1(c_1) &= \frac{c^2}{8M}\sqrt{c^2-2} + \frac{\sqrt{c^2-2}}{2} - \sqrt{2}\sqrt{c^2-1} \\
&= \frac{1}{16\sqrt{2}}\{16\sqrt{c_1^2-1} - 32\sqrt{2c_1^2-1} - 5c_1^2\sqrt{c_1^2-1}\,[2+3c_1^2(\sqrt{c_1^2-1}\,\pi-2)]\}, \\
r_2(c_1) &= \frac{\sqrt{2}c^2}{8M}\sqrt{c^2-1} + \frac{\sqrt{2}\sqrt{c^2-1}}{2} - \sqrt{c^2-2} \\
&= \frac{1}{16\sqrt{2}}\{16\sqrt{2c_1^2-1} - 32\sqrt{c_1^2-1} - 5c_1^2\sqrt{2c_1^2-1}\,[2+3c_1^2(\sqrt{c_1^2-1}\,\pi-2)]\}, \\
r_3(c_1) &= \frac{c^2}{8M}\sqrt{c^2-2} + \frac{\sqrt{2}\sqrt{c^2-1}}{2} - \sqrt{c^2-2} \\
&= \frac{1}{2}\sqrt{4c_1^2-2} - \sqrt{2c_1^2-2} - \frac{5}{32}c_1^2\sqrt{2c_1^2-2}\,[2+3c_1^2(\sqrt{c_1^2-1}\,\pi-2)], \\
r_4(c_1) &= \frac{\sqrt{2}c^2}{8M}\sqrt{c^2-1} + \frac{\sqrt{c^2-2}}{2} - \sqrt{2}\sqrt{c^2-1} \\
&= \frac{\sqrt{2}}{2}\sqrt{c_1^2-1} - \sqrt{4c_1^2-2} - \frac{5}{32}c_1^2\sqrt{4c_1^2-2}\,[2+3c_1^2(\sqrt{c_1^2-1}\,\pi-2)].
\end{aligned}$$

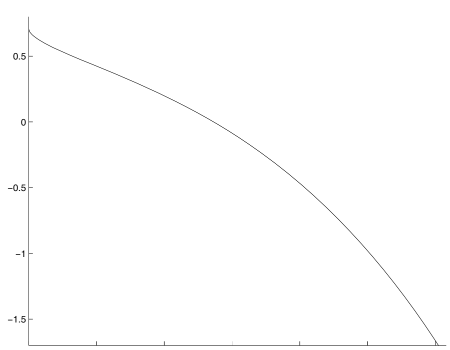
Then by [24], we easily obtain that function $r_2(c_1)$ has unique root $c_1^{(3)} = 1.1206387468394088\cdots$, the root of $r_3(c_1)$ is $c_1^{(4)} = 1.1363325434708866\cdots$, $r_1(c_1) < 0$, and $r_4(c_1) < 0$. The figure of $r_i(c_1),\ i = 1,2,3,4$, see Figure 3(1)–(4).



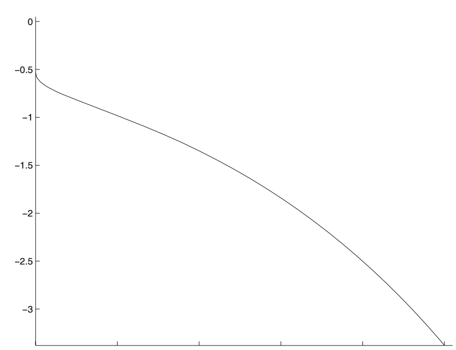
(1) The figure of $r_1(c_1)$



(2) The figure of $r_2(c_1)$



(3) The figure of $r_3(c_1)$



(4) The figure of $r_4(c_1)$

Figure 3

As $M > 0$, we can not determine the sign of $D^*(c_1)$ from the sign of $r_1(c_1), r_2(c_1)$; as $M < 0$ and $c_1 > c_1^{(4)}$, we can get $D^*(c_1) > 0$. Thus, we get the following table as $c_1 > c_1^{(4)}$.

The table to determine the qualitative analysis of system $(2.1)_\varepsilon$

c_1	$(c_1^{(4)}, c_1^{(5)})$	$(c_1^{(5)}, c_1^{(6)})$	$(c_1^{(6)}, +\infty)$
the sign of Δ_1	$-$	$+$	$+$
the sign of Δ_2	$-$	$-$	$-$
the sign of $\Delta_1 + \Delta_2$	$-$	$-$	$-$
the sign of f_1	$+$	$+$	$-$
the sign of f_2	$-$	$-$	$-$
the sign of $D^*(c_1)$	$+$	$+$	$+$

In the table, under assumptions $a_4 > 0$, and $\varepsilon > 0$ small enough, according to the previous analysis, we have that as $\Delta_i < 0$ (> 0), the homoclinic loop L_i^* is stable (unstable) inside; as $\Delta_1 + \Delta_2 < 0$ (> 0), the double-homoclinic loop $\Gamma^* = L_1^* \bigcup L_2^*$ is stable (unstable) outside; as $f_i < 0$ (> 0), the focus A_i is stable (unstable); and as $D^*(c_1) > 0$ (< 0), the trajectories are bounded (unbounded) on the left of $x = c$, where $i = 1, 2$.

Now, we complete the proof of the main result. For convenience, we give the assumptions that $a_4 > 0$ and $\varepsilon > 0$ small enough.

By (2.10), (3.1) and (3.2), Lemma 2.3 and according to the table, for $c_1 \in (c_1^{(4)}, c_1^{(5)})$, the trajectories of the system $(2.1)_\varepsilon$ are bounded on the left of $x = c$, the focus A_1 is unstable and A_2 is stable, but the homoclinic loop L_1^*, L_2^* and the double-homoclinic $\Gamma^* = L_1^* \bigcup L_2^*$ are all stable. We can get that there is a small unstable limit cycle $L_2^{(1)}$ inside L_2^*.

For $a_4 > 0$ fixed and small enough, according to Lemma 2.2, when $a_1 > K_1(\varepsilon, a_4)$, and $0 < |a_1 - K_1(\varepsilon, a_4)| \ll a_4$, the double-homoclinic loop Γ^* and homoclinic loop L_1^*, L_2^* change their stabilities into unstable (cf. [6–8]). Then there is a large stable limit cycle $\Gamma^{(1)}$ which appears near and outside the double-homoclinic loop Γ^*, with two small stable limit cycle $L_1^{(1)}, L_2^{(2)}$ appeared near inside L_1^*, L_2^* and outside $L_2^{(1)}$.

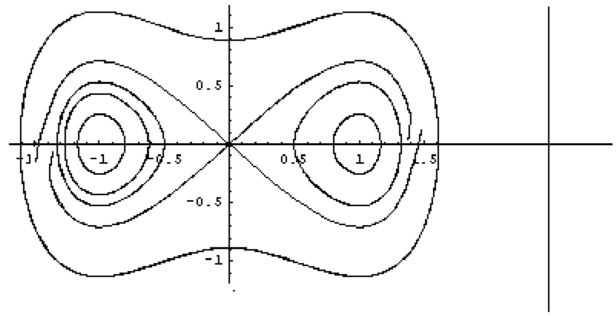
For fixed a_1, a_4 small, according to Lemma 2.1, when

$$0 < K_3(\varepsilon, a_1, a_4) - a_3 \ll |a_1 - K_1(\varepsilon, a_4)| \ll a_4,$$

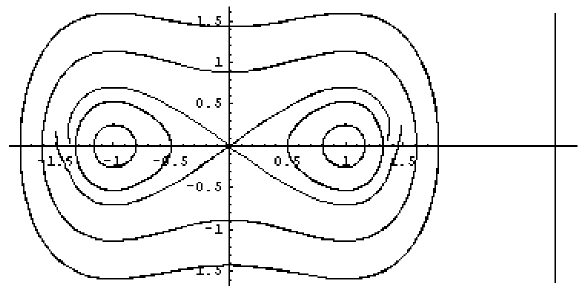
the homoclinic loop L_2^* breaks up and an unstable limit cycle $L_2^{(3)}$ appears near L_2^* and outside $L_2^{(2)}$; when $0 < a_3 - K_3(\varepsilon, a_1, a_4) \ll |a_1 - K_1(\varepsilon, a_4)| \ll a_4$, L_2^* breaks up and a large unstable limit cycle $\Gamma^{(2)}$ appears near Γ^* and inside $\Gamma^{(1)}$. Finally, for fixed a_1, a_3, a_4, if

$$0 < a_2 - K_2(\varepsilon, a_1, a_3, a_4) \ll |K_3(\varepsilon, a_1, a_4) - a_3| \ll |a_1 - K_1(\varepsilon, a_4)| \ll a_4$$

holds, then the homoclinic loop L_1^* breaks up, and a small stable limit cycle $L_1^{(2)}$ appears near L_1^* and outside $L_1^{(1)}$. The distributions of the system (1.3) are $1 + (3, 2)$ and $2 + (2, 2)$ (see Figure 4(1)–(2)).



(1) The figure as $c_1 \in (c_1^{(4)}, c_1^{(5)}) \bigcup (c_1^{(5)}, c_1^{(6)})$, and distribution of $1 + (3, 2)$



(2) The figure as $c_1 \in (c_1^{(4)}, c_1^{(5)}) \bigcup (c_1^{(5)}, c_1^{(6)})$, and distribution of $2 + (2, 2)$

Figure 4

Using the similar methods, when $c_1 \in (c_1^{(5)}, c_1^{(6)})$, we get the same result of the limit cycles and the distributions as $c_1 \in (c_1^{(4)}, c_1^{(5)})$, and when $c_1 \in (c_1^{(6)}, +\infty)$, we get five limit cycles and the distributions are $1 + (3, 1)$ and $2 + (2, 1)$.

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