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Global Entropy Solutions of the Cauchy Problem for Nonhomogeneous Relativistic Euler System**

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Abstract We analyze the 2×2 nonhomogeneous relativistic Euler equations for perfect fluids in special relativity. We impose appropriate conditions on the lower order source terms and establish the existence of global entropy solutions of the Cauchy problem under these conditions.

Keywords Relativistic Euler system, Entropy solutions, Riemann solutions, Glimm scheme
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1 Introduction

The relativistic Euler equations for a perfect fluid in two dimensional Minkowski space-time with source terms have the form:

$$\begin{cases} \partial_t\Big((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)=\Phi(\rho,v), \\ \partial_t\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\Big)=\Psi(\rho,v), \end{cases} \tag{1.1}$$

which models the balance laws of momentum and energy for relativistic fluids, where $\rho > 0$ is the proper energy density, p is the pressure, v is the particle speed, and c is the speed of light. The equation of state is

$$p = p(\rho),$$

where $p(\rho)$ is a smooth function of ρ and satisfies

$$p'(\rho) > 0, \quad p''(\rho) > 0.$$

In the corresponding physical region $\mathcal{V} = \{U : 0 \le \rho < \rho_{\max},\ |v| < c\}$ ($\rho_{\max} = \sup\{\rho : p'(\rho) \le c^2\}$), the system is strictly hyperbolic, and is genuinely nonlinear with respect to all the characteristics under certain assumptions on the pressure function p (see Lemma 2.1).

For the corresponding homogeneous systems (2×2 or 3×3), many mathematical problems have been studied, such as, the construction of the models, the local existence of smooth solutions, the global existence of BV solutions, the periodic solutions, the spherically symmetric

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solutions, the uniqueness and stability of solutions, the limit problems, and the kinetic schemes, etc. (see, e.g., [1, 4–6, 9–13, 15–22, 25–28, 30]; also consult Chen [2]).

The homogeneous system corresponding to (1.1) is

$$\begin{cases} \partial_t\Big((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)=0, \\ \partial_t\Big((p+\rho c^2)\dfrac{v}{c^2-v^2}\Big)+\partial_x\Big((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\Big)=0. \end{cases} \tag{1.2}$$

For the initial data

$$t=0: \quad \rho=\rho_0(x), \quad v=v_0(x), \tag{1.3}$$

where $\rho_0(x),\ v_0(x)\in\mathcal{V}$ are bounded functions with bounded variation, we study the Cauchy problem of (1.1) with the equation of state

$$p=\kappa^2\rho,$$

where $\kappa<c$ is the sound speed. In this case, system (1.1) reduces to

$$\begin{cases} \partial_t\Big(\rho\Big(\dfrac{\kappa^2+c^2}{c^2}\dfrac{v^2}{c^2-v^2}+1\Big)\Big)+\partial_x\Big(\rho\Big((\kappa^2+c^2)\dfrac{v}{c^2-v^2}\Big)\Big)=\Phi(\rho,v), \\ \partial_t\Big(\rho\Big((\kappa^2+c^2)\dfrac{v}{c^2-v^2}\Big)\Big)+\partial_x\Big(\rho\Big((\kappa^2+v^2)\dfrac{c^2v^2}{c^2-v^2}\Big)\Big)=\Psi(\rho,v). \end{cases} \tag{1.4}$$

The Cauchy problem of the non-relativistic version (the classical isentropic Euler system, i.e., $c=+\infty$) of (1.4) has been solved in [31]. They established the global existence of weak solutions. The purpose of this paper is to solve the Cauchy problem (1.3)–(1.4) based on the global existence of BV solutions for the homogeneous system (1.1) by Smoller-Temple [25] and adopting the methods of Ying-Wang [31] and Chen-Wagner [3]. We will impose appropriate conditions on the lower order source terms and establish the existence of the global entropy solutions under these conditions.

We also assume that the variation of $\rho_0(x)$ and $v_0(x)$ vanishes outside a bounded interval, namely, $\rho_0(x)=\rho_0(+\infty)$, $v_0(x)=v_0(+\infty)$ in a neighborhood of $x=+\infty$ and $\rho_0(x)=\rho_0(-\infty)$, $v_0(x)=v_0(-\infty)$ in a neighborhood of $x=-\infty$.

We organize this paper as follows. In §2, we discuss the conditions on the source terms and construct the approximate solutions by using the Glimm difference scheme and Picard iteration; In §3, we establish estimations on the approximate solutions, that is, we show that the approximate solutions are bounded functions of bounded variation and continuous in t in the sense of L^1; In §4, we prove that the limit solutions are the entropy solutions of the Cauchy problem (1.3)–(1.4).

2 The Primary Properties and the Difference Scheme

Problem (1.3)–(1.4) fits into the following Cauchy problem for a general system of conservation laws:

$$\begin{cases} U_t+F(U)_x=G(U), \\ t=0: U=U_0(x), \end{cases} \tag{2.1}$$

where

$$U=\Big(\rho\Big(\frac{\kappa^2+c^2}{c^2}\frac{v^2}{c^2-v^2}+1\Big),\rho\Big((\kappa^2+c^2)\frac{v}{c^2-v^2}\Big)\Big)^{\top}=(U_1,U_2)^{\top},$$
$$F(U)=\Big(\rho\Big((\kappa^2+c^2)\frac{v}{c^2-v^2}\Big),\rho\Big((\kappa^2+v^2)\frac{c^2v^2}{c^2-v^2}\Big)\Big)^{\top}=(F_1(U),F_2(U))^{\top},$$
$$G(U)=(\Phi(\rho,v),\Psi(\rho,v))^{\top}.$$

The corresponding homogeneous system is

$$U_t+F(U)_x=0. \tag{2.2}$$

We recall that the eigenvalues of the system are

$$\lambda=\frac{v-\kappa}{1-\frac{v\kappa}{c^2}},\quad \mu=\frac{v+\kappa}{1+\frac{v\kappa}{c^2}}. \tag{2.3}$$

It is easy to see that

Lemma 2.1 *System* (1.1) *is strictly hyperbolic if and only if*

$$\sqrt{p'(\rho)}<c, \tag{2.4}$$

in the case $p=\kappa^2\rho$, namely, $\kappa<c$.

Moreover, under (2.4)*, system* (1.1) *is genuinely nonlinear if*

$$p''(\rho)\geq -2\frac{(c^2-p')p'}{p+\rho c^2}.$$

We calculate that

$$dU\triangleq\frac{\partial(U_1,U_2)}{\partial(\rho,v)}=\begin{pmatrix}U_{1\rho} & U_{1v}\\ U_{2\rho} & U_{2v}\end{pmatrix}=\begin{pmatrix}\frac{c^4+\kappa^2v^2}{c^2(c^2-v^2)} & \frac{(\kappa^2+c^2)2\rho v}{(c^2-v^2)^2}\\ \frac{(\kappa^2+c^2)v}{c^2-v^2} & \frac{(\kappa^2+c^2)(c^2+v^2)\rho}{c^2-v^2}\end{pmatrix},$$

$$dU^{-1}=\begin{pmatrix}\frac{c^2(c^2+v^2)}{c^4-\kappa^2v^2} & -\frac{2c^2v}{c^4-\kappa^2v^2}\\ \frac{(c^2-v^2)c^2v}{(c^4-\kappa^2v^2)\rho} & \frac{(c^4+\kappa^2v^2)(c^2-v^2)}{(\kappa^2+c^2)(c^4-\kappa^2v^2)\rho}\end{pmatrix},$$

$$dF\triangleq\frac{\partial(F_1,F_2)}{\partial(\rho,v)}=\begin{pmatrix}F_{1\rho} & F_{1v}\\ F_{2\rho} & F_{2v}\end{pmatrix}=\begin{pmatrix}\frac{(\kappa^2+c^2)v}{c^2-v^2} & \frac{(\kappa^2+c^2)(c^2+v^2)\rho}{(c^2-v^2)^2}\\ \frac{(\kappa^2+v^2)c^2}{c^2-v^2} & \frac{(\kappa^2+c^2)2c^2v\rho}{(c^2-v^2)^2}\end{pmatrix}.$$

Then system (1.4) reduces to

$$dU\binom{\rho}{v}_t+dF\binom{\rho}{v}_x=G(U),$$

namely,

$$\binom{\rho}{v}_t+dU^{-1}dF\binom{\rho}{v}_x=dU^{-1}G(U)=\begin{pmatrix}\overline{\Phi}(\rho,v)\\ \overline{\Psi}(\rho,v)\end{pmatrix}, \tag{2.5}$$

where

$$
\begin{aligned}
dU^{-1}dF &= \begin{pmatrix} \frac{(c^2-\kappa^2)c^2v}{c^4-\kappa^2v^2} & \frac{(\kappa^2+c^2)c^2\rho}{c^4-\kappa^2v^2} \\ \frac{(c^2-v^2)^2\kappa^2c^2}{(c^4-\kappa^2v^2)(\kappa^2+c^2)\rho} & \frac{(c^2-\kappa^2)c^2v}{c^4-\kappa^2v^2} \end{pmatrix}, \\
dU^{-1}\begin{pmatrix}\Phi(\rho,v)\\ \Psi(\rho,v)\end{pmatrix} &= \begin{pmatrix}\overline{\Phi}(\rho,v)\\ \overline{\Psi}(\rho,v)\end{pmatrix}.
\end{aligned}
\tag{2.6}
$$

If we set

$$
w = \ln\frac{c-v}{c+v}, \tag{2.7}
$$

then we have the one-to-one mapping: $(\rho, v) \to (\rho, w)$, and

$$
\frac{\partial(\rho,v)}{\partial(\rho,w)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-2ce^w}{(1+e^w)^2} \end{pmatrix} \triangleq D.
$$

Then (2.5), thus (1.4), reduces to

$$
D\begin{pmatrix}\rho\\ w\end{pmatrix}_t + dU^{-1}dFD\begin{pmatrix}\rho\\ w\end{pmatrix}_x = \begin{pmatrix}\overline{\Phi}(\rho,v)\\ \overline{\Psi}(\rho,v)\end{pmatrix},
$$

namely,

$$
\begin{pmatrix}\rho\\ w\end{pmatrix}_t + D^{-1}dU^{-1}dFD\begin{pmatrix}\rho\\ w\end{pmatrix}_x = D^{-1}\begin{pmatrix}\overline{\Phi}(\rho,v)\\ \overline{\Psi}(\rho,v)\end{pmatrix} = \begin{pmatrix}\widetilde{\Phi}(\rho,w)\\ \widetilde{\Psi}(\rho,w)\end{pmatrix}, \tag{2.8}
$$

with initial data

$$
t = 0: \quad \rho = \rho_0(x), \quad w = w_0(x) = \ln\frac{c - v_0(x)}{c + v_0(x)}, \tag{2.9}
$$

where

$$
\begin{cases}
\overline{\Phi}(\rho,v) = \widetilde{\Phi}(\rho,w) = \widetilde{\Phi}\Big(\rho, \ln\dfrac{c-v}{c+v}\Big), \\
\overline{\Psi}(\rho,v) = -\dfrac{c^2-v^2}{2c}\widetilde{\Psi}(\rho,w) = -\dfrac{c^2-v^2}{2c}\widetilde{\Psi}\Big(\rho, \ln\dfrac{c-v}{c+v}\Big).
\end{cases}
\tag{2.10}
$$

The homogeneous form corresponding to (2.8) is

$$
\begin{pmatrix}\rho\\ w\end{pmatrix}_t + D^{-1}dU^{-1}dFD\begin{pmatrix}\rho\\ w\end{pmatrix}_x = 0. \tag{2.11}
$$

We assume that functions $\widetilde{\Phi}(\rho,w)$ and $\widetilde{\Psi}(\rho,w)$ satisfy the following conditions: $\forall\ 0 < \rho < +\infty$, $-\infty < w < +\infty$, there exists a constant $K > 0$, such that

$$
\begin{cases}
\widetilde{\Phi}(\rho,w), \widetilde{\Psi}(\rho,w) \in C^1, \\
|\widetilde{\Phi}(\rho',w) - \widetilde{\Phi}(\rho,w)| \le K|\rho'-\rho|, \\
|\widetilde{\Phi}(\rho,w') - \widetilde{\Phi}(\rho,w)| \le K\rho|w'-w|, \\
\widetilde{\Phi}(\rho,0) = 0, \\
|\widetilde{\Psi}(\rho',w') - \widetilde{\Psi}(\rho,w)\Big| \le K\Big(|w'-w| + \Big|\ln\dfrac{\rho'}{\rho}\Big|\Big).
\end{cases}
\tag{2.12}
$$

Remark 2.1 In order to illustrate the specific meaning of (2.12), we give two examples satisfying conditions (2.12).

Example 2.1 For bounded solution $(\rho, v) : -c < \underline{v} < v < \bar{v} < c,\ 0 < \underline{\rho} < \rho < \bar{\rho} < +\infty$, if

$$\begin{pmatrix} \Phi(\rho, v) \\ \Psi(\rho, v) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha U_2 \end{pmatrix},$$

then

$$\begin{pmatrix} \widetilde{\Phi}(\rho, w) \\ \widetilde{\Psi}(\rho, w) \end{pmatrix} = \begin{pmatrix} -\frac{2\alpha c^2 v(\kappa^2+c^2)\rho}{(c^4-\kappa^2v^2)(c^2-v^2)} \\ -\frac{2\alpha cv(c^4+\kappa^2v^2)}{(c^4-\kappa^2v^2)(c^2-v^2)} \end{pmatrix} \triangleq \begin{pmatrix} H(v)\rho \\ K(v) \end{pmatrix}.$$

Thus we can see that $H(v)$ and $K(v)$ and their derivatives are bounded. Denoting $K = \max\{|H(v)|,\ |H'(v)\frac{c^2-v^2}{2c}|,\ |K'(v)\frac{c^2-v^2}{2c}|\}$, we have

$$\begin{cases} |\widetilde{\Phi}(\rho', w) - \widetilde{\Phi}(\rho, w)| = |H(v)||\rho' - \rho| \le K|\rho' - \rho|, \\ \Big|\dfrac{\partial\widetilde{\Phi}}{\partial w}\Big| = \Big|\dfrac{\partial\widetilde{\Phi}}{\partial v}\dfrac{dv}{dw}\Big| = \Big|H'(v)\dfrac{c^2 - v^2}{2c}\Big| \le K, \\ \widetilde{\Phi}(\rho, 0) = 0, \\ |\widetilde{\Psi}(\rho', w) - \widetilde{\Psi}(\rho, w)| = 0 < K\rho, \\ \Big|\dfrac{\partial\widetilde{\Psi}}{\partial w}\Big| = \Big|\dfrac{\partial\widetilde{\Psi}}{\partial v}\dfrac{dv}{dw}\Big| = \Big|K'(v)\dfrac{c^2 - v^2}{2c}\Big| \le K. \end{cases}$$

Therefore, the conditions are satisfied.

Example 2.2 Since

$$\begin{pmatrix} \Phi(\rho, v) \\ \Psi(\rho, v) \end{pmatrix} = \begin{pmatrix} \alpha U_1 \\ \beta U_2 \end{pmatrix},$$

we can similarly verify (2.12) for bounded solution $(\rho, v) :\ -c < \bar{v} < v < \bar{v} < c,\ 0 < \underline{\rho} < \rho < \bar{\rho} < +\infty$.

If $\rho(x, t)$ and $w(x, t)$ are independent of the variable x, the problem (2.8)–(2.9) reduces to a Cauchy problem for ordinary differential equations:

$$\begin{cases} \dfrac{d\rho}{dt} = \widetilde{\Phi}(\rho, w),\ \dfrac{dw}{dt} = \widetilde{\Psi}(\rho, w), \\ t = 0 :\ \rho = \rho_0,\ w = w_0, \end{cases} \tag{2.13}$$

where ρ_0 and w_0 are constants.

Lemma 2.2 *Under conditions* (2.12)*,* $\forall\ T > 0$*, there exists a unique solution* (ρ, w) *of problem* (2.13) *for* $t \in [0, T]$*, and a constant* $M_1 > 0$*, such that,* $|w| + |\ln\rho| \le M_1$.

Proof Under the one-to-one mapping: $(\rho, w) \to (u, w) = (\ln\rho, w)$, (2.13) becomes

$$\begin{cases} \dfrac{du}{dt} = \widehat{\Phi}(u, w),\ \dfrac{dw}{dt} = \widehat{\Psi}(u, w), \\ t = 0 :\ u = u_0,\ w = w_0, \end{cases} \tag{2.14}$$

where $u_0 = \ln \rho_0$, and

$$\widehat{\Phi}(u,w) = e^{-u}\widetilde{\Phi}(e^u, w), \quad \widehat{\Psi}(u,w) = \widetilde{\Psi}(e^u, w).$$

According to (2.12), we have

$$\begin{cases} \Big|\dfrac{\partial\widehat{\Phi}}{\partial u}\Big| = \Big|\dfrac{\partial\widetilde{\Phi}(\rho,w)}{\partial\rho} - e^{-u}\widetilde{\Phi}(e^u,w)\Big| = \Big|\dfrac{\partial\widetilde{\Phi}}{\partial\rho} - \dfrac{\widetilde{\Phi}}{\rho}\Big| \le 2K, \\ \Big|\dfrac{\partial\widehat{\Phi}}{\partial w}\Big| = \Big|e^{-u}\dfrac{\partial\widetilde{\Phi}(e^u,w)}{\partial w}\Big| \le e^{-u}Ke^u = K, \\ |\widehat{\Psi}(u',w') - \widehat{\Psi}(u,w)| = |\widetilde{\Psi}(e^{u'},w') - \widetilde{\Psi}(e^u,w)| \le K\Big(\Big|\ln\dfrac{e^{u'}}{e^u}\Big| + |w'-w|\Big) \\ \qquad\qquad = K(|u'-u| + |w'-w|). \end{cases}$$

It then follows the existence and uniqueness, as well as the upper bound estimate.

Now we use a difference scheme based on the Glimm scheme (see [8]) to construct the approximate solutions (ρ^l, w^l) for problem (2.8)–(2.9) with mesh length l and h which are required to satisfy the condition

$$\frac{l}{h} > \sup \frac{v^l + \kappa}{1 + \frac{v^l\kappa}{c^2}}, \tag{2.15}$$

where $v^l = c\frac{1-e^{w^l}}{1+e^{w^l}}$.

For any given $T > 0$, we will state that $\rho^l \ge \delta_0 > 0$, where the constant δ_0 depends only on $\rho_0(x)$, $w_0(x)$, T, and K, so that it is possible to construct ρ^l, w^l.

For integers $n \ge 1$, we set

$$\begin{aligned} Y_n &= \{m:\ m \text{ is integer, and } m+n \text{ is even}\}, \\ A &= \prod_{n\ge1, m\in Y_n} [(m-1)l, (m+1)l] \times \{nh\}. \end{aligned}$$

We choose a point $\{a_{m,n}\} \in A$ randomly and define $a_{m,n} = \{ml, 0\}$, m even. For $0 < t < h$, $(m-1)l < x < (m+1)l$, m odd, we define

$$\begin{cases} \rho^l(x,t) = \rho_0^l(x,t) + \widetilde{\Phi}(\rho_0^l(x,t), w_0^l(x,t))t, \\ w^l(x,t) = w_0^l(x,t) + \widetilde{\Psi}(\rho_0^l(x,t), w_0^l(x,t))t, \end{cases} \tag{2.16}$$

where $\rho_0^l(x,t)$ and $w_0^l(x,t)$ are the solutions of (2.11) with initial data

$$\rho_0(x) = \begin{cases} \rho_0((m-1)l), \\ \rho_0((m+1)l), \end{cases} \quad w_0(x) = \begin{cases} w_0((m-1)l), & x < ml, \\ w_0((m+1)l), & x > ml. \end{cases}$$

Suppose that ρ^l, w^l have been defined for $t < nh$. We define

$$\begin{cases} \rho^l(x,t) = \rho_0^l(x,t) + \widetilde{\Phi}(\rho_0^l(x,t), w_0^l(x,t))(t-nh), \\ w^l(x,t) = w_0^l(x,t) + \widetilde{\Psi}(\rho_0^l(x,t), w_0^l(x,t))(t-nh) \end{cases} \tag{2.17}$$

for $nh < t < (n+1)h$, $ml < x < (m+2)l$, where $m \in Y_n$, $\rho_0^l(x,t)$ and $w_0^l(x,t)$ are the solutions of (2.11) with initial data $(t = nh)$

$$\rho_0^l(x) = \begin{cases} \rho^l(a_{m,n}, nh-0), \\ \rho^l(a_{m+2,n}, nh-0), \end{cases} \quad w_0^l(x) = \begin{cases} w^l(a_{m,n}, nh-0), & x < (m+1)l, \\ w^l(a_{m+2,n}, nh-0), & x > (m+1)l. \end{cases}$$

3 Estimates on the Difference Solutions

We recall that the Riemann invariants of the system are

$$\begin{cases} r = \dfrac{1}{2}\ln\dfrac{c+v}{c-v} + \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho = -\dfrac{1}{2}w + \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho, \\ s = \dfrac{1}{2}\ln\dfrac{c+v}{c-v} - \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho = -\dfrac{1}{2}w - \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho. \end{cases} \tag{3.1}$$

From [25], we know that there exists a C^1 function $0 < g < 1$ such that

$$\text{on the 1-shock wave curve, } \frac{ds}{dr} = g, \text{ namely, } \frac{d(s-s_0)}{d(r-r_0)} = g,$$

$$\text{on the 2-shock wave curve, } \frac{dr}{ds} = g, \text{ namely, } \frac{d(r-r_0)}{d(s-s_0)} = g,$$

that is to say, all shock wave curves in the r, s-plane have the same shape, i.e., the 1-shock wave curve starting from the point (r_0, s_0) is given by

$$s - s_0 = f(r - r_0), \quad r \le r_0, \tag{3.2}$$

and the 2-shock wave curve is given by

$$r - r_0 = f(s - s_0), \quad s \le s_0. \tag{3.3}$$

Besides, the 1-rarefaction wave curve and the 2-rarefaction wave curve are given by

$$s - s_0 = 0, \quad r > r_0, \tag{3.4}$$

and

$$r - r_0 = 0, \quad s > s_0, \tag{3.5}$$

respectively. Therefore, the 1-wave curve can be expressed in the form:

$$s - s_0 = f(r - r_0), \tag{3.6}$$

and the 2-wave curve can be expressed in the form:

$$r - r_0 = f(s - s_0), \tag{3.7}$$

where $f(x)$ is a C^2-function defined on the interval $(-\infty, +\infty)$ satisfying

$$f(x) \equiv 0, \quad \forall\, x > 0, \quad 0 \le f'(x) < 1, \quad f''(x) < 0, \quad \lim_{x \to -\infty} f'(x) = 1.$$

In order to estimate the total variation of difference solutions (ρ^l, w^l) defined by (2.16) and (2.17), we first consider the Riemann problem for system (2.8) with initial data

$$\rho_0(x) = \begin{cases} \rho_l, & x < 0, \\ \rho_r, & x > 0, \end{cases} \qquad w_0(x) = \begin{cases} w_l, & x < 0, \\ w_r, & x > 0, \end{cases} \tag{3.8}$$

the solutions of which are denoted by $\rho_0(x,t)$ and $w_0(x,t)$.

Consider three constant state regions (ρ_l, w_l), (ρ_m, w_m) and (ρ_r, w_r), where (ρ_l, w_l) and (ρ_m, w_m) are connected by a 1-wave, and (ρ_m, w_m) and (ρ_r, w_r) are connected by a 2-wave. Define r_l, s_l, r_m, s_m, r_r, s_r by means of (3.1), and set

$$\Delta r = r_m - r_l, \quad \Delta s = s_r - s_m. \tag{3.9}$$

Define

$$P(\rho_l, w_l, \rho_r, w_r) = -\min(0, \Delta r) - \min(0, \Delta s). \tag{3.10}$$

Obviously, $P \geq 0$. And we can similarly prove as in [23] that

$$P(\rho_i, w_i, \rho_j, w_j) \leq P(\rho_i, w_i, \rho_k, w_k) + P(\rho_k, w_k, \rho_j, w_j), \tag{3.11}$$

where ρ_i, w_i, ρ_k, w_k, ρ_j, w_j are arbitrary positive constants.

From (3.1) we know that

$$\begin{cases} r_0(x,t) = -\dfrac{1}{2}w_0 + \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho_0, \\ s_0(x,t) = -\dfrac{1}{2}w_0 - \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho_0, \end{cases}$$

and (3.6) and (3.7) imply that

$$\begin{aligned} &TV\{r_0(\,\cdot\,,t)\} + TV\{s_0(\,\cdot\,,t)\} \\ &\leq 4P(\rho_l, w_l, \rho_r, w_r) + |r_0(+\infty,t) - r_0(-\infty,t)| + |s_0(+\infty,t) - s_0(-\infty,t)|. \end{aligned} \tag{3.12}$$

We can define the approximate solutions of (2.8) with initial data (3.8) by difference scheme (2.16)–(2.17) as follows:

$$\begin{cases} \rho(x,t) = \rho_0(x,t) + \widetilde{\Phi}(\rho_0(x,t), w_0(x,t))t, \\ w(x,t) = w_0(x,t) + \widetilde{\Psi}(\rho_0(x,t), w_0(x,t))t. \end{cases} \tag{3.13}$$

Define

$$\delta r = r_r - r_l, \quad \delta s = s_r - s_l, \tag{3.14}$$

$$\begin{cases} \delta r_t = \Big\{ -\dfrac{1}{2}(w_r + \widetilde{\Psi}_r t) + \dfrac{c\kappa}{c^2+\kappa^2}\ln(\rho_r + \widetilde{\Phi}_r t)\Big\} \\ \qquad\quad -\Big\{ -\dfrac{1}{2}(w_l + \widetilde{\Psi}_l t) + \dfrac{c\kappa}{c^2+\kappa^2}\ln(\rho_l + \widetilde{\Phi}_l t)\Big\}, \\ \delta s_t = \Big\{ -\dfrac{1}{2}(w_r + \widetilde{\Psi}_r t) - \dfrac{c\kappa}{c^2+\kappa^2}\ln(\rho_r + \widetilde{\Phi}_r t)\Big\} \\ \qquad\quad -\Big\{ -\dfrac{1}{2}(w_l + \widetilde{\Psi}_l t) - \dfrac{c\kappa}{c^2+\kappa^2}\ln(\rho_l + \widetilde{\Phi}_l t)\Big\}, \end{cases} \tag{3.15}$$

where $\widetilde{\Phi}_r = \widetilde{\Phi}(\rho_r, w_r)$, $\widetilde{\Psi}_r = \widetilde{\Psi}(\rho_r, w_r)$, $\widetilde{\Phi}_l = \widetilde{\Phi}(\rho_l, w_l)$, $\widetilde{\Psi}_l = \widetilde{\Psi}(\rho_l, w_l)$.

Lemma 3.1 *If* $t \leq \frac{1}{2K}$, *then*

$$|\delta r_t - \delta r|, \quad |\delta s_t - \delta s| \leq \frac{5}{2}K\Big(1 + \frac{c^2+\kappa^2}{2c\kappa}\Big)(|\delta r| + |\delta s|)t.$$

Proof It follows from (3.13) and (3.14) that

$$\begin{aligned}
|\delta r_t - \delta r| &= \left| -\frac{1}{2}(\widetilde{\Psi}_r - \widetilde{\Psi}_l)t + \frac{c\kappa}{c^2+\kappa^2}\ln\frac{1+\big(\frac{\widetilde{\Phi}_r}{\rho_r}\big)t}{1+\big(\frac{\widetilde{\Phi}_l}{\rho_l}\big)t}\right| \\
&\le \frac{1}{2}|(\widetilde{\Psi}_r - \widetilde{\Psi}_l)|t + \frac{c\kappa}{c^2+\kappa^2}\left|\ln\frac{1+\big(\frac{\widetilde{\Phi}_r}{\rho_r}\big)t}{1+\big(\frac{\widetilde{\Phi}_l}{\rho_l}\big)t}\right| \\
&\le \frac{1}{2}K\Big(|w_r - w_l| + \Big|\ln\frac{\rho_r}{\rho_l}\Big|\Big)t + \frac{c\kappa}{c^2+\kappa^2}\left|\ln\frac{1+\big(\frac{\widetilde{\Phi}_r}{\rho_r}\big)t}{1+\big(\frac{\widetilde{\Phi}_l}{\rho_l}\big)t}\right| \\
&\le \frac{1}{2}K\Big(|w_r - w_l| + \Big|\ln\frac{\rho_r}{\rho_l}\Big|\Big)t + \frac{c\kappa}{c^2+\kappa^2}\frac{1}{1+\widehat{\Phi}(u^*,w^*)t}|\widehat{\Phi}(u_r,w_r) - \widehat{\Phi}(u_l,w_l)| \\
&\le \frac{1}{2}K\Big(|w_r - w_l| + \Big|\ln\frac{\rho_r}{\rho_l}\Big|\Big)t + \frac{c\kappa}{c^2+\kappa^2}4K(|u_r-u_l| + |w_r - w_l|)t \\
&\le \Big(\frac{1}{2} + \frac{4c\kappa}{c^2+\kappa^2}\Big)K\Big(|w_r - w_l| + \Big|\ln\frac{\rho_r}{\rho_l}\Big|\Big)t \\
&\le \frac{5}{2}K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)(|\delta r| + |\delta s|)t,
\end{aligned}$$

where $|\widehat{\Phi}| = \big|\frac{\widetilde{\Phi}}{\rho}\big| \le K$, and the point (u^*, w^*) lies on the segment between (u_l, w_l) and (u_r, w_r). The estimate on $|\delta s_t - \delta s|$ is similar.

From (3.6)–(3.7), (3.9), and (3.14), we know

$$\delta r = \Delta r + f(\Delta s), \quad \delta s = \Delta s + f(\Delta r). \tag{3.16}$$

The eigenvalues of matrix $\begin{pmatrix} 1 & f'(\Delta s) \\ f'(\Delta r) & 1 \end{pmatrix}$ are $1 \pm \sqrt{f'(\Delta s)f'(\Delta r)}$, and we have

$$\Delta r = p(\delta r, \delta s), \quad \Delta s = q(\delta r, \delta s), \tag{3.17}$$

from the fact that $\sqrt{f'(\Delta s)f'(\Delta r)} < 1$, where $p, q \in C^2$. It is easy to see that

$$\begin{aligned}
\frac{\partial p}{\partial \delta r} &= \frac{1}{1-f'(p)f'(q)}, \quad & \frac{\partial q}{\partial \delta r} &= \frac{-f'(p)}{1-f'(p)f'(q)}, \\
\frac{\partial p}{\partial \delta s} &= \frac{-f'(q)}{1-f'(p)f'(q)}, \quad & \frac{\partial q}{\partial \delta s} &= \frac{1}{1-f'(p)f'(q)}.
\end{aligned}$$

Now we consider the situation when only one wave is involved in the solutions $\rho_0(x,t)$, $w_0(x,t)$; if we change ρ_l, w_l, ρ_r and w_r to ρ_l^*, w_l^*, ρ_r^* and w_r^*, then r and s become r^* and s^*. We denote

$$\delta r^* = r_r^* - r_l^*, \quad \delta s^* = s_r^* - s_l^*.$$

Taking the similar procedures as in [31], we can state the following lemma while omitting the tedious proof.

Lemma 3.2 $P(\rho_l^*, w_l^*, \rho_r^*, w_l^*) \le P(\rho_l^*, w_l^*, \rho_r^*, w_l^*) + 3(|\delta r^* - \delta r| + |\delta s^* - \delta s|)$.

For integers $n \geq 1$, and $t = nh$, let

$$\begin{aligned}&F(nh+0,\rho^l,w^l)\\&=\sum_{m\in Y_n} P(\rho^l(a_{m,n},nh{+}0),\, w^l(a_{m,n},nh{+}0),\, \rho^l(a_{m+2,n},nh{+}0),\, w^l(a_{m+2,n},nh{+}0)).\end{aligned} \tag{3.18}$$

We denote

$$Z_n = I \cup \{a_{m,n} \mid m \in Y_n\},$$

where I is the set of all integers, then

$$Z_n = \{z_{i,n} : i \text{ integer}, \ z_{i,n} \leq z_{i+1,n}\},$$

and let

$$\begin{aligned}&F(nh-0,\rho^l,w^l)\\&=\sum_i P(\rho^l(z_{i,n},nh-0),\, w^l(z_{i,n},nh-0),\, \rho^l(z_{i+1,n},nh-0),\, w^l(z_{i+1,n},nh-0)).\end{aligned} \tag{3.19}$$

We consider the initial value problem (2.13) with the initial data $\rho_0(\pm\infty)$ and $w_0(\pm\infty)$, and denote the solutions by $\rho(\pm\infty,t)$ and $w(\pm\infty,t)$ respectively. From the definition (3.1) of Riemann invariants and Lemma 2.2, we know that

$$|r(\pm\infty,t)|, \quad |s(\pm\infty,t)| \leq \frac{1}{2}M_1 < M_1, \quad \forall\, t \in [0,T].$$

Lemma 3.3 *If* $h \leq \frac{1}{2K}$, *then*

$$\begin{aligned}&F((n+1)h+0,\rho^l,w^l)\\&\leq F(nh+0,\rho^l,w^l)\Big(1+60Kh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)+60KM_1\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)h.\end{aligned}$$

Proof From the definition (3.18)–(3.19), we have

$$F((n+1)h-0,\rho_0^l,w_0^l) = F(nh+0,\rho^l,w^l). \tag{3.20}$$

Let

$$r^l = -\frac{1}{2}w^l + \frac{c\kappa}{c^2+\kappa^2}\ln\rho^l, \quad s^l = -\frac{1}{2}w^l - \frac{c\kappa}{c^2+\kappa^2}\ln\rho^l.$$

From Lemmas 3.1 and 3.2, we have

$$\begin{aligned}&P(\rho^l(a_{m,n+1},(n+1)h-0),\, w^l(a_{m,n+1},(n+1)h-0),\\&\quad \rho^l(a_{m+2,n+1},(n+1)h-0),\, w^l(a_{m+2,n+1},(n+1)h-0))\\&\leq P(\rho_0^l(a_{m,n+1},(n+1)h-0),\, w_0^l(a_{m,n+1},(n+1)h-0),\\&\quad \rho_0^l(a_{m+2,n+1},(n+1)h-0),\, w_0^l(a_{m+2,n+1},(n+1)h-0))\\&\quad + 15K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)(|\delta r^l|+|\delta s^l|)h.\end{aligned}$$

Then we have

$$
\begin{aligned}
&F((n+1)h-0,\rho^l,w^l)\\
&\le F((n+1)h-0,\rho_0^l,w_0^l)+15K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\{TV\{r^l(\,\cdot\,,nh+0)\}+TV\{s^l(\,\cdot\,,nh+0)\}\}h\\
&=F(nh+0,\rho^l,w^l)+15K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\{TV\{r^l(\,\cdot\,,nh+0)\}+TV\{s^l(\,\cdot\,,nh+0)\}\}h,
\end{aligned}
$$

where we use (3.20). From (3.11), it holds that

$$F((n+1)h+0,\rho^l,w^l)\le F((n+1)h-0,\rho^l,w^l).$$

Thus, we obtain

$$
\begin{aligned}
&F((n+1)h+0,\rho^l,w^l)\\
&\le F(nh+0,\rho^l,w^l)+15K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\{TV\{r^l(\,\cdot\,,nh+0)\}+TV\{s^l(\,\cdot\,,nh+0)\}\}h. \quad (3.21)
\end{aligned}
$$

Besides, we have, as in (3.12), that

$$TV\{r^l(\,\cdot\,,nh+0)\}+TV\{s^l(\,\cdot\,,nh+0)\}\le 4F(nh+0,\rho^l,w^l)+4M_1. \quad (3.22)$$

Substituting (3.22) into (3.21), we finally arrive at

$$
\begin{aligned}
&F((n+1)h+0,\rho^l,w^l)\\
&\le F(nh+0,\rho^l,w^l)+60K\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)(F(nh+0,\rho^l,w^l)+M_1)h\\
&=F(nh+0,\rho^l,w^l)\Big(1+60Kh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)+60KM_1\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)h,
\end{aligned}
$$

which completes the proof.

Lemma 3.4 *For any $T>0$, if $h\le\frac{1}{2K}$, then there exist constants $M>0$ and $\delta_0>0$ depending only on $\rho_0(x)$, $w_0(x)$, K and T, such that*

$$
\begin{aligned}
&TV\{\rho^l(\,\cdot\,,t)\}+TV\{w^l(\,\cdot\,,t)\}+TV\{r^l(\,\cdot\,,t)\}+TV\{s^l(\,\cdot\,,t)\}\le M,\\
&|\rho^l(x,t)|+|w^l(x,t)|+|r^l(x,t)|+|s^l(x,t)|\le M,\\
&\rho^l(x,t)\ge\delta_0>0.
\end{aligned}
$$

Proof Let

$$F(0)=\sum_{m\in Y_n}P(\rho_0(ml),\,w_0(ml),\,\rho_0((m+2)l),\,w_0((m+2)l)).$$

We assume that $T=Nh$, set $a=1+60Kh\big(1+\frac{c^2+\kappa^2}{2c\kappa}\big)$.

If $h\le\frac{1}{2K}$, we have by induction, from Lemma 3.3, that

$$
\begin{aligned}
&F(nh+0,\rho^l,w^l)\\
&\le F(0)a^n+60KM_1h\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)(1+a+a^2+\cdots+a^{n-1})\\
&\le F(0)a^n+60KM_1nh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)a^n
\end{aligned}
$$

$$
\begin{aligned}
&\le \Big(F(0)+60KM_1nh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)\Big(1+60KT\frac{1}{N}\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)^N\\
&\le \Big(F(0)+60KM_1nh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)\exp\Big\{60KT\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big\}.
\end{aligned}
$$

From (3.10) and (3.16), we have

$$F(0)\le TV\{r_0\}+TV\{s_0\}.$$

This, together with (3.22), implies that

$$
\begin{aligned}
&TV\{r^l(\,\cdot\,,nh+0)\}+TV\{s^l(\,\cdot\,,nh+0)\}\\
&\le 4\Big(F(0)+60KM_1nh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)\exp\Big\{60KT\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big\}\\
&\le 4\Big(TVr_0+TVs_0+60KM_1nh\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big)\exp\Big\{60KT\Big(1+\frac{c^2+\kappa^2}{2c\kappa}\Big)\Big\}.
\end{aligned}
$$

It then follows from Lemma 3.1 that

$$TV\{r^l(\,\cdot\,,t)\}+TV\{s^l(\,\cdot\,,t)\}\le Q,$$

where $Q>0$ depends only on $\rho_0(x)$, $w_0(x)$, K and T. Therefore,

$$
\begin{aligned}
|r^l(x,t)|\le|r^l(\pm\infty,t)|+TV\{r^l(\,\cdot\,,t)\}\le M_1+Q,\\
|s^l(x,t)|\le|s^l(\pm\infty,t)|+TV\{s^l(\,\cdot\,,t)\}\le M_1+Q.
\end{aligned}
$$

Returning to ρ and v by Definition 3.1 of the Riemann invariants, we complete the proof of the lemma.

Remark 3.1 Lemma 3.4 tells us that, for any $T>0$, if $h>0$ is sufficiently small, then the total variations of the functions ρ^l, w^l, r^l and s^l are bounded uniformly for h and $\{a_{m,n}\}$; their upper and lower bounds, and the positive lower bound of ρ^l are also bounded uniformly.

Now we come back to the Cauchy problem of system (2.5) with initial data (1.3), and the Riemann problem with initial data

$$\rho_0(x)=\begin{cases}\rho_l, & x<0,\\ \rho_r, & x>0,\end{cases}\qquad v_0(x)=\begin{cases}v_l, & x<0,\\ v_r, & x>0.\end{cases}\tag{3.23}$$

From (2.10), we have

$$
\begin{cases}
\overline{\Phi}(\rho,v),\ \bar{\Psi}(\rho,v)\in C^1,\\
|\overline{\Phi}(\rho',v)-\overline{\Phi}(\rho,v)|=|\widetilde{\Phi}(\rho',w)-\widetilde{\Phi}(\rho,w)|\le K|\rho'-\rho|\le K'|\rho'-\rho|,\\
|\overline{\Phi}(\rho,v')-\overline{\Phi}(\rho,v)|=\Big|\widetilde{\Phi}\Big(\rho,\ln\dfrac{c-v'}{c+v'}\Big)-\widetilde{\Phi}\Big(\rho,\ln\dfrac{c-v}{c+v}\Big)\Big|\\
\qquad\qquad\le K\rho\Big|\ln\dfrac{c-v'}{c+v'}-\ln\dfrac{c-v}{c+v}\Big|\le KM'\rho|v'-v|\le K'\rho|v'-v|,\\
\overline{\Phi}(\rho,0)=0,\\
|\bar{\Psi}(\rho',v)-\bar{\Psi}(\rho,v)|=\dfrac{c^2-v^2}{2c}|\widetilde{\Phi}(\rho',w)-\widetilde{\Phi}(\rho,w)|\le KM''\Big|\ln\dfrac{\rho'}{\rho}\Big|\le K'\Big|\ln\dfrac{\rho'}{\rho}\Big|,\\
\Big|\dfrac{\partial\overline{\Psi}}{\partial v}\Big|=\Big|\dfrac{v}{c}\widetilde{\Psi}\Big(\rho,\ln\dfrac{c-v}{c+v}\Big)-\dfrac{c^2-v^2}{2c}\dfrac{\partial\widetilde{\Psi}}{\partial w}\Big(-\dfrac{2c}{c^2-v^2}\Big)\Big|\le KM'''\le K',
\end{cases}\tag{3.24}
$$

where $K' = \max\{K, KM', KM'', KM'''\} > 0$. Because

$$\begin{aligned}
&\frac{d\rho}{dt} = \overline{\Phi}(\rho, v) = \widetilde{\Phi}(\rho, w),\\
&\frac{dv}{dt} = \overline{\Psi}(\rho, v), \quad w = \ln\frac{c-v}{c+v} = h(v), \quad v = c\frac{1-e^w}{1+e^w} = g(w),\\
&\frac{dw}{dt} = \frac{(1+e^w)^2}{2ce^w}\frac{dv}{dt} = \frac{(1+e^w)^2}{2ce^w}\overline{\Psi}(\rho, v) = \widetilde{\Psi}(\rho, w),
\end{aligned}$$

we know from the Picard iteration, that

$$\begin{aligned}
\frac{w^{n+1}-w^n}{\Delta t} &= \widetilde{\Psi}(\rho^n, w^n),\\
\frac{v^{n+1}-v^n}{\Delta t} &= \frac{g(w^{n+1})-g(w^n)}{\Delta t} = g'(w^n)\frac{w^{n+1}-w^n}{\Delta t} + O(1)\frac{(w^{n+1}-w^n)^2}{\Delta t}\\
&= \overline{\Psi}(\rho^n, v^n) + O(1)\Delta t.
\end{aligned}$$

Then

$$v^{n+1} = v^n + \overline{\Psi}(\rho^n, v^n)\Delta t + O(1)(\Delta t)^2.$$

Therefore, we can construct ρ^l, v^l as constructing ρ^l, w^l in §2:

For $0 < t < h$, $(m-1)l < x < (m+1)l$, m odd, we define

$$\begin{cases} \rho^l(x,t) = \rho_0^l(x,t) + \overline{\Phi}(\rho_0^l(x,t), v_0^l(x,t))t,\\ v^l(x,t) = v_0^l(x,t) + \overline{\Psi}(\rho_0^l(x,t), v_0^l(x,t))t + O(1)t^2, \end{cases} \tag{3.25}$$

where $\rho_0^l(x,t)$, $v_0^l(x,t)$ are the solutions of (2.5) with initial value

$$\rho_0(x) = \begin{cases} \rho_0((m-1)l),\\ \rho_0((m+1)l), \end{cases} \quad v_0(x) = \begin{cases} v_0((m-1)l), & x < ml,\\ v_0((m+1)l), & x > ml. \end{cases}$$

Suppose that ρ^l, v^l have been defined for $t < nh$. Define

$$\begin{cases} \rho^l(x,t) = \rho_0^l(x,t) + \overline{\Phi}(\rho_0^l(x,t), v_0^l(x,t))(t-nh),\\ v^l(x,t) = v_0^l(x,t) + \overline{\Psi}(\rho_0^l(x,t), v_0^l(x,t))(t-nh) + O(1)(t-nh)^2 \end{cases} \tag{3.26}$$

for $nh < t < (n+1)h$, $ml < x < (m+2)l$, where $m \in Y_n$, and $\rho_0^l(x,t)$, $v_0^l(x,t)$ are the solutions of (2.5) with initial value ($t = nh$)

$$\rho_0^l(x) = \begin{cases} \rho^l(a_{m,n}, nh-0),\\ \rho^l(a_{m+2,n}, nh-0), \end{cases} \quad v_0^l(x) = \begin{cases} v^l(a_{m,n}, nh-0), & x < (m+1)l,\\ v^l(a_{m+2,n}, nh-0), & x > (m+1)l. \end{cases}$$

Lemma 3.5 *For any $T > 0$, and any $X > 0$, there exists a constant $L > 0$ depending only on T, X, K, $\rho_0(x)$ and $v_0(x)$, such that*

$$\int_{-X}^{X} [|\rho^l(x,t_2) - \rho^l(x,t_1)| + |v^l(x,t_2) - v^l(x,t_1)|]dx \le L(|t_2 - t_1| + h), \quad \forall\, t_1, t_2 \in [0,T].$$

Proof Suppose that $nh \le t_1 < (n+1)h < \cdots < t_2 \le (n+k+1)h$. We have

$$\int_{-X}^{X} [|\rho^l(x,t_2) - \rho^l(x,t_1)| + |v^l(x,t_2) - v^l(x,t_1)|]dx \le \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\begin{aligned}\mathrm{I}_1 = &\sum_{i=0}^{k+1} \int_{-X}^{X} [|\rho^l(x,(n+i)h+0) - \rho^l(x,(n+i)h-0)| \\ &+ |v^l(x,(n+i)h+0) - v^l(x,(n+i)h-0)|]dx, \\ \mathrm{I}_2 = &\sum_{i=0}^{k} \int_{-X}^{X} [|\rho^l(x,(n+i+1)h-0) - \rho^l(x,(n+i)h+0)| \\ &+ |v^l(x,(n+i+1)h-0) - v^l(x,(n+i)h+0)|]dx \\ &+ \int_{-X}^{X} [|\rho^l(x,t_1) - \rho^l(x,nh+0)| + |v^l(x,t_1) - v^l(x,nh+0)|]dx \\ &+ \int_{-X}^{X} [|\rho^l(x,t_2) - \rho^l(x,(n+k)h+0)| + |v^l(x,t_2) - v^l(x,(n+k)h+0)|]dx.\end{aligned}$$

It is easy to know that

$$\mathrm{I}_1 \le 4l\Big(\Big[\frac{t_2-t_1}{h}\Big] + 3\Big) \sup_{0\le t\le T} (TV\{\rho^l(\,\cdot\,,t)\} + TV\{v^l(\,\cdot\,,t)\}),$$

and that there is an even integer m such that $x \in [ml,(m+2)l]$ for any fixed x. Taking (3.24) into consideration, we have

$$\begin{aligned}&|\rho^l(x,(n+i+1)h-0) - \rho^l(x,(n+i)h+0)| \\ \le\ &|\rho_0^l(x,(n+i+1)h-0) - \rho_0^l(x,(n+i)h+0)| + |\overline{\Phi}^l|h \\ \le\ &TV_{[(m-1)l,(m+3)l]}\{\rho_0^l(\,\cdot\,,(n+i)h+0)\} + |\overline{\Phi}^l|h\end{aligned}$$

and

$$\begin{aligned}&|v^l(x,(n+i+1)h-0) - v^l(x,(n+i)h+0)| \\ \le\ &TV_{[(m-1)l,(m+3)l]}\{v_0^l(\,\cdot\,,(n+i)h+0)\} + |\overline{\Psi}^l|h + O(1)h^2,\end{aligned}$$

where

$$\begin{aligned}\overline{\Phi}^l &= \overline{\Phi}(\rho_0^l(x,(n+i+1)h-0), v_0^l(x,(n+i+1)h-0)), \\ \overline{\Psi}^l &= \overline{\Psi}(\rho_0^l(x,(n+i+1)h-0), v_0^l(x,(n+i+1)h-0)).\end{aligned}$$

We have similar results for

$$|\rho^l(x,t_1) - \rho^l(x,nh+0)| + |v^l(x,t_1) - v^l(x,nh+0)|$$

and

$$|\rho^l(x,t_2) - \rho^l(x,(n+k)h+0)| + |v^l(x,t_2) - v^l(x,(n+k)h+0)|.$$

From the above estimates, we obtain

$$\mathrm{I}_2 \le \Big(4l\Big(\sup_{0\le t\le T}[TV\{\rho^l(\,\cdot\,,t)\}+TV\{v^l(\,\cdot\,,t)\}]\Big)+2X(\sup|\overline{\Phi}^l|+\sup|\overline{\Psi}^l|)h+O(1)h^2\Big)$$
$$\cdot\Big(\Big[\frac{t_2-t_1}{h}\Big]+4\Big).$$

The lemma follows immediately from the above estimates and Lemma 3.4.

Remark 3.2 From (2.6),

$$\begin{pmatrix}\Phi(\rho,v)\\ \Psi(\rho,v)\end{pmatrix} = dU\begin{pmatrix}\overline{\Phi}(\rho,v)\\ \overline{\Psi}(\rho,v)\end{pmatrix},$$

where dU is defined in §2. Then

$$\begin{pmatrix}\Phi(\rho,v)\\ \Psi(\rho,v)\end{pmatrix} = \begin{pmatrix}U_{1\rho}\overline{\Phi}+U_{2\rho}\overline{\Psi}\\ U_{1v}\overline{\Phi}+U_{2v}\overline{\Psi}\end{pmatrix}.$$

Lemma 3.4 implies that $U_{1\rho}$, $U_{2\rho}$, U_{1v}, U_{2v}, $\overline{\Phi}$, and $\overline{\Psi}$ are bounded, and satisfy the Lipschitz conditions. Thus Φ,Ψ are bounded and satisfy the following Lipschitz conditions:

$$\begin{cases}|\Phi(\rho',v')-\Phi(\rho,v)|\le K''(|\rho'-\rho|+|v'-v|),\\ |\Psi(\rho',v')-\Psi(\rho,v)|\le K''(|\rho'-\rho|+|v'-v|),\end{cases} \tag{3.27}$$

where the constant K'' depends only on K and the bounds of ρ, v.

4 Existence of Entropy Solutions

We recall that an entropy-entropy flux pair for homogeneous system (2.2) is a pair of C^1 functions $(\eta(U),q(U))$ satisfying

$$\nabla\eta(U)\nabla F(U)=\nabla q(U).$$

Definition 4.1 *A bounded measurable function $U\in\mathcal{V}\cap\{\rho>0\}$ is an entropy solution of* (2.1) *in $\Pi_T:=[0,T)\times\mathbb{R}$ if $U=U(t,x)$ satisfies the following:*

(i) *$U(x,t)$ is a weak solution of* (2.1), *i.e.,* (2.1) *holds in the weak sense in Π_T: for any $\phi\in C_0^1(\Pi_T)$,*

$$\int_{\Pi_T}(U\partial_t\phi+F(U)\partial_x\phi)\,dx\,dt+\int_{-\infty}^{\infty}U_0(x)\phi(0,x)dx=0; \tag{4.1}$$

(ii) *The Lax entropy inequality*

$$\eta(U)_t+q(U)_x\le\nabla\eta(U)G(U) \tag{4.2}$$

holds in the sense of distributions in Π_T, i.e., for any nonnegative $\phi\in C_0^1(\Pi_T)$,

$$\int_0^{\infty}\int_{-\infty}^{\infty}(\eta(U)\phi_t+q(U)\phi_x+\nabla\eta(U)G(U)\phi)\,dx\,dt+\int_{-\infty}^{\infty}\eta(U_0(x))\phi(x,0)\,dx\ge0 \tag{4.3}$$

for any C^2 convex entropy pair $(\eta(U),q(U))$.

From (2.4), we see that

$$|\lambda|,\ |\mu| \leq \frac{|v|+\kappa}{1-\frac{|v|\kappa}{c^2}} \leq c\Big(\frac{c+\kappa}{c-\kappa}\Big).$$

We set $h_i = \frac{T}{2^i}$, $\frac{l_i}{h_i} = c\big(\frac{c+\kappa}{c-\kappa}\big)$, if i is sufficiently large, Lemma 3.4 implies that, the difference scheme (3.25)–(3.26) in §2 is realizable.

If i is sufficiently large, the estimation of Lemmas 3.4 and 3.5 holds. we consider the sequence $\{(\rho^{l_i}, v^{l_i}) : i = 1, 2, 3, \cdots\}$; there is a subsequence which converges in L^1, on the intervals $[-X, X]$ of any horizontal line, uniformly for $t \in [0, T]$, and the limit functions of ρ^{l_i} and v^{l_i} are the bounded measurable functions $\rho(x,t)$, $v(x,t)$. To prove that $\rho(x,t)$, $v(x,t)$ are the weak solutions of the Cauchy problem (1.4)–(1.3) for suitably chosen $\{a_{m,n}\}$, we need to show that

$$\int_{-\infty}^{+\infty}\int_0^T (U_1\varphi_t + F_1(U)\varphi_x + \Phi(\rho,v)\varphi)\,dx\,dt + \int_{-\infty}^{+\infty} U_1(x,0)\varphi(x,0)dx = 0, \tag{4.4}$$

$$\int_{-\infty}^{+\infty}\int_0^T (U_2\psi_t + F_2(U)\psi_x + \Psi(\rho,v)\psi)\,dx\,dt + \int_{-\infty}^{+\infty} U_2(x,0)\psi(x,0)dx = 0 \tag{4.5}$$

for any $\varphi(x,t),\ \psi(x,t) \in C_0^1(\Pi_T)$.

We consider the Riemann problem (1.4) and (3.23) again, assuming that there is a shock wave, say, 2-shock wave, for the sake of definiteness. Denote by σ the speed of the shock wave, the Rankine-Hugiont conditions are

$$\sigma[U_{10}] = [F_1(U_0)], \tag{4.6}$$

$$\sigma[U_{20}] = [F_2(U_0)], \tag{4.7}$$

where

$$U_{10} = U_1(\rho_0(x,t), v_0(x,t)), \qquad U_{20} = U_2(\rho_0(x,t), v_0(x,t)),$$
$$F_1(U_0) = F_1(U(\rho_0(x,t), v_0(x,t))), \quad F_2(U_0) = F_2(U(\rho_0(x,t), v_0(x,t))).$$

According to the difference scheme (3.25)–(3.26), we define for $nh < t < (n+1)h$,

$$\begin{cases} \rho(x,t) = \rho_0(x,t) + \overline{\Phi}(\rho_0(x,t), v_0(x,t))(t-nh), \\ v(x,t) = v_0(x,t) + \overline{\Psi}(\rho_0(x,t), v_0(x,t))(t-nh) + O(1)(t-nh)^2, \end{cases} \tag{4.8}$$

which are the approximate solutions of the Riemann problem (1.4)–(3.23).

Lemma 4.1 *If $0 < t < h \leq \frac{1}{2K'}$, then*

$$|\sigma[U_1] - [F_1(U)]| \leq Lt(|\rho_r - \rho_m| + |v_r - v_m|), \tag{4.9}$$

$$|\sigma[U_2] - [F_2(U)]| \leq Lt(|\rho_r - \rho_m| + |v_r - v_m|), \tag{4.10}$$

where L is a constant depending only on K, the upper and lower bounds of $\rho(x,t)$ and $v(x,t)$, and the positive lower bound of $\rho(x,t)$.

Proof First we prove (4.9).

$$
\begin{aligned}
[U_1] &= \Big[\rho\Big(\frac{\kappa^2+c^2}{c^2}\frac{v^2}{c^2-v^2}+1\Big)\Big] = \Big[\rho\Big(M_0\frac{v^2}{c^2-v^2}+1\Big)\Big] \\
&= (\rho_r+\overline{\Phi}_r t)\Big(M_0\frac{(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2}+1\Big) - (\rho_m+\overline{\Phi}_m t)\Big(M_0\frac{(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}+1\Big) \\
&= M_0\Big(\rho_r\frac{(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \rho_m\frac{(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big) \\
&\quad + M_0 t\Big[\frac{\overline{\Phi}_r(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big] + (\rho_r-\rho_m) + (\overline{\Phi}_r-\overline{\Phi}_m)t, \\
[F_1(U)] &= \Big[\rho(\kappa^2+c^2)\frac{v}{c^2-v^2}\Big] \\
&= c^2M_0\Big[(\rho_r+\overline{\Phi}_r t)\frac{(v_r+\overline{\Psi}_r t)}{c^2-(v_r+\overline{\Psi}_r t)^2} - (\rho_m+\overline{\Phi}_m t)\frac{(v_m+\overline{\Psi}_m t)}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big].
\end{aligned}
$$

Condition (4.6) is

$$
\sigma\Big(\rho_r\Big(M_0\frac{v_r^2}{c^2-v_r^2}+1\Big) - \rho_m\Big(M_0\frac{v_m^2}{c^2-v_m^2}+1\Big)\Big) = c^2M_0\Big(\rho_r\frac{v_r}{c^2-v_r^2} - \rho_m\frac{v_m}{c^2-v_m^2}\Big),
$$

and, together with (3.24), we have

$$
\begin{aligned}
&\sigma[U_1]-[F_1(U)] \\
&= \Big[\sigma M_0\Big(\rho_r\int_0^1\frac{2c^2(v_r+\overline{\Psi}_r t\eta)^2\overline{\Psi}_r t}{c^2-(v_r+\overline{\Psi}_r t\eta)^2}d\eta - \rho_m\int_0^1\frac{2c^2(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m t}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}d\eta\Big)\Big] \\
&\quad + \Big[\sigma M_0\Big(\frac{\overline{\Phi}_r t(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m t(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big)\Big] + [\sigma(\overline{\Phi}_r-\overline{\Phi}_m)t] \\
&\quad - c^2M_0\Big[\rho_r\int_0^1\frac{(c^2+(v_r+\overline{\Psi}_r t\eta)^2)\overline{\Psi}_r t}{(c^2-(v_r+\overline{\Psi}_r t\eta)^2)^2}d\eta - \rho_m\int_0^1\frac{(c^2+(v_m+\overline{\Psi}_m t\eta)^2)\overline{\Psi}_m t}{(c^2-(v_m+\overline{\Psi}_m t\eta)^2)^2}d\eta\Big] \\
&\quad - c^2M_0\Big[\frac{\overline{\Phi}_r t(v_r+\overline{\Psi}_r t)}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m t(v_m+\overline{\Psi}_m t)}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&|\sigma[U_1]-[F_1(U)]| \\
&\le \Big|2c^2\sigma M_0\Big(\rho_r\int_0^1\frac{(v_r+\overline{\Psi}_r t\eta)^2\overline{\Psi}_r t}{c^2-(v_r+\overline{\Psi}_r t\eta)^2}d\eta - \rho_m\int_0^1\frac{(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m t}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}d\eta\Big)\Big| \\
&\quad + \Big|\sigma M_0\Big(\frac{\overline{\Phi}_r t(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m t(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big)\Big| + |\sigma(\overline{\Phi}_r-\overline{\Phi}_m)t| \\
&\quad + \Big|c^2M_0\Big(\rho_r\int_0^1\frac{(c^2+(v_r+\overline{\Psi}_r t\eta)^2)\overline{\Psi}_r t}{(c^2-(v_r+\overline{\Psi}_r t\eta)^2)^2}d\eta - \rho_m\int_0^1\frac{(c^2+(v_m+\overline{\Psi}_m t\eta)^2)\overline{\Psi}_m t}{(c^2-(v_m+\overline{\Psi}_m t\eta)^2)^2}d\eta\Big)\Big| \\
&\quad + \Big|c^2M_0\Big(\frac{\overline{\Phi}_r t(v_r+\overline{\Psi}_r t)}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m t(v_m+\overline{\Psi}_m t)}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big)\Big| \\
&= \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5.
\end{aligned}
$$

We first consider I_1:

$$\begin{aligned}
\mathrm{I}_1 &= \Big|\Big(2c^2\sigma M_0\Big(\rho_r\int_0^1 \frac{(v_r+\overline{\Psi}_r t\eta)^2\overline{\Psi}_r t}{c^2-(v_r+\overline{\Psi}_r t\eta)^2}d\eta - \rho_m\int_0^1 \frac{(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m t}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}d\eta\Big)\Big)\Big| \\
&\le 2c^2|\sigma|M_0 t\Big|\rho_r\int_0^1 \Big(\frac{(v_r+\overline{\Psi}_r t\eta)^2\overline{\Psi}_r}{c^2-(v_r+\overline{\Psi}_r t\eta)^2} - \frac{(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}\Big)d\eta\Big| \\
&\quad + 2c^2|\sigma|M_0 t\Big|(\rho_r-\rho_m)\int_0^1 \frac{(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}d\eta\Big| \\
&\le 2c^2|\sigma|M_0 t\Big|\rho_r\int_0^1 \frac{(\overline{\Psi}_r-\overline{\Psi}_m)(v_r+\overline{\Psi}_r t\eta)^2}{c^2-(v_r+\overline{\Psi}_r t\eta)^2}d\eta\Big| \\
&\quad + 2c^2|\sigma|M_0 t\Big|\rho_r\int_0^1 \overline{\Psi}_m\Big(\frac{(v_r+\overline{\Psi}_r t\eta)^2}{c^2-(v_r+\overline{\Psi}_r t\eta)^2} - \frac{(v_m+\overline{\Psi}_m t\eta)^2}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}\Big)d\eta\Big| \\
&\quad + 2c^2|\sigma|M_0 t\Big|(\rho_r-\rho_m)\int_0^1 \frac{(v_m+\overline{\Psi}_m t\eta)^2\overline{\Psi}_m}{c^2-(v_m+\overline{\Psi}_m t\eta)^2}d\eta\Big| \\
&\le L_1 t(|\rho_r-\rho_m|+|v_r-v_m|).
\end{aligned}$$

Similarly, for I_2, we have

$$\begin{aligned}
\mathrm{I}_2 &= \Big|\sigma M_0\Big(\frac{\overline{\Phi}_r t(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{\overline{\Phi}_m t(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big)\Big| \\
&\le |\sigma|M_0 t\Big|\frac{(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2}(\overline{\Phi}_r-\overline{\Phi}_m)\Big| \\
&\quad + |\sigma|M_0 t\Big|\overline{\Psi}_m\Big(\frac{(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2} - \frac{(v_m+\overline{\Psi}_m t)^2}{c^2-(v_m+\overline{\Psi}_m t)^2}\Big)\Big| \\
&\le |\sigma|M_0 t\Big|\frac{(v_r+\overline{\Psi}_r t)^2}{c^2-(v_r+\overline{\Psi}_r t)^2}(\overline{\Phi}_r-\overline{\Phi}_m)\Big| \\
&\quad + c^2|\sigma|M_0 t\Big|\overline{\Psi}_m\frac{(v_r+v_m+\overline{\Psi}_r t+\overline{\Psi}_m t)(v_r-v_m+(\overline{\Psi}_r-\overline{\Psi}_m)t)}{(c^2-(v_r+\overline{\Psi}_r t)^2)(c^2-(v_m+\overline{\Psi}_m t)^2)}\Big| \\
&\le L_2 t(|\rho_r-\rho_m|+|v_r-v_m|).
\end{aligned}$$

The estimates on I_3, I_4 and I_5 are similar to I_1 and I_2, we omit them here. The proof of (4.10) is similar to (4.9). Thus we complete the proof of the lemma.

We have stated the above lemma by assuming that there is a 2-shock wave. Now we assume that there is a rarefaction wave, say, 1-rarefaction wave. The domain of this rarefaction wave is

$$\Omega = \Big\{(x,t): 0<t<h, \frac{v_l-\kappa}{1-\frac{v_l\kappa}{c^2}} < \frac{x}{t} < \frac{v_m-\kappa}{1-\frac{v_m\kappa}{c^2}}\Big\}.$$

Corresponding to Lemma 4.1, we have the following lemma about rarefaction waves:

Lemma 4.2 *If $h \le \frac{1}{2K'}$, then*

$$\begin{aligned}
&\Big|\int_\Omega (U_1\varphi_t + F_1(U)\varphi_x + \Phi(\rho,v)\varphi)dxdt + \int_{\partial\Omega} U_1\varphi dx - F_1(U)\varphi dt\Big| \\
&\le C_1 h^3 + C_2 h^2(|\rho_m-\rho_l|+|v_m-v_l|), \qquad (4.11)
\end{aligned}$$

$$
\begin{aligned}
&\Big|\int_\Omega (U_2\psi_t + F_2(U)\psi_x + \Psi(\rho,v)\psi)dxdt + \int_{\partial\Omega} U_2\psi dx - F_2(U)\psi dt\Big| \\
&\le C_1h^3 + C_2h^2(|\rho_m-\rho_l| + |v_m - v_l|) \qquad (4.12)
\end{aligned}
$$

for any $\varphi(x,t),\ \psi(x,t)\in C_0^1(\Pi_T)$*, and* C_1 *and* C_2 *are constants depending only on* K*, the upper and lower bounds of* $\rho(x,t), v(x,t), \varphi(x,t), \psi(x,t)$*, and the positive lower bound of* $\rho(x,t)$*.*

Proof Different from constructing ρ^l, v^l in §2, we construct the approximate solutions $\rho^{(q)}$, $v^{(q)}$ in the following way:

$$
s^{(k)} = k\frac{s_m - s_l}{q} + s_l, \quad k = 0,1,\cdots,q.
$$

We divide the domain Ω into $q+1$ parts using the line $x=\sigma_k t$, $k=1,\cdots,q$, and $\rho^{(k)}$, $v^{(k)}$ as the approximations of the functions $\rho_0(x,t)$, $v_0(x,t)$ in the sub-domain:

$$
\Omega^k = \Big\{(x,t) : t>0,\ \sigma_k < \frac{x}{t} < \sigma_{k+1}\Big\}, \quad k=0,1,\cdots,q,
$$

σ_k, $\rho^{(k)}$ and $v^{(k)}$ are defined as follows

$$
\begin{cases}
\sigma_0 = \dfrac{v_l-\kappa}{1-\frac{v_l\kappa}{c^2}}, \quad \sigma_{q+1} = \dfrac{v_m-\kappa}{1-\frac{v_m\kappa}{c^2}};\\
\rho^{(0)} = \rho_l, \quad v^{(0)} = v_l;\\
\dfrac12 \ln\dfrac{c+v^{(k)}}{c-v^{(k)}} - \dfrac{c\kappa}{c^2+\kappa^2}\ln\rho^{(k)} = s^{(k)}, \qquad k=1,\cdots,q\ ;\\
\sigma_{k+1}(U_1^{(k+1)} - U_1^{(k)}) = F_1^{(k+1)}(U) - F_1^{(k)}(U), \quad k=0,1,\cdots,q-1;\\
\sigma_{k+1}(U_2^{(k+1)} - U_2^{(k)}) = F_2^{(k+1)}(U) - F_2^{(k)}(U), \quad k=0,1,\cdots,q-1,
\end{cases} \tag{4.13}
$$

where

$$
\begin{aligned}
U_1^{(k)} &= \rho^{(k)}\Big(\frac{\kappa^2+c^2}{c^2}\frac{(v^{(k)})^2}{c^2-(v^{(k)})^2}+1\Big), \quad F_1^{(k)}(U) = \rho^{(k)}(\kappa^2+c^2)\frac{v^{(k)}}{c^2-(v^{(k)})^2};\\
U_2^{(k)} &= \rho^{(k)}(\kappa^2+c^2)\frac{v^{(k)}}{c^2-(v^{(k)})^2}, \qquad F_2^{(k)}(U) = \rho^{(k)}\Big((\kappa^2+c^2)\frac{(v^{(k)})^2}{c^2-(v^{(k)})^2}+\kappa^2\Big).
\end{aligned}
$$

We denote the approximate functions by $(\rho_0^{(q)}, v_0^{(q)})(x,t)$. Define

$$
\begin{cases}
\rho^{(q)}(x,t) = \rho_0^{(q)}(x,t) + \overline{\Phi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))t,\\
v^{(q)}(x,t) = v_0^{(q)}(x,t) + \overline{\Psi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))t + O(1)t^2.
\end{cases} \tag{4.14}
$$

$\rho^{(q)}(x,t)$ and $v^{(q)}(x,t)$ converge uniformly to $\rho(x,t)$ and $v(x,t)$ respectively as $q\to+\infty$, for $0<t<h$.

Using the Green's formula, we have

$$
\begin{aligned}
&\int_\Omega (U_1^{(q)}\varphi_t + F_1^{(q)}(U)\varphi_x + \Phi(\rho^{(q)}, v^{(q)})\varphi)\,dx\,dt + \int_{\partial\Omega} U_1^{(q)}\varphi dx - F_1^{(q)}(U)\varphi\,dt\\
&= \int_\Omega (\Phi(\rho^{(q)}, v^{(q)}) - \Phi(\rho_0^{(q)}, v_0^{(q)}))\varphi\,dx\,dt + \sum_{k=1}^{p}\int_0^h \{-[U_1^{(q)}]\sigma_k + [F_1^{(q)}(U)]\}_{x=\sigma_k t}\varphi\,dt\\
&= \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}
$$

From (3.27) and (4.14), we have

$$\begin{aligned}
&|\Phi(\rho^{(q)}, v^{(q)}) - \Phi(\rho_0^{(q)}, v_0^{(q)})| \\
&\le K''(|\rho^{(q)} - \rho_0^{(q)}| + |v^{(q)} - v_0^{(q)}|) \\
&= K''(|\overline{\Phi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))|t + |\overline{\Psi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))|t) \\
&\le K''(\max|\overline{\Phi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))| + \max|\overline{\Psi}(\rho_0^{(q)}(x,t), v_0^{(q)}(x,t))|)t \\
&\le Ct.
\end{aligned}$$

Therefore,

$$|\mathrm{J}_1| \le C\int_\Omega |\varphi| t dx dt \le C_1 h^3.$$

In view of (4.13)–(4.14), we have

$$\begin{aligned}
&\{-[U_1^{(q)}]\sigma_k + [F_1^{(q)}(U)]\}_{x=\sigma_k t} \\
= &\, c^2 M_0 \rho^{(k)} \int_0^1 \frac{(c^2 + (v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t\eta)^2)\overline{\Psi}(\rho^{(k)}, v^{(k)})t}{(c^2 - (v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t\eta)^2)^2} d\eta \\
&- c^2 M_0 \rho^{(k-1)} \int_0^1 \frac{(c^2 + (v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t\eta)^2)\overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t}{(c^2 - (v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t\eta)^2)^2} d\eta \\
&+ c^2 M_0 t \frac{\overline{\Phi}(\rho^{(k)}, v^{(k)})(v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t)}{c^2 - (v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t)^2} \\
&- c^2 M_0 t \frac{\overline{\Phi}(\rho^{(k-1)}, v^{(k-1)})(v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t)}{c^2 - (v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t)^2} \\
&- \sigma_k M_0 t \rho^{(k)} \int_0^1 \frac{2c^2(v^{(k)} + \overline{\Psi}^{(k)} t\eta)^2 \overline{\Psi}(\rho^{(k)}, v^{(k)})}{c^2 - (v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t\eta)^2} d\eta \\
&+ \sigma_k M_0 t \rho^{(k-1)} \int_0^1 \frac{2c^2(v^{(k-1)} + \overline{\Psi}^{(k-1)} t\eta)^2 \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})}{c^2 - (v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t\eta)^2} d\eta \\
&- \sigma_k M_0 t \Big(\frac{(v^{(k)} + \overline{\Psi}(\rho^{(k)}, v^{(k)})t)^2 \overline{\Phi}(\rho^{(k)}, v^{(k)})}{c^2 - (v^{(k)} + \overline{\Psi}\rho^{(k)}, v^{(k)})t)^2} \Big) \\
&+ \sigma_k M_0 t \Big(\frac{(v^{(k-1)} + \overline{\Psi}(\rho^{(k-1)}, v^{(k-1)})t)^2 \overline{\Phi}(\rho^{(k-1)}, v^{(k-1)})}{c^2 - (v^{(k-1)} + \overline{\Psi}\rho^{(k-1)}, v^{(k-1)})t)^2} \Big) \\
&- \sigma_k t(\overline{\Phi}(\rho^{(k)}, v^{(k)}) - \overline{\Phi}(\rho^{(k-1)}, v^{(k-1)})).
\end{aligned}$$

Hence, from (4.9), we obtain

$$|\{-[U_1^{(q)}]\sigma_k + [F_1^{(q)}(U)]\}_{x=\sigma_k t}| \le Ct(|\rho^{(k)} - \rho^{(k-1)}| + |v^{(k)} - v^{(k-1)}|).$$

Then

$$|\mathrm{J}_2| \le \sum_{k=1}^p \int_0^h ct\{|\rho^{(k)} - \rho^{(k-1)}| + |v^{(k)} - v^{(k-1)}|\} dt \le C_2 h^2(|\rho_m - \rho_l| + |v_m - v_l|).$$

Let $q \to +\infty$, and we obtain (4.11). The proof of (4.12) is similar.

Now we can prove (4.4) and (4.5). In fact,

$$\begin{aligned}
&\Big| \int\int_{0\le t\le T} (U_1^l \varphi_t + F_1^l(U)\varphi_x + \Phi(\rho, v)\varphi)\, dx\, dt + \int_{-\infty}^{+\infty} U_1^l(x,0)\varphi(x,0)\, dx \Big| \\
&\le W_1 \Big| \sum_{n=1}^{N} \int_{-\infty}^{+\infty} [U_1(\rho^l(x, nh-0), v^l(x, nh-0)) - U_1(\rho^l(x, nh+0), v^l(x, nh+0))] dx \Big| \\
&\quad + \frac{W_2}{h^2} C_1 h^3 + \frac{T}{h}\Big(C_2 + \frac{R^2}{2}\Big) h^2 \Big[\sup_{t\in[0,T]} TV\{\rho^l(\cdot, t)\} + \sup_{t\in[0,T]} TV\{v^l(\cdot, t)\} \Big] \\
&\quad + \Big| \int_{-\infty}^{+\infty} \varphi(x,0)[U_1(x,0) - U_1^l(x,0)] dx \Big|,
\end{aligned}$$

where $T = Nh$, W_1 is the upper bound of the function $\varphi(x,t)$, W_2 is a constant related to the measure of the support of $\varphi(x,t)$. When $h = \frac{T}{2^i}$, $i \to +\infty$, the right-hand side of the above inequality tends to zero for almost every $a_{m,n} \in A$ (see [8]). Thus (4.4) is proved. The proof of (4.5) is similar.

Adopting the method in [3], we can show that, the weak solution $U(x,t)$ also satisfies the entropy inequality (4.3) (see [29] for details).

We conclude that

Theorem 4.1 *Under conditions* (2.12)*, there exists a global entropy solution* $U(x,t)$ *of the Cauchy problem* (1.3)–(1.4) *on* $t \ge 0$.

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