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A Functional LIL for m-Fold Integrated Brownian Motion***

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Abstract Let $\{X_m(t), t \in R_+\}$ be an m-Fold integrated Brownian motion. In this paper, with the help of small ball probability estimate, a functional law of the iterated logarithm (LIL) for $X_m(t)$ is established. This extends the classic Chung type liminf result for this process. Furthermore, a result about the weighted occupation measure for $X_m(t)$ is also obtained.

Keywords m-Fold integrated Brownian motion, Functional law of the integrated logarithm, Small ball probability, Weighted occupation measure
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1 Introduction

Let $\{W(t), t \in R_+\}$ be a standard Brownian motion with $W(0) = 0$. A Gaussian process $\{X_m(t), t \in R_+\}$ is called m-fold integrated Brownian motion with positive integer m provided $X_0(t) = W(t)$ and

$$X_m(t) = \int_0^t X_{m-1}(s)ds, \quad m \geq 1.$$

That is,

$$X_m(t) = \int_0^t \frac{(t-s)^m}{m!} dW(s).$$

From the definition, it follows that X_m is a self-similar process with scaling property

$$X_m(ct) \overset{\mathrm{d}}{=} c^{(2m+1)/2} X_m(t),$$

where $X \overset{\mathrm{d}}{=} Y$ means X and Y have the same finite dimensional distributions. This process X_m is an interesting process and has been studied by many authors in different view points. For example, Watanabe [17] established the law of the iterated logarithm for $X_1(t)$. Wahba [15, 16] used $X_m(\cdot)$ to derive a correspondence between smoothing by splines and Bayesian estimation in certain stochastic models. Lachal [10, 11] considered the law of the iterated logarithm and the regular points for $X_m(t), m \geq 1$. Khoshnevisan and Shi [6] obtained the small ball probability (A small ball probability of a process Y is refered to the probability $P\{\sup_{0\leq s\leq 1} |Y(s)| \leq \varepsilon\}$) and

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Chung's law of the iterated logarithm for $X_1(t)$. Lin [13] studied the increment properties for $X_m(t), m \ge 1$. Recently, Chen and Li [2] obtained a general result of the small ball probability and Chung's law of the iterated logarithm for $X_m(t), m \ge 1$. They showed that for any integer $m \ge 1$,

$$\lim_{\varepsilon \to 0} \varepsilon^{2/(2m+1)} \log P\Big\{ \sup_{0 \le t \le 1} |X_m(t)| \le \varepsilon \Big\} = -\kappa_m \tag{1.1}$$

with

$$\frac{m+1}{2}\Big[(2m+2)\sin\frac{\pi}{2m+2}\Big]^{-(2m+2)/(2m+1)} \le \kappa_m \le \frac{2m+1}{2}\Big(\frac{\pi}{2}\Big)^{2/(2m+1)}\Big(2m\sin\frac{\pi}{2m}\Big)^{-2m/(2m+1)}$$

and $\lim_{m\to\infty}\frac{\kappa_m}{m} = \frac{1}{\pi}$. Furthermore,

$$\liminf_{t\to\infty}\Big(\frac{\log\log t}{t}\Big)^{(2m+1)/2} \sup_{0\le s\le t}|X_m(s)| = \kappa_m^{(2m+1)/2}. \tag{1.2}$$

Let

$$M(t) = \sup_{0\le s\le t}|X_m(s)| \quad \text{and} \quad \eta_n(t) = \frac{M(nt)}{(\kappa_m n/\log\log n)^{(2m+1)/2}},$$

where κ_m is defined as in (1.1). The main purpose of this paper is to establish a functional LIL for the $\{\eta_n(\cdot)\}$, then use this result to obtain the Chung's LIL and some weighted occupation measures for this process. This kind of functional LIL has ever been considered by Wichura [18] for standard Brownian motion, Chen, Kuelbs and Li [1] for the symmetric stable process, and Kuelbs and Li [8] for fractional Brownian motion. And it is different from that of classic Strassen's functional LIL.

For convenience, in this paper we define

$$\begin{aligned}\mathbf{U} = \{f : f \text{ maps } [0,\infty) \text{ to } [0,\infty] \text{ with } f(0)=0,\ \lim_{t\to\infty} f(t) = \infty \\ \text{and } f \text{ is right continuous, nondecreasing}\}\end{aligned}$$

and let $\mathbf{U}$ be endowed with the topology of weak convergence, i.e., pointwise convergence at all continuity points of the limit function. Then the weak topology on $\mathbf{U}$ is metrizable, separable and complete. More detail can be found in Chen, Kuelbs and Li [1].

For any sequence $\{f_n\} \subseteq \mathbf{U}$, we define $C(\{f_n\})$ as the cluster set of $\{f_n\}$. That is, all possible subsequential limits of $\{f_n\}$ in the weak topology. Assume that $\{f_n\}$ is relatively compact in the weak topology and $C(\{f_n\}) = A \subseteq \mathbf{U}$, then we write it as $\{f_n\} \Rightarrow A$.

The first result we obtain is the functional LIL for $X_m(t)$.

Theorem 1.1 *Let $\{X_m(t), t \in R_+\}$ be an m-fold integrated Brownian motion with positive integer m. Then*

$$P(\{\eta_n\} \Rightarrow \mathbf{K}) = 1, \tag{1.3}$$

where

$$\mathbf{K} = \Big\{ f : f \in \mathbf{U}, \int_0^\infty f^{-2/(2m+1)}(t)dt \le 1 \Big\}.$$

As a consequence of Theorem 1.1, we have the following Chung's LIL.

Corollary 1.1 *Let* $\{\eta_n(\,\cdot\,)\}$ *be defined as that in Theorem* 1.1. *Then*

$$\liminf_{n\to\infty}\eta_n(1)=1 \quad a.s.$$

Remark 1.1 A similar argument shows that if n is changed into T, then as $T\to\infty$, the conclusions of Theorem 1.1 and Corollary 1.1 are also true, and hence we have the conclusion (1.2); As $T\to 0$, if we write $\eta_n(x)$ as

$$\widetilde{\eta}_T(x)=\frac{M(Tx)}{(\kappa_m T/\log\log 1/T)^{(2m+1)/2}},$$

then the conclusions of Theorem 1.1 and Corollary 1.1 are true for $\widetilde{\eta}_T$.

The next results is about the weighted occupation measure for the m-fold Brownian motion. This is very similar to that of Chen et al. [1] for symmetric stable processes and Kuelbs and Li [8] for fractional Brownian motion. We have $\liminf\limits_{n\to\infty}\eta_n(1)=1$ by Corollary 1.1, but how fast does $\eta_n(\,\cdot\,)$ get away from zero function, say over interval $[0,1]$, and how many samples $\eta_n(\,\cdot\,)$, $n\le t$ fall into the interval $[0,c]$, $c\ge 1$? One measure of these questions is the weighted occupation measure

$$\varphi_c(t)=\frac{1}{t}\int_0^t I_{[0,c]}\Big(\eta_s(1)\theta\Big(\frac{s}{t}\Big)\Big)ds, \tag{1.4}$$

where $c\ge 1$, $\theta:(0,1]\to[1,\infty)$ with $\theta(1)=1$, $\theta(s)$ is non-increasing and $\lim\limits_{s\to 0^+}\theta(s)=\infty$.

Let

$$h(s)=\theta^{2/(2m+1)}(s)+\int_s^1\frac{\theta^{2/(2m+1)}(u)}{u}du, \quad 0\le s\le 1. \tag{1.5}$$

Then the range of $h(s)$ is all of $[1,\infty)$. For more detail about this, we refer to Chen et al. [1] and Kuelbs and Li [8].

Corollary 1.2 *Let* $\theta:(0,1]\to[1,\infty)$ *be defined as above. If* $h(s)$ *defined as in* (1.5) *is a strictly decreasing and continuous function from* $(0,1]$ *to* $[1,\infty)$, *then*

$$\limsup_{t\to\infty}\varphi_c(t)=1-s_c \quad a.s., \tag{1.6}$$

where $s=s_c$ *is the solution to* $h(s)=c^{2/(2m+1)}$, $c\ge 1$.

Example Let $\theta(s)=(1+\log\frac{1}{s})^{(2m+1)/2}$, $0<s\le 1$. Then

$$s_c=\exp\Big\{2-2\sqrt{1+\frac{c^{2/(2m+1)}-1}{2}}\Big\},$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t I_{[0,c]}\Big(\eta_s(1)\Big(1+\log\frac{t}{s}\Big)^{(2m+1)/2}\Big)ds=1-\exp\Big\{2-2\sqrt{1+\frac{c^{2/(2m+1)}-1}{2}}\Big\} \quad \text{a.s.}$$

Corollary 1.3

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t I_{[0,c]}\Big(\eta_t\Big(\frac{s}{t}\Big)\Big)ds=\begin{cases}1 & \text{if } c\ge 1,\\ c^{2/(2m+1)} & \text{if } 0\le c<1.\end{cases} \tag{1.7}$$

2 Proof of Theorem 1.1

In this section, we will show Theorem 1.1 with the application of the small ball probability estimates. In order to do this, we need the following lemmas.

Lemma 2.1 *Fix sequences* $\{t_i\}_{i=1}^l, \{a_i\}_{i=1}^l$ *and* $\{b_i\}_{i=1}^l$ *such that* $0 = t_0 < t_1 < \cdots < t_l$, *and* $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_l < b_l < \infty$. *Then*

$$\limsup_{\varepsilon \to 0} \varepsilon^{2/(2m+1)} \log P\{a_i\varepsilon \le M(t_i) \le b_i\varepsilon, 1 \le i \le l\} \le -\kappa_m \sum_{i=1}^l \frac{t_i - t_{i-1}}{b_i^{2/(2m+1)}}.$$

Proof Let $A_i = \Big\{\sup_{t_{i-1} \le t \le t_i} |X(t)| \le b_i\varepsilon\Big\},\ 1 \le i \le l$. Then

$$P\{a_i\varepsilon \le M(t_i) \le b_i\varepsilon,\ 1 \le i \le l\} \le P\Big(\bigcap_{i=1}^l A_i\Big). \tag{2.1}$$

For any $t \ge t_i,\ 0 \le i \le l-1$, we set $\xi_0 = 0$ and

$$\xi_i(t) = (m!)^{-1} \int_0^{t_i} (t-s)^m dW(s).$$

Then

$$\begin{aligned} P\Big(\bigcap_{i=1}^l A_i\Big) &= P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1} \le t \le t_l} |X(t)| \le b_l\varepsilon\Big) \\ &= P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1} \le t \le t_l} \Big|\xi_{l-1}(t) + \int_{t_{l-1}}^t \frac{(t-s)^m}{m!} dW(s)\Big| \le b_l\varepsilon\Big) \\ &= E\Big\{P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1} \le t \le t_l} \Big|\int_{t_{l-1}}^t \frac{(t-s)^m}{m!} dW(s) + \xi_{l-1}(t)\Big| \le b_l\varepsilon\Big)\Big|\xi_{l-1}(t)\Big\}. \end{aligned} \tag{2.2}$$

Since $\int_{t_{l-1}}^t \frac{(t-s)^m}{m!} dW(s)$ is independent of $A_i,\ 1 \le i \le l-1$ and $\xi_{l-1}(t)$, it follows from the Anderson inequality that the right-hand side of (2.2) is no more than

$$P\Big(\bigcap_{i=1}^{l-1} A_i\Big) P\Big(\sup_{t_{l-1} \le t \le t_l} \Big|\int_{t_{l-1}}^t \frac{(t-s)^m}{m!} dW(s)\Big| \le b_l\varepsilon\Big). \tag{2.3}$$

Clearly, $\int_a^{a+h} \frac{(a+h-s)^m}{m!} dW(s)$ and $X_m(h) = \int_0^h \frac{(h-s)^m}{m!} dW(s)$ have the same distribution. This yields that (2.3) is equal to

$$P\Big(\bigcap_{i=1}^{l-1} A_i\Big) P\Big(\sup_{0 \le t \le t_l - t_{l-1}} |X_m(t)| \le b_l\varepsilon\Big),$$

which in combination with (2.2) implies

$$P\Big(\bigcap_{i=1}^l A_i\Big) \le P\Big(\bigcap_{i=1}^{l-1} A_i\Big) P\Big(\sup_{0 \le t \le t_l - t_{l-1}} |X_m(t)| \le b_l\varepsilon\Big).$$

Iterating this argument and using the scaling property $X_m(ct) = c^{(2m+1)/2}X_m(t)$, we obtain

$$P\Big(\bigcap_{i=1}^{l} A_i\Big) \le \prod_{i=1}^{l} P\Big(\sup_{0\le t\le t_i-t_{i-1}} |X_m(t)| \le b_i\varepsilon\Big)$$
$$= \prod_{i=1}^{l} P\Big(\sup_{0\le t\le 1} |X_m(t)| \le \frac{b_i\varepsilon}{(t_i-t_{i-1})^{(2m+1)/2}}\Big). \tag{2.4}$$

Combining (1.1) with (2.4) yields

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log P\Big(\bigcap_{i=1}^{l} A_i\Big)$$
$$\le \lim_{\varepsilon\to 0} \sum_{i=1}^{l} \varepsilon^{2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} |X_m(t)| \le \frac{b_i\varepsilon}{(t_i-t_{i-1})^{(2m+1)/2}}\Big) = -\kappa_m \sum_{i=1}^{l} \frac{t_i-t_{i-1}}{b_i^{2/(2m+1)}},$$

which in combination with (2.1) draws the conclusion of Lemma 2.1.

The following lemma is a Gaussian correlation conjecture obtained by Li [12].

Lemma 2.2 *Let U be a centered Gaussian measure on separable Banach space E. Then for any two symmetric U-measurable convex sets A, B on E and any $0 < \lambda < 1$,*

$$U(A\cap B) \ge U(\lambda A)U(\sqrt{1-\lambda^2}B).$$

Lemma 2.3 *Let $(X_1, X_2, \cdots, X_N)$ be a centered Gaussian vector in R^N. Then for any positive numbers λ_i, $1 \le i \le N$,*

$$P\Big\{\bigcap_{i=1}^{N} (|X_i| \le \lambda_i)\Big\} \ge \prod_{i=1}^{N} P(|X_i| \le \lambda_i).$$

This lemma is due to Khatri [5] and Šidák [14], and is always called the Khatri-Šidák lemma, in which the covariance matrix of $(X_1, X_2, \cdots, X_n)$ is arbitrary.

The following lemma is a lower bound for the small ball probability in Lemma 2.1.

Lemma 2.4 *Let $\{t_i\}, \{a_i\}, \{b_i\}, 1 \le i \le l$ be as in Lemma* 2.1. *Then*

$$\liminf_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log P\{a_i\varepsilon \le M(t_i) \le b_i\varepsilon,\ 1 \le i \le l\} \ge -\kappa_m \sum_{i=1}^{l} \frac{t_i-t_{i-1}}{b_i^{2/(2m+1)}}. \tag{2.5}$$

Proof Let A_i, ξ_i be defined as in Lemma 2.1. By a similar argument of Kuelbs and Li [8], it is sufficient for the proof of Lemma 2.4 to show

$$\liminf_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log P\Big(\bigcap_{i=1}^{l} A_i\Big) \le -\kappa_m \sum_{i=1}^{l} \frac{t_i-t_{i-1}}{b_i^{2/(2m+1)}}. \tag{2.6}$$

For any $0 < \delta < \min\{b_i, 1 \le i \le l\}$ and $0 < \lambda < 1$, using Lemma 2.2 we have

$$\begin{aligned}
&P\Big(\bigcap_{i=1}^{l} A_i\Big) = P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1}\le t\le t_l}\Big|\int_{t_{l-1}}^{t}\frac{(t-s)^m}{m!}dW(s)+\xi_{l-1}(t)\Big| \le b_l\varepsilon\Big)\\
&\ge P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1}\le t\le t_l}\Big|\int_{t_{l-1}}^{t}\frac{(t-s)^m}{m!}dW(s)\Big| \le (b_l-\delta)\varepsilon, \sup_{t_{l-1}\le t\le t_l}|\xi_{l-1}(t)| \le \delta\varepsilon\Big)\\
&= P\Big(\sup_{t_{l-1}\le t\le t_l}\Big|\int_{t_{l-1}}^{t}\frac{(t-s)^m}{m!}dW(s)\Big| \le (b_l-\delta)\varepsilon\Big)P\Big(\bigcap_{i=1}^{l-1} A_i, \sup_{t_{l-1}\le t\le t_l}|\xi_{l-1}(t)| \le \delta\varepsilon\Big)\\
&\ge P\Big(\sup_{0\le t\le t_l-t_{l-1}}|X_m(t)| \le (b_l-\delta)\varepsilon\Big)\cdot P\Big(\bigcap_{i=1}^{l-1}\sup_{t_{i-1}\le t\le t_i}|X(t)| \le b_i\lambda\varepsilon\Big)\\
&\quad\cdot P\Big(\sup_{t_{l-1}\le t\le t_l}|\xi_{l-1}(t)| \le \sqrt{1-\lambda^2}\,\delta\varepsilon\Big). \qquad (2.7)
\end{aligned}$$

Repeating this estimate, we get

$$P\Big(\bigcap_{i=1}^{l} A_i\Big) \ge H_1H_2, \qquad (2.8)$$

where

$$\begin{aligned}
H_1 &= \prod_{i=1}^{l} P\Big(\sup_{0\le t\le t_i-t_{i-1}}|X_m(t)| \le (b_i-\delta)\lambda^{l-i}\varepsilon\Big),\\
H_2 &= \prod_{i=1}^{l} P\Big(\sup_{t_{i-1}\le t\le t_i}|\xi_{i-1}(t)| \le \sqrt{1-\lambda^2}\,\delta\lambda^{l-i}\varepsilon\Big).
\end{aligned}$$

Next we turn to estimate the lower bounds of H_1 and H_2. By (1.1) and the scaling property of $X_m(t)$, it is easy to obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}\log H_1 &= \sum_{i=1}^{l}\lim_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}\log P\Big(\sup_{0\le t\le 1}|X_m(t)| \le \frac{\lambda^{l-i}(b_i-\delta)\varepsilon}{(t_i-t_{i-1})^{(2m+1)/2}}\Big)\\
&= -\kappa_m\sum_{i=1}^{l}\frac{t_i-t_{i-1}}{[\lambda^{l-i}(b_i-\delta)]^{2/(2m+1)}}. \qquad (2.9)
\end{aligned}$$

Note that

$$\begin{aligned}
\xi_{i-1}(t) &= \frac{1}{m!}\int_0^{t_{i-1}}(t-t_{i-1}+t_{i-1}-s)^m dW(s)\\
&= \sum_{k=0}^{m}\frac{C_m^k(t-t_{i-1})^k}{m!}\int_0^{t_{i-1}}(t_{i-1}-s)^{m-k}dW(s)\\
&= \sum_{k=0}^{m}\frac{(t-t_{i-1})^k}{k!}X_{m-k}(t_{i-1}).
\end{aligned}$$

By Lemma 2.3, it follows that

$$\begin{aligned}
H_2 &= \prod_{i=1}^{l} P\Big(\sup_{t_{i-1}\le t\le t_i} |\xi_{i-1}(t)| \le \sqrt{1-\lambda^2}\,\delta\lambda^{l-i}\varepsilon\Big) \\
&= \prod_{i=1}^{l} P\Big(\sum_{k=0}^{m} \frac{(t_i-t_{i-1})^k}{k!}|X_{m-k}(t_{i-1})| \le \sqrt{1-\lambda^2}\,\delta\lambda^{l-i}\varepsilon\Big) \\
&\ge \prod_{i=1}^{l} P\Big\{\bigcap_{k=0}^{m}\Big(|X_{m-k}(t_{i-1})| \le \frac{k!\sqrt{1-\lambda^2}\,\delta\lambda^{l-i}\varepsilon}{m(t_i-t_{i-1})^k}\Big)\Big\} \\
&\ge \prod_{i=1}^{l}\prod_{k=0}^{m} P\Big(|X_{m-k}(t_{i-1})| \le \frac{k!\sqrt{1-\lambda^2}\,\delta\lambda^{l-i}\varepsilon}{m(t_i-t_{i-1})^k}\Big).
\end{aligned} \tag{2.10}$$

Since for any $1 \le k \le m$, $EX_{m-k}(t_{i-1}) = 0$ and

$$EX_{m-k}^2(t_{i-1}) = \frac{1}{(m-k)!}\int_0^{t_{i-1}} (t_{i-1}-s)^{2(m-k)}ds = \frac{t_{i-1}^{2(m-k)+1}}{(m-k)!(2m-2k+1)}, \tag{2.11}$$

it follows that

$$\begin{aligned}
H_2 &\ge \prod_{i=1}^{l}\prod_{k=0}^{m} P\Big(\frac{\sqrt{(m-k)!(2m-2k+1)}\,|X_{m-k}(t_{i-1})|}{t_{i-1}^{(m-k)+1/2}} \\
&\qquad\qquad \le \frac{k!\sqrt{(m-k)!(2m-2k+1)(1-\lambda^2)}\,\delta\lambda^{l-i}\varepsilon}{mt_{i-1}^{(m-k)+1/2}(t_i-t_{i-1})^k}\Big) \\
&= \prod_{i=1}^{l}\prod_{k=0}^{m} P(|N(0,1)| \le A_{ik}\varepsilon),
\end{aligned}$$

where $N(0,1)$ denotes the standard normal variable and

$$A_{ik} = A(\lambda, t_i, t_{i-1}, \delta, k) = \frac{k!\sqrt{(m-k)!(2m-2k+1)(1-\lambda^2)}\,\delta\lambda^{l-i}}{mt_{i-1}^{(m-k)+1/2}(t_i-t_{i-1})^k}.$$

Then

$$H_2 \ge \prod_{i=1}^{l}\prod_{k=0}^{m} P(|N(0,1)| \le A_{ik}\varepsilon) = \prod_{i=1}^{l}\prod_{k=0}^{m}\int_{-A_{ik}\varepsilon}^{A_{ik}\varepsilon}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \ge \prod_{i=1}^{l}\prod_{k=0}^{m}\frac{2A_{ik}\varepsilon e^{-(A_{ik}\varepsilon)^2/2}}{\sqrt{2\pi}},$$

which implies

$$\lim_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}\log H_2 = \lim_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}\Big[\sum_{i=1}^{l}\sum_{k=0}^{m}\Big(\log\frac{2A_{ik}\varepsilon}{\sqrt{2\pi}} - \frac{A_{ik}^2\varepsilon^2}{2}\Big)\Big] = 0. \tag{2.12}$$

Therefore, by (2.8), (2.9) and (2.12), we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}\log P\Big(\bigcap_{i=1}^{l} A_i\Big) &\ge \liminf_{\varepsilon\to 0}\varepsilon^{2/(2m+1)}(\log H_1 + \log H_2) \\
&\ge -\kappa_m\sum_{i=1}^{l}\frac{t_i-t_{i-1}}{[\lambda^{l-i}(b_i-\delta)]^{2/(2m+1)}}.
\end{aligned}$$

Letting $\lambda \nearrow 1$ and $\delta \searrow 0$, we get (2.6). This completes the proof of Lemma 2.4.

Proof of Theorem 1.1 By a similar argument of Chen, Kuelbs, Li [1] or Kuelbs, Li [8], in order to show Theorem 1.1, it is enough to establish the following three propositions

$$P(\{\eta_n\} \text{ is relatively compact in } \mathbf{U}) = 1, \tag{2.13}$$

$$P(C(\{\eta_n\}) \subseteq \mathbf{K}) = 1, \tag{2.14}$$

$$P(\mathbf{K} \subseteq C(\{\eta_n\})) = 1. \tag{2.15}$$

Applying (1.1) and the small ball probability estimate in Lemma 2.1 and along the proof of Chen, Kuelbs and Li [8], (2.13) and (2.14) are easy to obtain. However, the proof of (2.15) is different from theirs.

From the arguments of Chen, et al. [1] (see also Kuelbs and Li [8]), in order to show (2.15), it is enough to prove, for any fixed $f \in \mathbf{U}$ with $\int_0^\infty f^{-2/(2m+1)}(t)dt < 1$ and every weak neighborhood N_f of f, that

$$P\{\eta_n \in N_f, \text{i.o.}\} = 1. \tag{2.16}$$

To establish (2.16), we define

$$t_f^* = \sup\{t : f(t) < \infty\}.$$

Since $f \in \mathbf{U}$, f is right continuous and $f(0) = 0$. This implies that $t_f^* = \infty$ or $0 < t_f^* < \infty$.

In the case of $t_f^* = \infty$, a typical neighborhood of f is of the form $N_f = \bigcap\limits_{i=1}^{p} G_i$, where

$$G_i = \{g : f(t_i) - \theta < g(t_i) < f(t_i) + \theta\}, \quad \theta > 0$$

and $0 = t_0 < t_1 < t_2 < \cdots < t_p$.

Otherwise, if $0 < t_f^* < \infty$, then a typical neighborhood of f can be written as $N_f = (\bigcap\limits_{i=1}^{p} G_i) \bigcap (\bigcap\limits_{j=1}^{q} \widetilde{G}_{p+j})$, where $G_i,\ 1 \le i \le p$ is defined as above and

$$\widetilde{G}_{p+j} = \{g : g(t_{p+j}) > m_j\}, \quad m_j > 0,\ 1 \le j \le q.$$

The proof for either case is similar. Therefore, we only consider here the case of $t_f^* = \infty$, that is, $N_f = \bigcap\limits_{i=1}^{p} G_i$.

Let

$$\beta = 1 - \int_0^\infty f^{-2/(2m+1)}(t)dt > 0, \quad n_l = \exp\{l^{1+\beta}\}.$$

For any large l satisfying $n_{l-1}t_p/n_l < t_1$ (In fact, there exists a $l_0 \in N$ such that for any $l > l_0$, $n_{l-1}t_p/n_l < t_1$.) define

$$A_l = \Big\{ \sup_{\frac{n_{l-1}t_p}{n_l} \le t \le t_i} \Big(\frac{\kappa_m n_l}{\log\log n_l}\Big)^{-(2m+1)/2} \Big| \int_{n_{l-1}t_p}^{n_l t} \frac{(n_l t - s)^m}{m!} dW(s) \Big| \in \Big(f(t_i) - \frac{\theta}{2}, f(t_i) + \frac{\theta}{2}\Big), 1 \le i \le p \Big\},$$

$$B_l = \Big\{ \sup_{\frac{n_{l-1}t_p}{n_l} \le t \le t_i} \Big(\frac{\kappa_m n_l}{\log\log n_l}\Big)^{-(2m+1)/2} \Big| \int_0^{n_{l-1}t_p} \frac{(n_l t - s)^m}{m!} dW(s) \Big| \le \frac{\theta}{2},\ 1 \le i \le p \Big\},$$

$$C_l = \{\eta_{n_l}(t_i) \in (f(t_i) - \theta, f(t_i) + \theta),\ 1 \le i \le p\}.$$

Then $A_l \cap B_l \subseteq C_l$ and this yields

$$P(C_l, \text{i.o.}) \geq P(A_l \cap B_l, \text{i.o.}) \geq P(A_l, \text{i.o.}) - P(B_l^c, \text{i.o.}). \tag{2.17}$$

Note that

$$\int_{n_{l-1}t_p}^{n_l t} \frac{(n_l t - s)^m}{m!} dW(s)$$

and $X_m(n_l t - n_{l-1}t_p)$ have the same distribution. Hence, by the scaling property of $X_m(t)$ and Lemma 2.4, we have that for any $0 < \gamma < \beta^2/(1-\beta^2)$,

$$\begin{aligned} P(A_l) &= P\Big\{\sup_{0 \leq u \leq n_l t_i - n_{l-1}t_p} \Big(\frac{\kappa_m n_l}{\log\log n_l}\Big)^{-(2m+1)/2} |X_m(u)| \in \Big(f(t_i) - \frac{\theta}{2}, f(t_i) + \frac{\theta}{2}\Big),\ i \leq p\Big\} \\ &= P\Big\{\sup_{0 \leq u \leq t_i - \frac{n_{l-1}t_p}{n_l}} \Big(\frac{\kappa_m}{\log\log n_l}\Big)^{-(2m+1)/2} |X_m(u)| \in \Big(f(t_i) - \frac{\theta}{2}, f(t_i) + \frac{\theta}{2}\Big),\ i \leq p\Big\} \\ &\geq \exp\Big\{ - \Big(\frac{t_1 - n_{l-1}t_p/n_l}{(f(t_1) + \theta/2)^{2/(2m+1)}} + \sum_{i=2}^{p} \frac{t_i - t_{i-1}}{(f(t_i) + \theta/2)^{2/(2m+1)}}\Big)(1+\gamma)\log\log n_l\Big\}. \end{aligned}$$

Since $f(t)$ is non-decreasing and non-negative function, it follows that

$$\begin{aligned} & - \Big(\frac{t_1 - n_{l-1}t_p/n_l}{(f(t_1) + \theta/2)^{2/(2m+1)}} + \sum_{i=2}^{p} \frac{t_i - t_{i-1}}{(f(t_i) + \theta/2)^{2/(2m+1)}}\Big) \\ \geq & - \int_0^\infty \frac{1}{(f(t) + \theta/2)^{2/(2m+1)}} dt \geq - \int_0^\infty f^{-2/(2m+1)}(t) dt = -(1-\beta), \end{aligned}$$

which implies that

$$P(A_l) \geq \exp\{-(1-\beta)(1+\gamma)\log l^{1+\beta}\} = l^{-(1+\beta)(1-\beta)(1+\gamma)}.$$

Because $0 < \gamma < \beta^2/(1-\beta^2)$, we get $(1-\beta^2)(1+\gamma) < 1$. This yields $\sum_{l=1}^{\infty} P(A_l) = \infty$. Therefore, by the independence of $A_l,\ l = 1, 2 \cdots$, and the Borel-Cantelli lemma, it follows that

$$P(A_l, \text{i.o.}) = 1. \tag{2.18}$$

By some elementary computation, we have

$$B_l = \Big\{\sum_{k=0}^{m} \frac{[(n_l - n_{l-1})t_p]^k |X_{m-k}(n_{l-1}t_p)|}{k!(\kappa_m n_l/\log\log n_l)^{(2m+1)/2}} \leq \frac{\theta}{2}\Big\},$$

which implies that

$$\begin{aligned} P(B_l^c) &= P\Big\{\sum_{k=0}^{m} \frac{[(n_l - n_{l-1})t_p]^k |X_{m-k}(n_{l-1}t_p)|}{k!(\kappa_m n_l/\log\log n_l)^{(2m+1)/2}} \geq \frac{\theta}{2}\Big\} \\ &\leq \sum_{k=0}^{m} P\Big\{\frac{[(n_l - n_{l-1})t_p]^k |X_{m-k}(n_{l-1}t_p)|}{k!(\kappa_m n_l/\log\log n_l)^{(2m+1)/2}} \geq \frac{\theta}{2m}\Big\}. \end{aligned} \tag{2.19}$$

By (2.11), the right-hand side of (2.19) is equal to

$$\begin{aligned}&\sum_{k=0}^{m} P\Big\{|N(0,1)| \ge \frac{\theta[(m-k)!(2m-2k+1)]^{1/2}k!(\kappa_m n_l/\log\log n_l)^{(2m+1)/2}}{2m(n_{l-1}t_p)^{m-k+1/2}[(n_l-n_{l-1})t_p]^k}\Big\}\\ &\le \sum_{k=0}^{m} P\Big\{|N(0,1)| \ge \frac{\kappa_m^{m+1/2}}{2mt_p^{m+1/2}}(n_l/n_{l-1})^{1/4}\Big\}\\ &\le (m+1)P\Big\{|N(0,1)| \ge \frac{\kappa_m^{m+1/2}}{2mt_p^{m+1/2}}\exp\{l^\beta\}\Big\}. \qquad (2.20)\end{aligned}$$

Obviously, the right-hand side of (2.20) is summable. This in combination with (2.19) and the Borel-Cantelli lemma implies that

$$P(B_l^c, \text{i.o.}) = 0. \qquad (2.21)$$

By (2.17), (2.18) and (2.21), we have $P(C_l, \text{i.o.}) = 1$. That is,

$$P\{\eta_{n_l} \in N_f, \text{i.o.}\} = 1.$$

This implies (2.15), and the proof is completed.

Proof of Corollary 1.1 The proof is similar to that of Chen, Kuelbs and Li [1], here we will omit the details.

Proof of Corollary 1.2 and Corollary 1.3 In order to show Corollary 1.2 and Corollary 1.3, we need the following two facts.

Fact 2.1 Let $F_c(f) = \int_0^1 I_{[0,c]}(f(u)r(u))du$ and

$$G_c(t) = \int_0^1 I_{[0,c]}\Big(\eta_t(u)r(u)\Big(\frac{\log\log tu}{\log\log t}\Big)^{(2m+1)/2}\Big)du,$$

where $r : (0,1] \to [0,\infty)$ is measurable. Then for each $c > 0$,

$$\limsup_{t\to\infty} G_c(t) \le \sup_{f\in\mathbf{K}} F_c(f) \quad \text{a.s.} \qquad (2.22)$$

Furthermore, if $\sup_{f\in\mathbf{K}} F_c(f)$ is left continuous at c, then the two sides in (2.22) are equal.

Along the proof line of Chen, Kuelbs and Li [1], we obtain the conclusion of Fact 2.1.

The following fact is due to Kuelbs and Li [8].

Fact 2.2 Let g be a real-valued, non-negative and continuous function on $(0,1]$ with $0 < g(1) < 1$. Suppose that $tg(t)$ is non-increasing on $(0,1]$ and $\lim_{t\to 0} tg(t) > 1$. Then

$$\sup_{f\in\mathbf{F}} \int_0^1 I_{\{t:f(t)\ge g(t)\}}(x)dx = 1 - u_0,$$

where $\mathbf{F}$ is the set of non-negative, non-increasing, right continuous functions f on $(0,1]$ with $\int_0^1 f(t)dt \le 1$ and $u = u_0$ is the solution of the equation

$$ug(u) + \int_u^1 g(v)dv = 1.$$

By Fact 2.1 and Fact 2.2, Corollary 1.2 can be drawn as that of Kuelbs and Li [8] did. By Fact 2.1 and a similar argument of Chen, Kuelbs and Li [1], Corollary 1.3 is followed.

3 The Functional LIL for Integrated Kiefer Process

Let $W(x,y)$ be a two parameter Wiener process. A Kiefer process $K(x,y), 0 \le x \le 1, 0 \le y < \infty$ is defined by

$$K(x,y) = W(x,y) - xW(1,y).$$

In this section, we will use the method in Theorem 1.1 to study the functional LIL for integrated Kiefer process and then with the help of strong approximation result of a Kiefer process to draw the Chung LIL for integrated empirical process.

Theorem 3.1 *Let $\{K(x,y), 0 \le x \le 1, 0 \le y < \infty\}$ be a Kiefer process. Define $Y_0(x,y) = K(x,y)/\sqrt{y}$, $Y_m(x,y) = \int_0^x Y_{m-1}(s,y)ds$ and $\widetilde{M}_n(x) = \sup_{0 \le t \le x} |Y_m(t,n)|$,*

$$\widetilde{\eta}_n(x) = \frac{\widetilde{M}_n(x)}{(\kappa_m/\log\log n)^{(2m+1)/2}}.$$

Then

$$P(\{\widetilde{\eta}_n(x)\} \Rightarrow \widetilde{\mathbf{K}}) = 1,$$

where

$$\widetilde{\mathbf{K}} = \Big\{f : f \in \widetilde{\mathbf{U}},\ \int_0^1 f^{-2/(2m+1)}(t)dt \le 1\Big\}$$

and

$$\widetilde{\mathbf{U}} = \{f : f \text{ maps } [0,1] \text{ to } [0,\infty] \text{ with } f(0) = 0,\ f \text{ is right continuous, nondecreasing}\}.$$

Proof Noting that

$$K(t,y) = W(t,y) - tW(1,y),$$

where $\{W(x,y), 0 \le x, y < \infty\}$ is a two-parameter (standard) Brownian motion, we have

$$Y_m(t,y) = \int_0^t Y_{m-1}(x,y)dx = \int_0^t \frac{(t-s)^m}{\sqrt{y}\,m!} dW(s,y) - \frac{t^m}{\sqrt{y}\,m!} W(1,y).$$

It is easy to see that for any $y > 0$,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0 \le t \le 1} \Big|\frac{t^m}{\sqrt{y}\,m!} W(1,y)\Big| \le \varepsilon\Big) \\ &= \lim_{\varepsilon \to 0} \varepsilon^{-2/(2m+1)} P(|N(0,1)| \le m!\varepsilon) = 0. \end{aligned} \tag{3.1}$$

Hence, by the independence of $\int_0^t \frac{(t-s)^m}{\sqrt{y}\,m!} dW(s,y)$ and $\frac{t^m}{\sqrt{y}\,m!} W(1,y)$, it follows from (1.1) that for any $\delta > 0$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} |Y_m(t,y)| \le \varepsilon\Big) \\
&\ge \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \Big(\log P\Big(\sup_{0\le t\le 1} \Big|\int_0^t \frac{(t-s)^m}{\sqrt{y}m!} dW(s,y)\Big| \le (1-\delta)\varepsilon\Big) \\
&\quad + \log P\Big(\sup_{0\le t\le 1} \Big|\frac{t^m}{\sqrt{y}m!} W(1,y)\Big| \le \delta\varepsilon\Big)\Big) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} |X_m(t)| \le (1-\delta)\varepsilon\Big) = -\kappa_m (1-\delta)^{2/(2m+1)}. \qquad (3.2)
\end{aligned}$$

On the other hand, by (1.1)

$$\begin{aligned}
&-\kappa_m(1+\delta)^{2/(2m+1)} \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} \Big|\int_0^t \frac{(t-s)^m}{\sqrt{y}m!} dW(s,y)\Big| \le (1+\delta)\varepsilon\Big) \\
&\ge \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \Big(\log P\Big(\sup_{0\le t\le 1} |Y_m(t,y)| \le \varepsilon\Big) + \log P\Big(\sup_{0\le t\le 1} \Big|\frac{t^m}{\sqrt{y}m!} W(1,y)\Big| \le \delta\varepsilon\Big)\Big) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \Big(\log P\Big(\sup_{0\le t\le 1} |Y_m(t,y)| \le \varepsilon\Big) + \log P(|N(0,1)| \le m!\delta\varepsilon)\Big) \\
&= \lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} |Y_m(t,y)| \le \varepsilon\Big). \qquad (3.3)
\end{aligned}$$

Let $\delta \searrow 0$. Then (3.2) and (3.3) imply

$$\lim_{\varepsilon\to 0} \varepsilon^{-2/(2m+1)} \log P\Big(\sup_{0\le t\le 1} |Y_m(t,y)| \le \varepsilon\Big) = -\kappa_m. \qquad (3.4)$$

Furthermore, by Lemma 2.1, Lemma 2.4 and (3.1), we have that for any fixed sequences $\{t_i\}_{i=1}^l, \{a_i\}_{i=1}^l$ and $\{b_i\}_{i=1}^l$ such that $0 = t_0 < t_1 < \cdots < t_l \le 1$, and $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_l < b_l < \infty$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2/(2m+1)} \log P\{a_i\varepsilon \le \widetilde{M}_n(t_i) \le b_i\varepsilon, 1 \le i \le l\} = -\kappa_m \sum_{i=1}^{l} \frac{t_i - t_{i-1}}{b_i^{2/(2m+1)}}. \qquad (3.5)$$

Similar to the proof of Theorem 1.1, to show Theorem 3.1, it is enough to establish the following three propositions

$$P(\{\widetilde{\eta}_n\} \text{ is relatively compact in } \mathbf{U}) = 1, \qquad (3.6)$$

$$P(C(\{\widetilde{\eta}_n\}) \subseteq \widetilde{\mathbf{K}}) = 1, \qquad (3.7)$$

$$P(\widetilde{\mathbf{K}} \subseteq C(\{\eta_n\})) = 1. \qquad (3.8)$$

By (3.3) and (3.5) and along the proof of Chen, Kuelbs and Li [8], (3.6) and (3.7) are easy to obtain. The proof of (3.8) is very similar to that of Theorem 1.1; we only need to revise the definitions of the sets A_l, B_l, C_l and define

$$\widetilde{A}_l = \Big\{ \sup_{0\le t\le t_i} \Big(\frac{\kappa_m}{\log\log n_l}\Big)^{-(2m+1)/2} \Big| \int_0^t \frac{(t-s)^m}{m!\sqrt{n}} d(W(s,n) - W(s, n_{l-1}t_p)) - \frac{t^m(W(1,n) - W(1,n_{l-1}t_p))}{m!\sqrt{n}} \Big| \in \Big(f(t_i) - \frac{\theta}{2}, f(t_i) + \frac{\theta}{2}\Big),\ i \le p\Big\},$$

$$\widetilde{B}_l = \Big\{ \sup_{0\le t\le t_i} \Big(\frac{\kappa_m}{\log\log n_l}\Big)^{-(2m+1)/2} \Big| \int_0^t \frac{(t-s)^m}{m!\sqrt{n}} dW(s, n_{l-1}t_p) + \frac{t^m W(1, n_{l-1}t_p)}{m!\sqrt{n}} \Big| \le \frac{\theta}{2},\ 1 \le i \le p\Big\},$$

$$\widetilde{C}_l = \Big\{ \widetilde{\eta}_{n_l}(t_i) \in (f(t_i) - \theta, f(t_i) + \theta),\ 1 \le i \le p\Big\}.$$

Then along the proof of Theorem 1.1, we can get the conclusion of Theorem 3.1.

Theorem 3.2 *Let* $\{\alpha_n(t), 0 \le t \le 1\}$ *be the empirical process based on the first* n *observations of independent variables with the uniform distribution in* $(0,1)$. *Define* $\Lambda_0(t,n) = \alpha_n(t)$ *and*

$$\Lambda_m(t,n) = \int_0^t \Lambda_{m-1}(x,n)dx.$$

Then

$$\liminf_{n\to\infty} (\log\log n)^{(2m+1)/2} \sup_{o\le t\le 1} |\Lambda_m(t,n)| = \kappa_m \quad a.s.$$

Proof By Theorem 3.1, we have

$$\liminf_{n\to\infty} (\log\log n)^{(2m+1)/2} \sup_{o\le t\le 1} |Y_m(t,n)| = \kappa_m \quad \text{a.s.} \tag{3.9}$$

By a strong approximation result of Kiefer [7] (see also [3, Theorem 4.3.2]) and (3.9), Theorem 3.2 follows, and the proof is completed.

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