Chin. Ann. Math.
27B(4), 2006, 441–458
DOI: 10.1007/s11401-004-0365-0

Chinese Annals of Mathematics, Series B

Maslov P-Index Theory for a Symplectic Path with Applications**

Chungen LIU*

Abstract The Maslov P-index theory for a symplectic path is defined. Various properties of this index theory such as homotopy invariant, symplectic additivity and the relations with other Morse indices are studied. As an application, the non-periodic problem for some asymptotically linear Hamiltonian systems is considered.

Keywords Hamiltonian system, Symplectic path, Maslov P-index, Non-periodic boundary problem
2000 MR Subject Classification 58F05, 58E05, 34C25, 58F10

1 Introduction

Morse index theory has been playing a very important role in the nonlinear problems such as existence, multiplicity and stability problems of closed geodesics in a Riemannian manifold. During the past almost 30 years, the studying of the existence and multiplicity of periodic solutions of nonlinear Hamiltonian was one of the important directions in the field of Hamiltonian dynamics. In this period a great number of research papers appeared in this and related areas, and many aspects of critical point theory were applied to the variational study of Hamiltonian systems. It is natural to apply the Morse theoretical method to the problems involving various solutions of nonlinear Hamiltonian systems. Since 1980, two different index theories for periodic solutions of nonlinear Hamiltonian systems have appeared. One index theory was developed by I. Ekeland in 1980's for convex Hamiltonian systems. A beautiful systematic treatment of his index theory was given in his celebrated book [6]. Another index theory is a classification of general linear Hamiltonian system with periodic coefficients (without convexity). This index theory began with the work of H. Amann and E. Zehnder in [1]. They established the corresponding index theory for linear Hamiltonian systems with constant coefficients. After that many mathematicians worked on this problem (cf. [2, 8]).

The linearized system of a nonlinear Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)) \tag{1.1}$$

at a solution $x(t)$ is a linear Hamiltonian system

$$\dot{z}(t) = JB(t)z(t), \tag{1.2}$$

Manuscript received August 26, 2004. Revised April 20, 2005.
*School of Mathematics and LPMC, Nankai University, Tianjin 300071, China. E-mail: liucg@nankai.edu.cn
**Project supported by the National Natural Science Foundation of China (No.10531050) and FANEDD.

where $B(t) = H''(t, x(t))$, and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix. I_n is the identity in $\mathbb{R}^n$. The fundamental solution of this linear system with continuous symmetric coefficients is a path in the symplectic matrix group $\mathrm{Sp}(2n)$ starting from the identity. Here the symplectic group is defined by

$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(\mathbb{R}^{2n}) \,|\, M^T JM = J\},$$

where M^T denotes the transpose of M. We define the set of symplectic paths by

$$\mathcal{P}(2n) = \{\gamma \in C([0,1], \mathrm{Sp}(2n) \,|\, \gamma(0) = I_{2n})\}.$$

For any $\gamma \in \mathcal{P}(2n)$, we define $\nu(\gamma) = \dim \ker_{\mathbb{C}}(\gamma(1) - I)$. If $\nu(\gamma) = 0$, we say that the symplectic path γ is degenerate, and non-degenerate otherwise.

In their celebrated paper [5], C. Conley and E. Zehnder defined an index $i(\gamma) \in \mathbb{Z}$ for any non-degenerate path $\gamma \in \mathcal{P}(2n)$ with $n \geq 2$, i.e., the so called Conley-Zehnder index. This index theory was further defined for the non-degenerate paths in $\mathrm{Sp}(2)$ in [9]. The index theory for degenerate linear Hamiltonians was defined by Y. Long in [10] and C. Viterbo in [14]. Then in [11] this index theory was further extended to any paths in $\mathcal{P}(2n)$. For any path $\gamma \in \mathcal{P}(2n)$, we call the index pair

$$(i(\gamma), \nu(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2n\}$$

the Maslov-type index. If $\gamma(t)$ is the fundamental solution of the linear system (1.2), we denote the index pair of γ also by $(i(B), \nu(B))$. It is a classification of the linear Hamiltonian systems. If $x(t)$ is a 1-periodic solution of the nonlinear Hamiltonian system (1.1) with $H(t+1, x) = H(t, x)$ for any (t, x) and $B(t) = H''(t, x(t))$, we denote $(i(x), \nu(x)) = (i(B), \nu(B))$ in this case.

The main purpose of this paper is to generalize the Maslov-type index theory for symplectic paths to a so called Maslov P-index theory which is suitable to the problem of a nonlinear Hamiltonian system with a "P" boundary condition.

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), \\ x(1) = Px(0). \end{cases} \tag{1.3}$$

We will define the Maslov P-index pair $(i_P(\gamma), \nu_P(\gamma))$ for any symplectic path $\gamma \in \mathcal{P}(2n)$, where the nullity part $\nu_P(\gamma)$ is simply defined by

$$\nu_P(\gamma) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(1) - P),$$

and the definition of the rotation part $i_P(\gamma)$ is not so simple as the definition of the nullity part. It is defined by several steps in Section 2 below (from Definition 2.2 to 2.6). This index theory possesses many beautiful properties such as the homotopy invariant, the symplectic additivity and so on. We set out them in Section 3 below. For the purpose of using this index theory to the studying of the P-boundary problem of a nonlinear Hamiltonian system, we should bring forth the relation of the Morse index theory of the direct action functional restricted to some finite dimensional subspace by the methods of the Galerkin approximation or the methods of saddle point reduction with the Maslov P-index theory. The relation of Maslov P-index theory with the Morse index theory by using the Galerkin approximation methods is formulated in Section 4 below, and that by using the saddle point reduction methods is formulated in Section 5 of this paper. If we consider the second order Hamiltonian systems, the Morse index of the direct functional at a solution with P-boundary condition is finite and it is the Maslov P-index of the symplectic path (up to a constant), which is the fundamental solution of the linearization of the system. This is proved in Section 6. As natural applications, in Section 7, we consider the existence and multiplicity problem of the asymptotically linear Hamiltonian with P-boundary condition.

2 Maslov *P*-Index Theory for a Symplectic Path

We consider the following problem

$$\begin{cases} \dot{z}(t) = JB(t)z, \\ z(1) = Pz(0), \end{cases} \tag{2.1}$$

where $P \in \mathrm{Sp}(2n) = \{M \in \mathrm{Gl}(\mathbb{R}^{2n}) \mid M^T JM = J\}$, $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, and $B \in C(\mathbb{R}, \mathcal{L}(\mathbb{R}^{2n}))$ is a symmetric matrix function with $B(t+1) = (P^T)^{-1}B(t)P^{-1}$. Suppose $\gamma(t)$ is the fundamental solution of the linear Hamiltonian system $\dot{z}(t) = JB(t)z(t)$, i.e., $\gamma(0) = I_{2n}$ the identity matrix, and $\dot{\gamma}(t) = JB(t)\gamma(t)$. Then $\gamma \in C([0,1], \mathrm{Sp}(2n))$ is a symplectic path with $\gamma(0) = I_{2n}$. In this section we will define an index theoty for a symplectic path starting from the identity with the P-boundary condition. We denote the set of symplectic paths starting from identity by $\mathcal{P}(2n) = \{\gamma \in C([0,1], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\}$. For any path $\gamma \in \mathcal{P}(2n)$, we now define the index pair $(i_P(\gamma), \nu_P(\gamma))$ of γ with P-boundary condition. Firstly, we define

$$\nu_P(\gamma) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(1) - P).$$

We will define $i_P(\gamma)$ by several steps. If $P = I_{2n}$, the boundary condition is exactly the periodic condition. In this case we define $i_I(\gamma)$ to be the Maslov-type index of the symplectic path γ (cf. [11, 12]).

We now introduce some notations. For $M \in \mathrm{Sp}(2n)$, define

$$\begin{aligned} D_1(M) &= (-1)^{n-1} \det(M - I_{2n}), \\ \mathrm{Sp}(2n)^{\pm} &= \{M \in \mathrm{Sp}(2n) \mid \pm D_1(M) < 0\}, \\ \mathrm{Sp}(2n)^* &= \mathrm{Sp}(2n)^+ \cup \mathrm{Sp}(2n)^-, \quad \mathrm{Sp}(2n)^0 = \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^*. \end{aligned}$$

For any two $2k_i \times 2k_i$ matrices of square form, $M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ with $i = 1, 2$, the $\diamond$-product (or symplectic direct sum) of M_1 and M_2 is defined to be the $2(k_1 + k_2) \times 2(k_1 + k_2)$ matrix

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product of M.

Let $D(a) = \mathrm{diag}(a, a^{-1}) \in \mathrm{Sp}(2)$ for $a \in \mathbb{R} \setminus \{0\}$, and define

$$M_n^+ = D(2)^{\diamond n}, \quad M_n^- = D(-2) \diamond D(2)^{\diamond(n-1)}.$$

We have $M_n^+ \in \mathrm{Sp}(2n)^+$ and $M_n^- \in \mathrm{Sp}(2n)^-$.

As well known, every $M \in \mathrm{Sp}(2n)$ has unique polar decomposition $M = AU$ with $A = (MM^T)^{1/2}$, $U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$ and $u = u_1 + \sqrt{-1}\,u_2$ is a unitary matrix. If $\gamma(t) = A(t)U(t)$, $t \in [0,1]$ is a continuous symplectic path, there exists a continuous real function $\Delta(t)$ satisfying $\det u(t) = \exp(\sqrt{-1}\Delta(t))$. We define $\Delta_1(\gamma) = \Delta(1) - \Delta(0) \in \mathbb{R}$ which depends only on γ. Particularly, if $\{\gamma(0), \gamma(1)\} \subseteq \{M_n^+, M_n^-\}$, we have $\frac{1}{\pi}\Delta_1(\gamma) \in \mathbb{Z}$.

Definition 2.1 *Given two paths γ_0 and $\gamma_1 \in C([0,1], \mathrm{Sp}(2n))$, we say that they are homotopic if there exists a map $\delta \in C([0,1]\times[0,1], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$ and both $\dim\ker(\delta(s,0) - I_{2n})$ and $\dim\ker(\delta(s,1) - I_{2n})$ are constant for $s \in [0,1]$. We write $\gamma_0 \sim \gamma_1$.*

For any symplectic path $\gamma \in C([0,1], \mathrm{Sp}(2n))$ with $\{\gamma(0), \gamma(1)\} \subset \mathrm{Sp}(2n)^*$, there exists a symplectic path $\gamma_1 \in C([0,1], \mathrm{Sp}(2n))$ with $\{\gamma_1(0), \gamma_1(1)\} \subset \{M_n^+, M_n^-\}$ such that $\gamma \sim \gamma_1$. One can construct γ_1 by connecting $\gamma(0)$ and $\gamma(1)$ respectively with one of the element of $\{M_n^+, M_n^-\}$ within $\mathrm{Sp}(2n)^*$. It is easy to see that $\frac{1}{\pi}\Delta_1(\gamma_1) \in \mathbb{Z}$ is independent of the choice of homotopic path γ_1. In fact, for any two homotopic paths $\gamma_0 \sim \gamma_1$ with fixed end points belonging to $\{M_n^+, M_n^-\}$, there holds $\frac{1}{\pi}\Delta_1(\gamma_0) = \frac{1}{\pi}\Delta_1(\gamma_1)$.

Definition 2.2 *For a path $\gamma \in C([0,1], \mathrm{Sp}(2n))$ and a matrix $P \in \mathrm{Sp}(2n)$ with $\{P^{-1}\gamma(0), P^{-1}\gamma(1)\} \subset \mathrm{Sp}(2n)^*$, the Maslov P-index of γ is defined by*

$$i_P(\gamma) = \frac{1}{\pi}\Delta_1(\gamma_1),$$

where $\gamma_1 \in C([0,1], \mathrm{Sp}(2n))$ with $\{\gamma_1(0), \gamma_1(1)\} \subset \{M_n^+, M_n^-\}$ such that $P^{-1}\gamma \sim \gamma_1$.

Roughly speaking, the P-index of γ in some sense is the algebraic intersection number of the symplectic path γ with the P-singular set $\mathrm{Sp}(2n)_P^0 = \{M \in \mathrm{Sp}(2n) \mid \det(M - P) = 0\}$.

Definition 2.3 *For a path $\gamma \in C([0,1], \mathrm{Sp}(2n))$ and a matrix $P \in \mathrm{Sp}(2n)$ with $P^{-1}\gamma(1) \in \mathrm{Sp}(2n)^0$ and $P^{-1}\gamma(0) \in \mathrm{Sp}(2n)^*$, the Maslov P-index of γ is defined by*

$$i_P(\gamma) = \inf\{i_P(\beta) \mid \beta \in \mathcal{P}(2n),\ P^{-1}\beta(1) \in \mathrm{Sp}(2n)^* \text{ and } \beta \text{ is sufficiently } C^0 \text{ close to } \gamma\}.$$

Here the infimum is achieved by a path which is adjacent to γ with a very short path from $\gamma(1)$ to a nearby point belonging to $\mathrm{Sp}(2n)^*$. We note that the space $\mathrm{Sp}(2n)^*$ possesses two path connected components $\mathrm{Sp}(2n)^+$ and $\mathrm{Sp}(2n)^-$, and the singular set $\mathrm{Sp}(2n)^0$ is codimension 1 smooth submanifold of $\mathrm{Sp}(2n)$. $\frac{d}{dt}\big|_{t=0}(M\exp(Jt))$, with $M \in \mathrm{Sp}(2n)^0$, forms a transverse structure of $\mathrm{Sp}(2n)^0$ in $\mathrm{Sp}(2n)$ (cf. [12]).

Definition 2.4 *For a path $\gamma \in \mathcal{P}(2n)$ and a matrix $P \in \mathrm{Sp}(2n) \setminus \{I_{2n}\}$ with $P^{-1}\gamma(0) \in \mathrm{Sp}(2n)^0$, the Maslov P-index of γ is defined by*

$$i_P(\gamma) = \max\{i_P(\beta) \mid \beta \in C([0,1], \mathrm{Sp}(2n)), P^{-1}\beta(0) \in \mathrm{Sp}(2n)^*,\ \beta \text{ is sufficiently } C^0 \text{ close to } \gamma\}.$$

Here the maximum is achieved by a path which is adjacent to γ with a very short path from $\gamma(0)$ to a nearby point belonging to $\mathrm{Sp}(2n)^*$ but with the opposite direction.

Definition 2.5 *For a path $\gamma \in \mathcal{P}(2n)$ and $P = I_{2n}$ with $\gamma(1) \in \mathrm{Sp}(2n)^*$, the Maslov-type index of γ is defined by*

$$i_1(\gamma) = i_I(\gamma_1),$$

where γ_1 *is a path defined by*

$$\gamma_1(t) = \begin{cases} D(2-2t)^{\diamond n}, & t \in \Big[0, \dfrac{1}{2}\Big], \\ \gamma(2t-1), & t \in \Big(\dfrac{1}{2}, 1\Big]. \end{cases}$$

We recall that here $D(a) = \mathrm{diag}(a, a^{-1})$ and $i_I(\gamma_1)$ is defined by Definition 2.2.

Definition 2.6 *For a path* $\gamma \in \mathcal{P}(2n)$ *and* $P = I_{2n}$ *with* $\gamma(1) \in \mathrm{Sp}(2n)^0$*, the Maslov-type index of* γ *is defined by*

$$i_1(\gamma) = \inf\{i_1(\beta) \,|\, \beta \in \mathcal{P}(2n),\ \beta(1) \in \mathrm{Sp}(2n)^*,\ \beta \text{ is sufficiently } C^0 \text{ close to } \gamma\}.$$

Remark For any symplectic path $\gamma \in \mathcal{P}(2n)$ and any symplectic matrix $P \in \mathrm{Sp}(2n)$, the Maslov P-index $i_P(\gamma)$ of γ is well defined by Definitions 2.2–2.6.

Definition 2.7 *If* $\gamma \in \mathcal{P}(2n)$ *is the fundamental solution of the linear Hamiltonian system* (2.1)*, we say that the Maslov* P*-index of* γ *is the index of the linear Hamiltonian systems of* (2.1) *and denote it by* $(i_P(B), \nu_P(B))$.

Consider a nonlinear Hamiltonian system

$$\dot{x} = JH'(t, x),$$

where $H \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfying $H(t+1, Px) = H(t, x)$, $\forall (t, x)$ and $H'(t, x)$ is the gradient of H with respect to the variables x. Here $P \in \mathrm{Sp}(2n)$ is a fixed matrix. It is clear that $P^T H''(t+1, Px)P = H''(t, x)$. We consider a solution of the Hamiltonian system with the condition $x(1+t) = Px(t)$, i.e., a solution of

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), \\ x(1) = Px(0). \end{cases} \tag{2.2}$$

Linearizing the Hamiltonian system at the solution x we get a linear Hamiltonian system

$$\dot{y} = JB(t)y,$$

where $B(t) = H''(t, x(t))$ satisfies $B(1+t) = (P^{-1})^T B(t) P^{-1}$. Suppose $\gamma(t)$ is the fundamental solution of this linear Hamiltonian systems. Then we say that the Maslov P-index of γ is the index of the solution x of the nonlinear Hamiltonian system (2.2), and denote it by $(i_P(x), \nu_P(x))$.

3 The Properties of the Maslov *P*-Index Theory

We define the following two notations

$$\mathrm{Sp}(2n)_P^0 = \{M \in \mathrm{Sp}(2n) \,|\, \det(M - P) = 0\}, \quad \mathrm{Sp}(2n)_P^* = \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)_P^0.$$

The following results follow from [12] by some modifications. The proofs are omitted.

Proposition 3.1 *For $P \in \mathrm{Sp}(2n)$, let $\gamma \in \mathcal{P}(2n)$ with $\gamma(1) \in \mathrm{Sp}(2n)_P^0$. Then for any path α and $\beta \in \mathcal{P}(2n)$ which are sufficiently C^0-close to γ with $\alpha(1) \in \mathrm{Sp}(2n)_P^*$ and $\beta(1) \in \mathrm{Sp}(2n)_P^*$, there holds*

$$|i_P(\beta) - i_P(\alpha)| \le \nu_P(\gamma).$$

The equality can be achieved by two paths with the end points $\alpha(1)$ and $\beta(1)$ in different path connected components of the space $\mathrm{Sp}(2n)_P^*$.

We denote the set of non-degenerated paths by $\mathcal{P}_P^*(2n)$, i.e.,

$$\mathcal{P}_P^*(2n) = \{\gamma \in \mathcal{P}(2n) \,|\, \gamma(1) \in \mathrm{Sp}(2n)_P^*\}, \quad \mathcal{P}_P^0(2n) = \{\gamma \in \mathcal{P}(2n) \,|\, \gamma(1) \in \mathrm{Sp}(2n)_P^0\}.$$

We have the following symplectic additivity for the Maslov P-index theory.

Proposition 3.2 *For $P \in \mathrm{Sp}(2n)$, suppose $\gamma_j \in \mathcal{P}_{P_i}^*(2n_j)$ for $j = 0, 1$. Then $\gamma_0 \diamond \gamma_1 \in \mathcal{P}_P^*(2n_0 + 2n_1)$ and*

$$i_P(\gamma_0 \diamond \gamma_1) = i_{P_0}(\gamma_0) + i_{P_1}(\gamma_1), \tag{3.1}$$

where $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ and $P = P_0 \diamond P_1$, $P_j \in \mathrm{Sp}(2n_j)$.

Definition 3.3 *Given two paths γ_0 and $\gamma_1 \in \mathcal{P}(2n)$, we say that they are P-homotopic if there exists a map $\delta \in C([0,1], \mathcal{P}(2n))$ such that $\delta(0, \cdot\,) = \gamma_0(\,\cdot\,)$, $\delta(1, \cdot\,) = \gamma_1(\,\cdot\,)$ and* $\dim\ker(\delta(s,1) - P)$ *are constant for $s \in [0,1]$. We write $\gamma_0 \sim_P \gamma_1$.*

We have the following homotopy invariant for the Maslov P-index theory.

Proposition 3.4 *Let $P \in \mathrm{Sp}(2n)$. If $\gamma_j \in \mathcal{P}_P^*(2n)$ for $j = 0, 1$, then $i_P(\gamma_0) = i_P(\gamma_1)$ if and only if $\gamma_0 \sim_P \gamma_1$.*

We will prove a more general result in Theorem 4.7.

4 Galerkin Approximation for the *P*-Boundary Problem of Hamiltonian Systems

For given $P \in \mathrm{Sp}(2n)$, we consider the following problem

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), \\ x(1) = Px(0). \end{cases} \tag{4.1}$$

In the following, we always suppose that the Hamiltonian function H satisfies the following conditions:

(H1) $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and

$$H(t+1, Px) = H(t, x), \quad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{4.2}$$

(H2) There exist constants $a > 0$ and $p > 1$ such that

$$|H''(t, x)| \le a(1 + |x|^p), \quad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{4.3}$$

The symplectic group $\mathrm{Sp}(2n)$ is a Lie group, and its Lie algebra is $\mathrm{sp}(2n) = \{M \in \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}) \mid JM + M^T J = 0\}$. For any $M \in \mathrm{sp}(2n)$, it is well known that the one parameter curve $\exp(tM) = e^{tM}$ in $\mathrm{Sp}(2n)$ is a Lie sub-group of $\mathrm{Sp}(2n)$, and $\mathrm{Sp}(2n) = \exp(\mathrm{sp}(2n))$, i.e., the exponential map $\exp : \mathrm{sp}(2n) \to \mathrm{Sp}(2n)$ is surjective (not injective). So for $P \in \mathrm{Sp}(2n)$, there is a matrix $M \in \mathrm{sp}(2n)$ such that $P = e^M$ and $e^{tM} \in \mathrm{Sp}(2n)$. Recall that $W^{1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ is the subspace of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ which consists of all elements

$$z(t) = \sum_{k\in\mathbb{Z}} \exp(2k\pi tJ)a_k, \quad a_k \in \mathbb{R}^{2n},$$

satisfying

$$\|z\|^2_{1/2,2} := \sum_{k\in\mathbb{Z}} (1 + |k|)|a_k|^2 < +\infty.$$

This space is a Hilbert space with the norm $\|\cdot\|_{1/2,2}$ and the inner product $\langle\cdot,\cdot\rangle_{1/2,2}$. Taking the P-boundary space $W = e^{tM} W^{1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$, we define in this space the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$ such that it is a Hilbert space. For $z_i(t) = e^{tM}\xi_i(t),\ i = 1, 2$, define

$$\|z_1\| = \|\xi\|_{1/2,2}, \quad \langle z_1, z_2\rangle = \langle \xi_1, \xi_2\rangle_{1/2,2}.$$

We define the operator $A : W \to W$ such that

$$\langle Ax, y\rangle = \int_0^1 (-J\dot{x}(t), y(t))\,dt, \quad \forall\, x, y \in W. \tag{4.4}$$

A is a bounded self-adjoint operator with finite dimensional kernel N, and the restriction $A|_{N^\perp}$ is invertible. Define the functional on W by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \int_0^1 H(t, x)\,dt, \quad \forall\, x \in W. \tag{4.5}$$

Then $f \in C^2(W, \mathbb{R})$ and a critical point of f corresponds to a solution of problem (4.1). If $x = x(t)$ is a critical point of f, the second variation of f at x is given by

$$(f''(x)h, h) = \int_0^1 [(-J\dot{h}, h) - (H''(t, x)h, h)]\,dt = \langle (A - B)h, h\rangle, \quad \forall\, h \in W, \tag{4.6}$$

where $B : W \to W$ is defined by

$$\langle Bz_1, z_2\rangle = \int_0^1 (B(t)z_1(t), z_2(t))\,dt, \quad \forall\, z_1, z_2 \in W,\ B(t) = H''(t, x(t)). \tag{4.7}$$

Let $x = x(t) = e^{tM} \sum\limits_{k\in\mathbb{Z}} \exp(2k\pi tJ)a_k$. Then by direct computation, we get

$$-J\dot{x}(t) = e^{-tM^T}(-JM + 2k\pi) \sum_{k\in\mathbb{Z}} \exp(2k\pi tJ)a_k. \tag{4.8}$$

We note that $-JM$ is a symmetric matrix. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$ are the eigenvalues of JM^T. We have $\sigma(A) = \{(2k\pi + \lambda_j)/(1 + |k|)\}$. We define an operator $\widetilde{M} : W \to W$ by

$$\langle \widetilde{M}x, y\rangle = \int_0^1 (-JMx(t), y(t))\,dt, \quad \forall\, x, y \in W. \tag{4.9}$$

We have

$$\langle (A - \widetilde{M})x, y\rangle = \int_0^1 (-J\dot{\xi}(t), \eta(t))\,dt, \tag{4.10}$$

where $x(t)=e^{tM}\xi(t),\ y(t)=e^{tM}\eta(t),\ \xi,\eta\in W^{1/2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^{2n})$. $\sigma(A-\widetilde{M}\,)=\{\frac{2k\pi}{1+|k|}\}$, the multiplicity of any eigenvalue is $2n$ and the corresponding eigenspace is $e^{tM}\exp(2k\pi tJ)\mathbb{R}$. Let $W_m=e^{tM}\sum\limits_{k=-m}^{m}\exp(2k\pi tJ)\mathbb{R}$, and $P_m: W\to W_m$ the projection operator. Then the sequence $\Gamma=\{P_m\,|\,m\in\mathbb{N}\}$ is a Galerkin approximation scheme with respect to $A-\widetilde{M}$, i.e., it satisfies the following three conditions:

(1) $W_m=P_mW$ is finite dimensional space for all $m\in\mathbb{N}$.

(2) $P_m\to I$ strongly as $m\to\infty$.

(3) $[P_m,A-\widetilde{M}\,]:=P_m(A-\widetilde{M}\,)-(A-\widetilde{M}\,)P_m=0$.

Lemma 4.1 *Let $\Gamma=\{P_m\,|\,m\in\mathbb{N}\}$ be defined above. There exists $m_0>0$ such that*

$$\dim\ker(P_m(A-B)P_m)\le\dim\ker(A-B),\quad\forall\,m\ge m_0. \tag{4.11}$$

Proof (4.11) follows from the above condition (2) and the fact that the value 0 is isolated in $\{0\}\cup\sigma(A-B)$. The details are similar to the proof of Lemma 2.2 in [7].

For $P\in\mathrm{Sp}(2n)$, we consider two linear Hamiltonian systems

$$\dot z(t)=JB(t)z(t),\quad z(t)\in\mathbb{R}^{2n} \tag{4.12}$$

with $B(t+1)=(P^T)^{-1}B(t)P^{-1}$, and

$$\dot z(t)=J(-JM)z(t),\quad z(t)\in\mathbb{R}^{2n} \tag{4.13}$$

with $e^M=P$, and so $(P^T)^{-1}(-JM)P^{-1}=-JM$.

We suppose $\gamma=\gamma(t)$ and $\gamma_1=\gamma_1(t)=e^{tM}$ are the fundamental solutions of the linear Hamiltonian systems (4.12) and (4.13) respectively. Denoting $\gamma_2=\gamma_2(t)=e^{-tM}\gamma(t)$, we have the following result.

Theorem 4.2 *Let symplectic paths γ,γ_1 and $\gamma_2\in\mathcal{P}(2n)$ be defined as above, there holds*

$$i_P(\gamma)-i_P(\gamma_1)=i_1(\gamma_2)+n. \tag{4.14}$$

Thus the number $i_1(\gamma_2)+i_P(\gamma_1)$ depends only on P and not on the choice of M.

Proof We define three symplectic paths $\gamma_3,\gamma_5\in\mathcal{P}(2n)$ and $\gamma_4\in C([0,1],\mathrm{Sp}(2n))$ by

$$\gamma_3(t)=e^{-tM},\quad\gamma_4(t)=P^{-1}\gamma(t),$$

and $\gamma_5=\gamma_3*\gamma_4$ defined by

$$\gamma_5(t)=\begin{cases}\gamma_3(2t), & 0\le t\le\dfrac12,\\[2mm] \gamma_4(2t-1), & \dfrac12\le t\le 1.\end{cases}$$

Then we have $\gamma_5\sim_I\gamma_2$ in $\mathcal{P}(2n)$. In fact, the homotopy is defined by

$$\delta(s,t)=\begin{cases}\gamma_3(2st), & 0\le t\le\dfrac12,\\[2mm] e^{[(2s-2)t+1-2s]M}\gamma(2t-1), & \dfrac12\le t\le 1.\end{cases}$$

By Theorem 6.2.3 of [12], we have

$$i_1(\gamma_2) = i_1(\gamma_5). \tag{4.15}$$

By definition, we have

$$i_1(\gamma_5) = i_1(\gamma_3) + i_P(\gamma). \tag{4.16}$$

We note that $P^{-1}\gamma_1(t) = \gamma_3(1-t)$, so the two paths γ_1 and γ_3 are different only with two opposite orientations. So if $P^{-1} \in \mathrm{Sp}(2n)^*$, by Definitions 2.3 and 2.5, we get

$$i_1(\gamma_3) = -i_P(\gamma_1) - n. \tag{4.17}$$

If $P^{-1} \in \mathrm{Sp}(2n)^0$, then by Definitions 2.4 and 2.6, we also have (4.17). Now (4.14) follows from (4.15), (4.16) and (4.17).

We recall that the operators A, B and $\widetilde{M}$ are defined by (4.4), (4.7) and (4.9) respectively. $\Gamma = \{P_m \,|\, m \in \mathbb{N}\}$ is the Galerkin approximation scheme with respect to $A - \widetilde{M}$. γ and $\gamma_1 \in \mathcal{P}(2n)$ are defined above. We have the following result.

Theorem 4.3 *For $0 < d \le \frac{1}{4}\|((A-B)|_{Im(A-B)})^{-1}\|^{-1}$, there exists an $m_0 > 0$ such that for $m \ge m_0$, there holds*

$$\begin{aligned} &\dim m_d^+(P_m(A-B)P_m) = m + 2n + i_P(\gamma_1) - i_P(\gamma) - \nu_P(\gamma), \\ &\dim m_d^0\,(P_m(A-B)P_m) = \nu_P(\gamma), \\ &\dim m_d^-\,(P_m(A-B)P_m) = m - i_P(\gamma_1) + i_P(\gamma), \end{aligned} \tag{4.18}$$

where $m_d^+(\,\cdot\,)$, $m_d^-(\,\cdot\,)$, $m_d^0(\,\cdot\,)$ denote the eigenspaces corresponding to the eigenvalue λ belonging to $[d, +\infty)$, $(-\infty, d]$ and $(-d, d)$ respectively.

Proof We recall that

$$\nu_P(\gamma_1) = \dim_{\mathbb{R}} \ker_{\mathbb{C}}(\gamma_1(1) - P) = 2n. \tag{4.19}$$

Let $x(t) = e^{tM}\xi(t)$, $\xi \in W^{1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$. Then we have

$$\begin{aligned} \langle (A-B)x, x\rangle &= \int_0^1 [(-J\dot{x}(t), x(t)) - (B(t)x(t), x(t))]\,dt \\ &= \int_0^1 [(-J\dot{\xi}(t), \xi(t)) - (\widetilde{B}(t)\xi(t), \xi(t))]\,dt, \end{aligned} \tag{4.20}$$

where $\widetilde{B}(t) = JM + e^{-tM^T}B(t)e^{tM}$.

Consider the following linear Hamiltonian systems

$$\dot{z}(t) = J\widetilde{B}(t)z(t), \quad z(t) \in \mathbb{R}^{2n}. \tag{4.21}$$

Suppose $\widetilde{\gamma}(t)$ is the fundamental solution of (4.21). Then by direct computation, we have

$$\widetilde{\gamma}(t) = e^{-tM}\gamma(t) = \gamma_2(t). \tag{4.22}$$

By Theorem 2.1 of [7] (see also [12, Corollary 7.1.10]), there exists an $m_0 > 0$ such that

$$\begin{aligned} &\dim m_d^+(P_m(A-B)P_m) = m + n - i_1(\gamma_2) - \nu_1(\gamma_2), \\ &\dim m_d^0\,(P_m(A-B)P_m) = \nu_1(\gamma_2), \\ &\dim m_d^-\,(P_m(A-B)P_m) = m + n + i_1(\gamma_2). \end{aligned} \tag{4.23}$$

We note that in [7], the constant $d = \frac{1}{4}\|((A-B)|_{Im(A-B)})^{-1}\|^{-1}$, but if we check the proof of Theorem 2.3 in [7], the proof is still go through if we replace the constant by any $d \in (0, d_0]$, $d_0 = \frac{1}{4}\|((A-B)|_{Im(A-B)})^{-1}\|^{-1}$. Now (4.18) follows from (4.14), (4.23) and $\nu_1(\gamma_2) = \nu_P(\gamma)$.

In general, for a continuous symmetric matrix function $B_0(t)$ with $P^T(B_0(t+1)P = B_0(t)$, we define an operator B_0 in W as above. $(i_P(B_0), \nu_P(B_0))$ is the index pair defined in Definition 2.7. Then we have

$$\nu_P(B_0) = \dim\ker(A - B_0).$$

Let $\cdots \le \lambda_2' \le \lambda_1' < 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $A - B_0$, and let $\{e_j'\}$ and $\{e_j\}$ be the eigenvactors of $A - B_0$ corresponding to $\lambda_j'$and λ_j respectively.

For $m \ge 0$, set $W_0 = \ker(A - B_0)$, $W_m = W_0 \oplus \mathrm{span}\{e_1', \cdots, e_m'\} \oplus \mathrm{span}\{e_1, \cdots, e_m\}$, and P_m to be the orthogonal projection from W to W_m. Then $\Gamma = \{P_m\}$ is an approximation scheme with respect to the operator $A - B_0$. With this approximation scheme and the same notations as in Theorem 4.3, we have the following result.

Theorem 4.4 *There exists an $m_0 > 0$ such that for $m \ge m_0$, there holds*

$$\begin{aligned} &\dim m_d^+(P_m(A-B)P_m) = m + i_P(B_0) - i_P(B) + \nu_P(B_0) - \nu_P(B),\\ &\dim m_d^0(P_m(A-B)P_m) = \nu_P(B),\\ &\dim m_d^-(P_m(A-B)P_m) = m - i_P(B_0) + i_P(B). \end{aligned} \tag{4.24}$$

Proof Just as in the proof of Theorem 4.3, we have

$$\langle (A - B_0)x, x\rangle = \int_0^1 [(-J\dot\xi(t), \xi(t)) - (\widetilde{B}_0(t)\xi(t), \xi(t))]\, dt.$$

So the operator $A - B_0$ defined in W corresponds to the operator $-J\frac{d}{dt} - \widetilde{B}_0$ defined in $W^{1/2,2}(S^1, \mathbb{R}^{2n})$. With the same reason, $A - B$ defined in W corresponds to the operator $-J\frac{d}{dt} - \widetilde{B}$ defined in $W^{1/2,2}(S^1, \mathbb{R}^{2n})$. Thus by Theorem 2.3 of [6], we have

$$\begin{aligned} &\dim m_d^+(P_m(A-B)P_m) = m + i_1(\widetilde{B}_0) - i_1(\widetilde{B}\,) + \nu_1(\widetilde{B}_0) - \nu_P(\widetilde{B}\,),\\ &\dim m_d^0(P_m(A-B)P_m) = \nu_1(\widetilde{B}\,),\\ &\dim m_d^-(P_m(A-B)P_m) = m - i_1(\widetilde{B}_0) + i_1(\widetilde{B}\,). \end{aligned} \tag{4.25}$$

Now by (4.14),

$$i_1(\widetilde{B}_0) = i_P(B_0) - i(\gamma_1) - n, \quad i_1(\widetilde{B}) = i_P(B) - i(\gamma_1) - n.$$

we get (4.24).

As a direct consequence, we have the following monotonicity result.

Corollary 4.5 *Suppose continuous symmetric matrix functions $B_j(t)$, $j = 1, 2$ with $P^T B_j(t+1)P = B_j(t)$ satisfy*

$$B_1(t) < B_2(t), \ \textit{i.e.,}\ B_2(t) - B_1(t) \textit{ is positive definite for all } t \in [0, 1]. \tag{4.26}$$

Then there holds

$$i_P(B_1) + \nu_P(B_1) \le i_P(B_2). \tag{4.27}$$

Proof Just as done in Theorem 4.4, corresponding to $B_j(t)$, we have the operator $B_j : W \to W$. Let $\Gamma = \{P_m\}$ be the approximation scheme with respect to the operator $A - B_1$. Then by (4.24), there exists an $m_0 > 0$ such that if $m \geq m_0$ there holds

$$\dim m_d^-(P_m(A - B_2)P_m) = m - i_P(B_1) + i_P(B_2),$$

where we choose $0 < d < \frac{1}{2}\|B_2 - B_1\|$. Since $A - B_2 = (A - B_1) - (B_2 - B_1)$ and $B_2 - B_1$ is positive definite in $W_m = P_m W$ and $\langle (B_2 - B_1)x, x\rangle \geq 2d\|x\|$, we have $\langle (P_m(A - B_2)P_m)x, x\rangle \leq -2d\|x\|$ with $x \in W_0 \oplus \mathrm{span}\{e_1', \cdots, e_m'\}$ the direct summation of kernel and negative eigenspace of the operator $A - B_1$ in W_m. It implies that

$$m + \nu_P(B_1) \leq m_d^-(P_m(A - B_2)P_m) = m - i_P(B_1) + i_P(B_2).$$

We note that with the same result (4.27) one can replace the condition (4.26) by condition

$$\langle (B_2 - B_1)x, x\rangle \geq \delta\|x\|, \quad \forall\, x \in W, \tag{4.28}$$

where $\delta > 0$ is a constant number. Even for the periodic boundary case, i.e., $P = I$, the result (4.27) is new.

By definition, we have

$$i_P(0) = \begin{cases} 0, & P \neq I, \\ -n, & P = I, \end{cases} \qquad \nu_P(0) = \dim\ker(I - P).$$

So we have the following result.

Corollary 4.6 *Suppose continuous symmetric matrix function $B(t)$ satisfies $B(t) > 0$ and $P^T B(t+1)P = B(t)$. Then there holds*

$$\begin{aligned} i_P(B) &\geq \dim\ker(I - P), \quad P \neq I, \\ i_1(B) &\geq n. \end{aligned} \tag{4.29}$$

Theorem 4.7 *Let $P \in \mathrm{Sp}(2n)$. For any two paths γ_0 and $\gamma_1 \in \mathcal{P}(2n)$, if $\gamma_0 \sim_P \gamma_1$ on $[0, 1]$, then*

$$i_P(\gamma_0) = i_P(\gamma_1), \quad \nu_P(\gamma_0) = \nu_P(\gamma_1). \tag{4.30}$$

Proof We note that $\gamma_0 \sim_P \gamma_1$ if and only if $\overline{\gamma}_0 \sim_I \overline{\gamma}_1$, where $\overline{\gamma}_j(t) = e^{-tM}\gamma_j(t)$ for $j = 0, 1$. Now (4.30) follows from (4.14) and Theorem 6.2.3 of [12] for $\omega = 1$.

5 The Saddle Point Reduction and Morse Index

Let $S^1 = \mathbb{R}/\mathbb{Z}$. We equip the Hilbert space $L^2 = L^2(S^1, \mathbb{R}^{2n})$ with the usual norm

$$\|x\|_{L^2} = \Big(\int_0^1 |x(t)|^2\Big)^{1/2}, \quad \forall\, x \in L^2.$$

Denote $L = e^{tM}L^2(S^1, \mathbb{R}^{2n}) = \{x \mid x(t) = e^{tM}\xi(t),\ \xi \in L^2\}$ and define the inner product on L by

$$\langle x, y\rangle_L = \int_0^1 (e^{-tM^T}\xi(t), e^{tM}\eta(t))\, dt, \quad \forall\, x(t) = e^{tM}\xi(t),\ y(t) = e^{tM}\eta(t),\ \xi, \eta \in L^2.$$

With this inner product, L is a Hilbert space and we have

$$\langle x, y\rangle_L = \langle \xi, \eta\rangle_{L^2}, \quad \forall\, x(t) = e^{tM}\xi(t),\ y(t) = e^{tM}\eta(t),\ \xi, \eta \in L^2. \tag{5.1}$$

Denote $W = e^{tM}W^{1,2}(S^1, \mathbb{R}^{2n}) = \{x \mid x(t) = e^{tM}\xi(t),\ \xi \in W^{1,2}(S^1, \mathbb{R}^{2n})\}$.

In the Hilbert space L we define an operator A by

$$\langle Ax, y\rangle_L = \int_0^1 (-J\dot{x}, y)\, dt, \quad \forall\, x, y \in L. \tag{5.2}$$

Then the domain of A is $\mathrm{dom}A = W$. The range of A is closed and the resolution of A is compact. We define an operator $\widetilde{M} : L \to L$ by

$$\langle \widetilde{M}x, y\rangle_L = \int_0^1 (-JMx(t), y(t))\, dt, \quad \forall\, x, y \in L. \tag{5.3}$$

Under the norm of the space L, the spectrum of the operator $A - \widetilde{M}$ is $\sigma(A - \widetilde{M}) = 2\pi\mathbb{Z}$. It is a point spectrum, i.e., it contains only eigenvalues, and the multiplicity of every eigenvalue is $2n$. The eigensubspace of $A - \widetilde{M}$ belonging to the eigenvalue $2k\pi$ is

$$E_k = e^{tM} \exp(2k\pi tJ)\mathbb{R}^{2n} = e^{tM}((\cos 2k\pi t)I + (\sin 2k\pi t)J)\mathbb{R}^{2n}.$$

Especially, $\ker(A - \widetilde{M}\,) = e^{tM}\mathbb{R}^{2n}$. In the following of the this section, we denote $A_\infty = A - \widetilde{M}$.

We consider the Hamiltonian systems (4.1) with condition (H1) in Section 4 and

(H3) There exists a constant $C(H) > 0$ such that

$$|H''(t, x)| \le C(H), \quad \forall\, (t, x) \in [0, 1] \times \mathbb{R}^{2n}. \tag{5.4}$$

Define a functional on the space L by

$$g(x) = \int_0^1 H(t, x(t))\, dt. \tag{5.5}$$

By conditions (H1) and (H3), we have $g \in C^1(L, \mathbb{R})$, and

$$g'(x) = H'(t, x). \tag{5.6}$$

$g'(x)$ is Gadeaux differentiable, its Gadeaux derivative is

$$dg'(x)y = H''(t, x(t))y, \tag{5.7}$$

and there exists a constant $c(H) > 0$ such that

$$\|dg'(x)\|_{\mathcal{L}(\mathcal{L})} \le c(H). \tag{5.8}$$

Define the action functional by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle_L - g(x) = \frac{1}{2}\langle A_\infty x, x\rangle_L - g_\infty(x), \quad \forall\, x \in \mathrm{dom}A = W, \tag{5.9}$$

where $g_\infty(x) = -\frac{1}{2}\langle \widetilde{M}x, x\rangle_L + g(x)$. Under conditions (H1) and (H3), $f \in C^1(W, \mathbb{R})$, f' is Gadeaux differentiable. The critical points of f are solutions of the problem (4.1).

Let $P_0 : L \to E_0 = e^{tM}\mathbb{R}^{2n}$ be the projection map. Define

$$A_0 x = A_\infty x + P_0 x, \quad \forall\, x \in W. \tag{5.10}$$

Without loss of generality, we suppose the constant in (5.8) satisfies $c(H) \notin \sigma(A_0)$ and $c(H) > 1$. Denote by $\{E_\lambda\}$ the spectral resolution of the selfadjoint operator A_0, and define the projections on the Hilbert space L by

$$\mathcal{P} = \int_{c(H)}^{c(H)} dE_\lambda, \quad \mathcal{P}^+ = \int_{c(H)}^{+\infty} dE_\lambda, \quad \mathcal{P}^- = \int_{-\infty}^{-c(H)} dE_\lambda. \tag{5.11}$$

Then the Hilbert space L possesses an orthogonal decomposition

$$L = L^+ \oplus L^- \oplus Z, \tag{5.12}$$

where $Z = \mathcal{P}L$ is a finite dimensional space, and $L^\pm = \mathcal{P}^\pm L$. With standard arguments as in [12, 1, 2], we have the following result.

Theorem 5.1 *Suppose the function H satisfies the conditions* (H1) *and* (H3). *Then there exists a functional $a \in C^2(Z, \mathbb{R})$ and an injection map $u \in C^1(Z, L)$ such that $u : Z \to W$ satisfies the following conditions:*

(1) *The map u has the form $u(z) = w(z) + z$, where $\mathcal{P}w(z) = 0$.*

(2) *The functional a satisfies*

$$\begin{aligned} a(z) &= f(u(z)), \\ a'(z) &= A_\infty z - P g'_\infty(u(z)) = A_\infty u(z) - g'_\infty(u(z)), \\ a''(z) &= (A_\infty P - P dg'_\infty(u(z)))u'(z) = [A_\infty - dg'_\infty(u(z))]u'(z). \end{aligned}$$

And a' is globally Lipschitz continuous.

(3) *$z \in Z$ is a critical point of a, i.e., $a'(z) = 0$, if and only if $u(z)$ is a critical point of f.*

(4) *If $g(u) = \langle Bu, u\rangle_L = \int_0^1 (\overline{B}u(t), u(t))\, dt$ for all $u \in L$, where $\overline{B}$ is a constant symmetric matrix defined on $\mathbb{R}^{2n}$ satisfying $P^T\overline{B}P = \overline{B}$, then $a(z) = \frac{1}{2}\langle (A - B)z, z\rangle_L$.*

(5) $\dim\ker a''(z) = \nu_P(\gamma)$, *where γ is the fundamental solution of the linear Hamiltonian systems $\dot{y} = JH''(t, u(z)(t))y$.*

Particularly, for the symmetric matrix continuous function $B(t)$ satisfying $B(t+1) = (P^T)^{-1}B(t)P^{-1}$, we define a symmetric operator B on L by

$$\langle Bx, y\rangle_L = \int_0^1 (B(t)x(t), y(t))\, dt, \quad \forall\, x, y \in L, \tag{5.13}$$

and define

$$f(x) = \frac{1}{2}\langle (A - B)x, x\rangle_L, \quad \forall\, x \in W. \tag{5.14}$$

The critical points of f are solutions of the following problem

$$\begin{cases} \dot{x} = JB(t)x, \\ x(1) = Px(0). \end{cases} \tag{5.15}$$

By Theorem 5.1, we obtain a subspace

$$Z = \Big\{ x \,\Big|\, x(t) = \sum_{|k|\le k_0} e^{tM} \exp(2k\pi tJ) a_k,\ a_k \in \mathbb{R}^{2n} \Big\}$$

with a sufficiently large $k_0 \in \mathbb{N}$, an injection map $u \in C^\infty(Z, L)$, and a smooth functional $a \in C^\infty(Z, \mathbb{R})$ defined by

$$a(z) = f(u(z)), \quad \forall z \in Z. \tag{5.16}$$

Let $2d = \dim Z$. Note that the origin of Z as a critical point of a corresponds to the origin of L as a critical point of f. Denote by m^* for $* = +, 0$ and $-$ the positive, null, and negative Morse indices of the functional a at the origin respectively, i.e., the total multiplicities of positive, zero, and negative eigenvalues of the $2d \times 2d$ matrix $a''(0)$ respectively. We have the following result.

Theorem 5.2 *There holds*

$$\begin{cases} m^- = d + i_P(\gamma) - i_P(\gamma_1) - n, \\ m^0 = \nu_P(\gamma), \\ m^+ = d - i_P(\gamma) + i_P(\gamma_1) + n - \nu_P(\gamma). \end{cases} \tag{5.17}$$

Proof By the computation in (4.20), we have

$$f(x) = \frac{1}{2}\langle (A-B)x, x\rangle_L = \int_0^1 [(-J\dot{\xi}(t), \xi(t)) - (\widetilde{B}(t)\xi(t), \xi(t))]\, dt, \tag{5.18}$$

where $x(t) = e^{tM}\xi(t) \in W$. The eigenvalue of A_∞ in L corresponds to the eigenvalue of $-J\frac{d}{dt}$ in $L^2 = L^2(S^1, \mathbb{R}^{2n})$. The solutions of (5.15) corresponds to the 1-periodic solutions of the systems (4.21). The fundamental solution of (4.21) is $\gamma_2(t)$ defined by (4.22). So by Theorem 6.1.1 of [12] for $\omega = 1$, we have

$$\begin{cases} m^- = d + i_1(\gamma_2), \\ m^0 = \nu_1(\gamma_2), \\ m^+ = d - i_1(\gamma_2) - \nu_1(\gamma_2). \end{cases} \tag{5.19}$$

Now (5.17) follows from (4.15)–(4.17).

We consider problem (4.1) with H satisfying condition (H1) in Section 4 and (H3) above. Recall that the functional $f(x)$ is defined in (5.9). By Theorem 5.1, there exist the corresponding functional $a : Z \to \mathbb{R}$ and an injection $u : Z \to L$ such that $a(z) = f(u(z))$. Suppose $z = z(t) \in Z$ is a critical point of a. Then $x = x(t) = u(z)(t)$ is a solution of problem (4.1). Suppose $m^*(z)$ with $* = 0, \pm$ are the Morse index of a at z. We have the following result.

Theorem 5.3 *Under the above conditions and notations, there holds*

$$\begin{cases} m^-(z) = d + i_P(x) - i_P(\gamma_1) - n, \\ m^0\,(z) = \nu_P(x), \\ m^+(z) = d - i_P(x) + i_P(\gamma_1) + n - \nu_P(x). \end{cases} \tag{5.20}$$

Proof (5.20) follows from (2) in Theorem 5.1, and that (5.19) is true for this case if we replace $B(t)$ in (5.14) by $H''(t, x(t))$ (cf. [13, Theorem 7.1.1]).

6 Relations with the Morse Index of Second Order Systems

Given a matrix $U \in \mathrm{SO}(n)$, we consider the following problem

$$\begin{cases} \ddot{x} + \nabla V(t, x) = 0, \\ x(1) = Ux(0), \quad \dot{x}(1) = U\dot{x}(0), \end{cases} \tag{6.1}$$

where $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ satisfies $V(t+1, Ux) = V(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Let D be an $n \times n$ matrix satisfying $e^D = U$ and $D + D^T = 0$. And Let $W = e^{tD} W^{1,2}(S^1, \mathbb{R}^n)$ with the inner product

$$\langle x, y \rangle_W = \int_0^1 [(x(t), y(t)) + (\dot{x}(t), \dot{y}(t))]\, dt.$$

We define a functional $F : W \to \mathbb{R}$ by

$$F(x) = \int_0^1 \Big[\frac{1}{2}(\dot{x}(t), \dot{x}(t)) - V(t, x(t))\Big]\, dt, \quad \forall\, x \in W. \tag{6.2}$$

The critical points of F are solutions of problem (6.1). Suppose $x \in W$ is a critical point of F. The Hessian of F at x is given by

$$(F''(x)y, z) = \int_0^1 [(\dot{y}(t), \dot{z}(t)) - (\Delta V(t, x(t))y(t), z(t))]\, dt. \tag{6.3}$$

The linearized system of (6.1) at x is given by the linear second order systems

$$\ddot{y} + \Delta V(t, x(t))y = 0. \tag{6.4}$$

We rewrite this systems into a first order linear Hamiltonian systems

$$\dot{z} = JB(t)z, \quad z \in \mathbb{R}^{2n}, \tag{6.5}$$

where $z = (\dot{y}, y)^T$ and $B(t) = \begin{pmatrix} -I & 0 \\ 0 & -\Delta V(t, x(t)) \end{pmatrix}$. It is easy to check that

$$B(t+1) = (P^{-1})^T B(t) P^{-1}, \tag{6.6}$$

where the $2n \times 2n$ symplectic matrix is defined by

$$P = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}. \tag{6.7}$$

Suppose $\gamma = \gamma(t)$ is the fundamental solution of (6.5). Then the index $(i_P(\gamma), \nu_P(\gamma))$ with P-boundary condition is well defined. We denote the Morse index and nullity of F at x by $m^-(x)$ and $m^0(x)$, i.e., the total multiplicities of all the negative eigenvalues and zeros of $F''(x)$ respectively.

Theorem 6.1 *Under the above conditions, there holds*

$$m^-(x) = i_P(\gamma) - i_P(\gamma_1) - n, \quad m^0(x) = \nu_P(\gamma), \tag{6.8}$$

where $\gamma_1(t) = \begin{pmatrix} e^{tD} & 0 \\ 0 & e^{tD} \end{pmatrix}$.

Proof From (6.3), we have

$$(F''(x)y, z) = \int_0^1 [(-\dot{\xi} + D^T\xi, \dot{\eta}) + (D\dot{\xi}, \eta) + ((D^2 + \widetilde{C})\xi, \eta)]\, dt, \quad \xi, \eta \in W^{1,2}(S^1, \mathbb{R}^n) \tag{6.9}$$

with $y(t) = e^{tD}\xi(t)$ and $z(t) = e^{tD}\eta(t)$, $\widetilde{C}(t) = e^{-tD}\Delta V(t, x(t))e^{tD}$. From (6.9), we obtain the Sturm system:

$$-\frac{d}{dt}(-\dot{\xi} - D\xi) + D\dot{\xi} + (D^2 + \widetilde{C})\xi = 0. \tag{6.10}$$

Setting $-\dot{\xi} - D\xi = \zeta$ and $z = (\zeta, \xi)^T$, we rewrite (6.10) into the following first order linear Hamiltonian system

$$\dot{z} = J\widetilde{B}(t)z, \quad z \in \mathbb{R}^{2n}, \tag{6.11}$$

with the matrix $\widetilde{B}$ defined by

$$\widetilde{B}(t) = \begin{pmatrix} -I & -D \\ D & -\widetilde{C}(t) \end{pmatrix} = JM + \gamma_1(-t)B(t)\gamma_1(t), \quad M = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}. \tag{6.12}$$

Suppose $\widetilde{\gamma} = \widetilde{\gamma}(t)$ is the fundamental solution of the system (6.11). Then by Theorem 7.3.1 of [12], we have

$$m^-(x) = i_1(\widetilde{\gamma}), \quad m^0(x) = \nu_1(\widetilde{\gamma}). \tag{6.13}$$

By direct computation (see (4.21), (4.22)), we get $\widetilde{\gamma}(t) = \gamma_1(-t)\gamma(t) = \gamma_2(t)$ which is defined in Theorem 4.2. Then by (4.14), we get (6.13).

7 Applications: Asymptotically Linear Hamiltonian Systems with Non-periodic Boundary Condition

To the author's knowledge, the most beautiful (direct) application of the Maslov-type index theory for the case of periodic Hamiltonian systems is to study the existence and multiplicity of solutions of asymptotically linear Hamiltonian systems. The study on the periodic solutions of asymptotically linear Hamiltonian systems in global sense started from 1980 in [1]. Since then many contributions on this problem have appeared (cf. [2, 3, 7, 8, 12, 14]). In this section, we consider the non-periodic (P-boundary) solutions of asymptotically linear Hamiltonian systems. We note that the key point is the index developed in this paper.

We consider the following Hamiltonian systems with non-periodic boundary condition:

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)), \\ x(1) = Px(0), \end{cases} \tag{7.1}$$

where $H \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ satisfies $H(t+1, Px) = H(t, x)$, $\forall\, (t, x)$, and $H'(t, x)$ is the gradient of H with respect to the variables x. It is clear that

$$P^T H''(t+1, Px)P = H''(t, x). \tag{7.2}$$

We turn to study the existence and multiplicity of solutions of asymptotically linear Hamiltonian systems. A Hamiltonian system is called asymptotically linear at infinity if

(H_∞) There exists a continuous symmetric matrix function $B_\infty(t)$ such that

$$P^T B_\infty(t+1)P = B_\infty(t) \quad \text{for all } t \in \mathbb{R}$$

and

$$\|H'(t,x) - B_\infty(t)x\|_{\mathbb{R}^{2n}} = o(\|x\|_{\mathbb{R}^{2n}}) \tag{7.3}$$

uniformly in t, as $\|x\|_{\mathbb{R}^{2n}} \to \infty$.

We remind that for a matrix function $B = B(t)$ satisfying $P^T B(t+1)P = B(t)$, the index pair $(i_P(B), \nu_P(B))$ is defined by Definition 2.7. In the following result, we suppose $\nu_P(B_\infty) = 0$ and denote $i_\infty = i_P(B_\infty)$.

Theorem 7.1 *Suppose H satisfies conditions* (H1), (H3), (H_∞) (*see* (4.2), (5.4) *and* (7.3)). *Then problem* (7.1) *possesses at least one solution x_0. Let $B(t) = H''(t, x_0(t))$ and $(i_0, \nu_0) = (i_P(B), \nu_P(B))$. If*

$$i_\infty \notin [i_0, i_0 + \nu_0], \tag{7.4}$$

problem (7.1) *possesses at least two solutions. Furthermore, if x_0 is not pseudo-degenerated, and*

$$i_\infty \notin [i_0 - 2n, i_0 + \nu_0 + 2n], \tag{7.5}$$

problem (7.1) *possesses at least three solutions.*

Remark Suppose that x_0 is a critical point of f defined by (5.9). By Theorem 5.1, $z_0 = \mathcal{P}x$ is a critical point of a. Suppose the critical set of a is isolated. Then z_0 is a isolated invariant set of the gradient flow of a. By Conley homotopic index theory, we get the Conley homotopic index $h(z_0)$, and its Poincaré polynomial

$$p(t, h(z_0)) = t^{m^+(z_0)} \sum_{j=0}^{m^0(z_0)} a_j t^j, \quad a_j \in \{0, 1, 2, \cdots\},\ a_0 = 0 \text{ or } a_{m^0(z_0)} = 0,$$

$m^*(z_0)$, $* = 0, \pm$ are defined in (5.20) (cf. [1, 4]). We say that x_0 is pseudo-degenerated if $p(t, h(z_0)) = 0$ or contains the factor $(1+t)$. Theorem 7.1 should be compared with the main result of [10] (see also [12, Theorem 7.2.2] and [2, Theorem 4.1.3]) where they considered the periodic solutions of asymptotically Hamiltonian systems. The main ingredients of the the proof are the Maslov-type index theory, the Poincaré polynomial of the Conley homotopic index of isolated invariant set, and the saddle point reduction methods. The Conley homotopic index theory can be used here for the proof of Theorem 7.1. Now by using the index theory and the saddle point methods developed in Section 5, we can prove Theorem 7.1 as done in [12] and [10]. We omit the details.

By the index theory and the Galerkin approximation methods developed in Section 4, similar to [7] and [3], we have the following result.

Theorem 7.2 *Suppose H satisfies conditions* (H1), (H2), (H_∞) (*see* (4.2), (4.3) *and* (7.3)) *and*

(H4) *There exists continuous symmetric matrix function $B_0(t)$ such that for all $t \in \mathbb{R}$, $P^T B_0(t+1)P = B_0(t)$ and*

$$H'(t,x) = B_0(t)x + o(|x|) \quad \text{as } |x| \to 0 \text{ uniformly in } t. \tag{7.6}$$

(H5) *For $h(t,x) = H(t,x) - \frac{1}{2}(B_\infty(t)x, x)$ with $(t,x) \in \mathbb{R} \times \mathbb{R}^{2n}$, either*

$$h(t,x) \to 0, \quad |h'(t,x)| \to 0 \quad \text{as } |x| \to +\infty \text{ uniformly in } t, \tag{7.7}$$

or

$$h(t,x) \to \pm\infty, \quad |h'(t,x)| = O(1) \quad \textit{as } |x| \to \infty \textit{ uniformly in } t. \tag{7.8}$$

Then problem (7.1) *possesses a nontrivial solution, provided that*

$$[i_P(B_0), i_P(B_0) + \nu_P(B_0)] \cap [i_P(B_\infty), i_P(B_\infty) + \nu_P(B_\infty)] = \emptyset. \tag{7.9}$$

Remark Theorem 7.2 should be compared with results in [3] and [7] (see also [12]), where the periodic solutions were studied. The main ingredients of the proof given in [7] are the Maslov-type index theory, the Galerkin approximation methods (the results stated in (4.23) and (4.25)), and the critical point theory, specially, the saddle point theorem.

Similarly, we have developed the index theory which is parallel to that of Maslov-type index in the periodic case and is suitable to use in our case here. We have also developed the Galerkin approximation methods in Theorem 4.3. The critical point theory can also be used here to find as many critical points as that in [7] and [3]. So we can modify the proof given in [7] or [3] to give a proof of Theorem 7.2. We omit the details for simplicity.

References

[1] Amann, H. and Zehnder, E., Nontrivial solutions for a class of non-resonance problems and applications to nonlinear differential equations, *Ann. Scoula Norm. Sup. Pisa. Cl. Sci. Series* 4, **7**, 1980, 539–603.

[2] Chang, K. C., Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Basel, 1993.

[3] Chang, K. C., Liu, J. Q. and Liu, M. J., Nontrivial periodic solutions for strong resonance Hamiltonian systems, *Ann. Inst. H. Poincaré Anal. Non. linéaire*, **14**(1), 1997, 103–117.

[4] Conley, C., Isolated Invariant Sets and the Morse Index, CBMS Reg. Conf. Series in Math., **38**, Amer. Math. Soc., 1978.

[5] Conley, C. and Zehnder, E., Maslov-type index theory for flows and periodic solutions for Hamiltonian systems, *Commum. Pure Appl. Math.*, **37**, 1984, 207–253.

[6] Ekeland, I., Convexity Methods in Hamiltonian Mechanics, Springer-Verlag, Berlin, 1990.

[7] Fei, G. and Qiu, Q., Periodic solutions of asymptotically linear Hamiltonian systems, *Chin. Ann. Math.*, **18B**(3), 1997, 359–372.

[8] Li, S. and Liu, J., Morse theory and asymptotically linear Hamiltonian systems, *J. Diff. Equa.*, **78**, 1989, 53–73.

[9] Long, Y. and Zehnder, E., Morse index theory for forced oscillations of asymptotically linear Hamiltonian systems, Stoc. Proc. Phys. and Geom., S. Albeverio et al. (eds.), World Sci., 1990, 528–563.

[10] Long, Y., Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems, *Science in China*, **33**, 1990, 1409-1419.

[11] Long, Y., A Maslov-type index theory for symplectic paths, *Top. Meth. Nonl. Anal.*, **10**, 1997, 47–78.

[12] Long, Y., Index Theory for Symplectic Paths with Applications, Progress in Mathematics, Vol. 207, Birkhäuser Verlag, 2002.

[13] Long, Y., Index Theory of Hamiltonian systems with Applications (in Chinese), Science Press, Beijing, 1993.

[14] Viterbo, C., A new obstruction to embedding Lagrangian tori, *Invent. Math.*, **100**, 1990, 301–320.