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# A Lower Bound on Unknotting Number***

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**Abstract** In this paper the authors use a modified Wirtinger presentation to give a lower bound on the unknotting number of a knot in $S^3$.

**Keywords** Unknotting number, Knot, Commutator subgroup
**2000 MR Subject Classification** 57M05, 57M25

## 1 Introduction

Let $K$ be a knot in $S^3$, and for each regular projective diagram $D_K$ of $K$, it is possible to make a crossing change: transform overcrossing to undercrossing or vice versa. Denote by $u(D_K)$ the smallest number of such changes required in $D_K$ to obtain the unknot. The unknotting number $u(K)$ of a knot $K$ is the smallest number of such changes required to obtain the unknot, the minimum taken over all regular projections. Note that $u(K)$ can not always be realized on the minimal crossing regular projective diagram of $K$. According to the classical definition, we could after each change do an ambient isotopy, then perform next change in the new projection, etc., and continue in this manner until the unknot is obtained. According to the standard definition, we must perform all changes in a single (fixed) projection of $K$. These two definitions are equivalent (see [1, p.58]).

Let $G$ be the fundamental group of $S^3 - N(K)$, and $G'$ be the commutator subgroup of $G$. Then $G'$ is the normal subgroup of $G$ such that

$$G/G' = Z.$$

$G$ is the semi-product of $Z$ and $G'$, that is, there is a homomorphism from $Z$ to $\mathrm{Aut}(G')$ as in [2]. If there are $n$ elements, say $x_1, x_2, \cdots x_n$, in $G'$ such that $G'$ is the normal closure of $x_1, x_2, \cdots x_n$ in $G$, that is,

$$G' = \langle x_1, x_2, \cdots, x_n \rangle^G,$$

and if $n$ is minimal along all such presentations of $G'$ in $G$, then we define $a(K) = n$. Since $K$ is trivial in $S^3$ iff $\pi_1(S^3 - N(k))$ is infinite cyclic, so

$$a(K) = 0 \text{ iff } K \text{ is trivial in } S^3.$$

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In this paper, we show that $a(K)$ is a lower bound of $u(K)$, that is,

**Theorem 1** $u(K) \geq a(K)$.

Let $K_1$, $K_2$ be two nontrivial knots in $S^3$, and

$$K = K_1 \sharp K_2$$

be the connected sum of $K_1$ and $K_2$. By an observation, we have the following proposition:

**Proposition 2** $a(K_1 \sharp K_2) \geq \max\{a(K_1), a(K_2)\}$.

## 2 Modified Wirtinger Presentation and the Proof of the Theorem

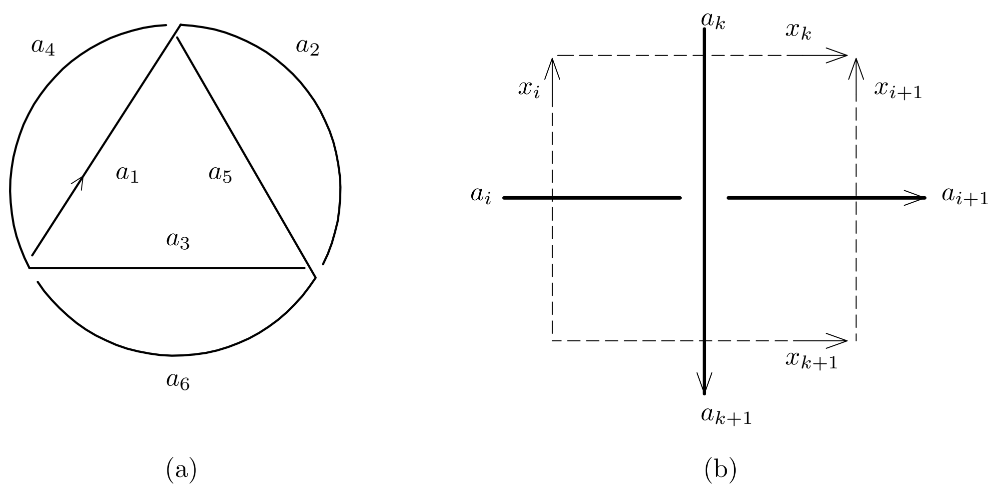


Figure 1

As we all know, Wirtinger gave a presentation of the fundamental group of the complement of a knot in $S^3$ as in [3]. Now we give a modified Wirtinger presentation as follows.

Let $K$ be a knot in $S^3$ with a given orientation and $D(K)$ be a projection diagram of $K$ with $n$ crossings, say $v_1, \ldots, v_n$. Now the $n$ crossings separate $D(K)$ into $2n$ arcs, we denote the $2n$ arcs by $a_1, a_2, \cdots, a_{2n}$, so that $a_i$ connects with $a_{i-1}$ and $a_{i+1}$ (mod $2n$) as in Figure 1(a). Then we obtain a presentation of $\pi_1(S^3 - \operatorname{int} N(K))$ such that each arc $a_i$ induces a generator, denoted by $x_i$, and each crossing $v_j$ induces two relations $C_j$ and $A_j$, where $C_j$ is $x_{k+1}x_{i+1} = x_i x_k$ , and $A_j$ is $x_k = x_{k+1}$ for some $i$ and $k$ as in Figure 1(b).

Comparing with the Wirtinger presentation, note that in Wirtinger presentation, $n$ crossings separate $D(K)$ into just $n$ arcs, and $C_j$ is the proper relation given by the crossing in the Wirtinger presentation, and $A_j$ is just the relation that identifies $x_k$ and $x_{k+1}$. Since they are the same generator in the Wirtinger presentation. So this modified Wirtinger presentation gives a presentation of $\pi_1(S^3 - \operatorname{int} N(K))$ such that

$$\pi_1(S^3 - \operatorname{int} N(K)) = \langle x_1, x_2, \cdots, x_{2n} \mid C_1, C_2, \cdots, C_n, A_1, A_2, \cdots, A_n \rangle.$$

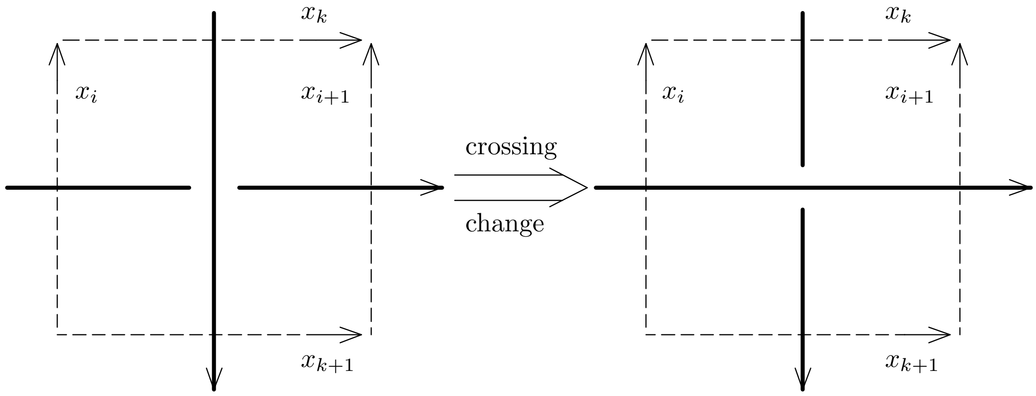


Figure 2

**Proof of Theorem 1** Let $B_j$ be the relation $x_i = x_{i+1}$ for the crossing $v_j$ in which $C_j$ is $x_{k+1}x_{i+1} = x_i x_k$ and $A_j$ is $x_k = x_{k+1}$. Then, after doing a crossing change on $v_j$, $A_j$ transforms to $B_j$, but $C_j$ is the same, as in Figure 2. Since, after doing $u(D(K)) = u$ times crossing changes, we obtain a trivial knot, so

$$\langle x_1, x_2, \cdots, x_{2n} \mid C_1, \cdots, C_n, B_{j_1}, \cdots, B_{j_u}, A_{j_{u+1}}, \cdots, A_{j_n} \rangle$$

is an infinite cyclic group, where

$$\{j_1, \cdots, j_n\} = \{1, 2, \cdots, n\}.$$

Since $[x_i]$ is the generator of

$$H_1(S^3 - N(K)) = G/G' = Z,$$

so

$$\langle x_1, \cdots, x_{2n} \mid C_1, \cdots, C_n, B_{j_1}, \cdots, B_{j_u}, A_{j_1}, \cdots, A_{j_u}, \cdots, A_{j_n} \rangle$$

is also the infinite cyclic group. Since $B_{j_1}, B_{j_2}, \cdots, B_{j_u} \in G'$, $G' = \langle B_{j_1}, \cdots, B_{j_u} \rangle^G$, hence

$$u(D_K) \geq a(K) \quad \text{for any } D_K.$$

That means $u(K) \geq a(K)$.

**Proof of Proposition 2** Let $\pi_1(S^3 - N(K_1)) = G_1$, and $\pi_1(S^3 - N(K_2)) = G_2$. Then

$$\pi_1(S^3 - N(K_1 \sharp K_2)) = G = \langle G_1 * G_2 \mid ab^{-1} \rangle,$$

where $a$ is the generator of $H_1(S^3 - N(K_1))$, and $b$ is the generator of $H_1(S^3 - N(K_2))$. Let $a(K_1) = t$, $a(K_2) = s$. We may assume that $t \geq s$. Now suppose that

$$G_1' = \langle r_1^1, r_2^1 \cdots, r_t^1 \rangle^{G_1}, \quad G_2' = \langle r_1^2, r_2^2 \cdots, r_s^2 \rangle^{G_2}.$$

Suppose that there are $\alpha$ elements $r_1, r_2 \cdots, r_\alpha$ in $G^{'}$ such that

$$G' = \langle r_1, \cdots, r_\alpha \rangle^G.$$

Note that $G^{'} = G_1^{'} * G_2^{'}$ as in [2], and

$$\langle G \mid r_1^2, r_2^2, \cdots, r_s^2 \rangle = G_1, \quad \langle G \mid r_1^2, \cdots, r_s^2, r_1, \cdots, r_\alpha \rangle = \langle G^1 \mid r_1^\star, \cdots, r_\alpha^\star \rangle = Z,$$

where $r_i^\star$ is the element in $G_1^{'}$ induced by $r_i$. So $\alpha \geq t$.

## 3 Remark

Y. Nakanishi [4] gave a lower bound of unknotting number, that is, he defined a knot invariant $m(K)$, where $M_K$ is the universal Abelian covering space of $S^3 - N(K)$, $\Lambda = Z[t, t^{-1}]$. Then $H_1(M_K)$ has a $\Lambda$-module structure, $m(K)$ is the minimal number of generators of $H_1(M_K)$ as $\Lambda$-module.

In fact, it is easy to see $a(K) \geq m(K)$. But for many knots, for example, double knots with twisting number zero, $m(K)$ is zero. Since $a(K)$ is not zero for nontrivial knots, so $a(K)$ is strictly larger than $m(K)$ for many cases.

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