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Existence and Asymptotic Behavior of Radially Symmetric Solutions to a Semilinear Hyperbolic System in Odd Space Dimensions***

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Abstract This paper is concerned with a class of semilinear hyperbolic systems in odd space dimensions. Our main aim is to prove the existence of a small amplitude solution which is asymptotic to the free solution as $t \to -\infty$ in the energy norm, and to show it has a free profile as $t \to +\infty$. Our approach is based on the work of [11]. Namely we use a weighted L^∞ norm to get suitable a priori estimates. This can be done by restricting our attention to radially symmetric solutions. Corresponding initial value problem is also considered in an analogous framework. Besides, we give an extended result of [14] for three space dimensional case in Section 5, which is prepared independently of the other parts of the paper.

Keywords Semilinear wave equations, Asymptotic behavior, Radially symmetric solution
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1 Introduction

This paper is concerned with the following system of semilinear wave equations:

$$\begin{cases} \partial_t^2 u_1 - c_1^2 \Delta u_1 = F(u_2) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \partial_t^2 u_2 - c_2^2 \Delta u_2 = G(u_1) & \text{in } \mathbb{R}^n \times \mathbb{R}, \end{cases} \tag{1.1}$$

where $n \ge 2$, c_1 and c_2 are positive constants,

$$F(u_2) = |u_2|^p \text{ or } |u_2|^{p-1}u_2, \quad G(u_1) = |u_1|^q \text{ or } |u_1|^{q-1}u_1 \quad \text{with } 1 < p \le q.$$

In previous papers [13, 14], we studied the above system when $n = 2$ or $n = 3$, and proved the existence of a global solution of the Cauchy problem for sufficiently small initial data, provided $\Gamma > 0$ and $p^* > 0$. Here p^* and Γ are defined as follows:

$$p^* = \frac{n-1}{2}p - \frac{n+1}{2}, \quad q^* = \frac{n-1}{2}q - \frac{n+1}{2}, \tag{1.2}$$

$$\alpha = pq^* - 1, \quad \beta = qp^* - 1, \quad \Gamma = \alpha + p\beta. \tag{1.3}$$

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Moreover, under the same assumption as above, we proved the following: Let $u_i^-(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R})$, $i = 1, 2$ be solutions of the homogeneous wave equations

$$\partial_t^2 u_i - c_i^2 \Delta u_i = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{1.4}$$

with small initial data at $t = 0$. Then there exists uniquely a small amplitude solution $(u_1, u_2) \in (C^2(\mathbb{R}^n \times \mathbb{R}))^2$ of (1.1) which is asymptotic to (u_1^-, u_2^-) as $t \to -\infty$ in the energy norm. In addition, there exists uniquely a pair of solutions $(u_1^+, u_2^+) \in (C^2(\mathbb{R}^n \times \mathbb{R}))^2$ of (1.4) which is asymptotic to (u_1, u_2) as $t \to \infty$ in the energy norm. Namely, one can define a scattering operator on a dense set of a neighborhood of 0 in the energy space by

$$\begin{aligned} &(u_1^-(x,0), u_2^-(x,0), \partial_t u_1^-(x,0), \partial_t u_2^-(x,0)) \\ &\longmapsto (u_1^+(x,0), u_2^+(x,0), \partial_t u_1^+(x,0), \partial_t u_2^+(x,0)). \end{aligned} \tag{1.5}$$

Thus we are interested mainly in the case where $n \geq 4$ in the present article. (As for the case of the single equation $\partial_t^2 u - \Delta u = |u|^p$, see [8, 17–19].)

First we focus on the Cauchy problem for (1.1) in $\mathbb{R}^n \times (0, \infty)$. In [4], the problem was studied when $c_1 = c_2$, and the following condition was introduced besides $\Gamma > 0$, in order to show the existence of a global small solution for $n \geq 4$:

$$\frac{p-1}{pq-1} > \frac{n-1}{2(n+1)}, \quad \text{i.e.,} \quad q < \frac{2(n+1)}{n-1} - \frac{n+3}{(n-1)p}. \tag{1.6}$$

Since solutions of the Cauchy problem generically blow up in finite time if $\Gamma < 0$ even though the initial data are small enough (see [3, 4, 6, 7]), and the blow-up occurs also when $\Gamma = 0$ and either $n = 2$ or $n = 3$ (see [1, 5, 15, 16]), we need to assume $\Gamma > 0$ for the global existence. While, it is an open problem whether (1.6) is an optimal condition to prove the existence result or not.

In the present paper we prove the condition (1.6) can be relaxed by $q < \frac{n+1}{n-3}$, as long as the solution is radially symmetric and n is odd (for the details, see Theorems 4.1 and 4.2 below). Indeed, (1.6) with $p \leq q$ yields $1 < q < \frac{n+3}{n-1}$. This means that the admissible region for the exponents p, q $(1 < p \leq q)$ determined by $\Gamma > 0$ and $q < \frac{n+1}{n-3}$ is larger than that determined by $\Gamma > 0$ and (1.6), since $\frac{n+3}{n-1} < \frac{n+1}{n-3}$. Besides, we show that the condition $p^* > 0$ assumed in [14] for $n = 3$ can be removed, as in [3] where the case of $c_1 = c_2$ was handled (see also Theorem 5.1 below).

Next we turn our attention to asymptotic behavior as $t \to \pm\infty$ of solutions to (1.1). The aim here is to extend the result obtained in [13, 14] to the case where n is odd. Actually, we can define the operator (1.5) when n is odd, provided that $u_i^-(x,0)$ and $\partial_t u_i^-(x,0)$ are radially symmetric and that $\Gamma > 0$ and

$$q^* \leq 1, \quad \text{i.e.,} \quad q \leq \frac{n+3}{n-1} \quad \text{if } c_1 = c_2, \tag{1.7}$$

$$q^* < q, \quad \text{i.e.,} \quad q < \frac{n+1}{n-3} \quad \text{if } c_1 \neq c_2 \tag{1.8}$$

(for the details, see Theorems 3.1 and 3.2 below).

The proof of these results is based on the basic estimates given by Theorems 2.1 and 2.2 below. Those estimates are the refinements of the corresponding estimates obtained by [10, 11] in which the single equation $\partial_t^2 u - \Delta u = |u|^p$ was considered. In order to treat the system (1.1) with the possibly unequal propagation speeds, we need to extend the previous estimates as in the theorems.

Although in the present article we restrict ourselves to the case of odd space dimensions $n = 2m+3$ with m a nonnegative integer, we can also obtain the analogous results in even space dimensions $n = 2m + 2$, by strengthening the approach of [12]. (The details will be published elsewhere.)

The plan of this paper is as follows. In the next section we derive a priori estimates for radially symmetric solutions of the linear inhomogeneous wave equations in odd space dimensions, which will play a crucial role in dealing with the system (1.1). Section 3 is devoted to the study of the asymptotic behavior of radially symmetric solutions to the system (1.1). The Cauchy problem for (1.1) in $\mathbb{R}^n \times [0, \infty)$ is discussed in Sections 4. We formulate the Cauchy problem for the case of $n = 3$, independent of the other sections, and extend the result of [14] to the case where $p^* \le 0$ in Sections 5.

2 Linear Wave Equations

This section is concerned with radially symmetric solutions of linear wave equations. First we consider the homogeneous wave equation

$$u_{tt} - c^2\Big(u_{rr} + \frac{n-1}{r}u_r\Big) = 0 \quad \text{in } \Omega, \tag{2.1}$$

where c is a positive constant, $\Omega = \{(r,t) \in \mathbb{R}^2;\ r > 0\}$, $n = 2m+3$ and m is a positive integer. Let $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$. Then it is shown in [11] that a solution $u(r,t) \in C^2(\Omega)$ of the equation (2.1) satisfying

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for } r > 0 \tag{2.2}$$

is given by

$$\begin{aligned} u(r,t) &= K_c[f,g](r,t) \quad \text{for } (r,t) \in \Omega, \\ K_c[f,g](r,t) &= \frac{1}{c}\int_{|r-ct|}^{|r+ct|} g(\lambda)K(\lambda,r,ct)d\lambda + \frac{1}{c}\frac{\partial}{\partial t}\int_{|r-ct|}^{|r+ct|} f(\lambda)K(\lambda,r,ct)d\lambda. \end{aligned} \tag{2.3}$$

Here we have set

$$K(\lambda,r,t) = \frac{(-1)^m}{2m!}\Big(\frac{\lambda}{r}\Big)^{2m+1}\Big(\frac{\partial}{\partial\lambda}\frac{1}{2\lambda}\Big)^m \phi^m(\lambda,r,t) \tag{2.4}$$

with

$$\phi(\lambda,r,t) = r^2 - (\lambda - t)^2. \tag{2.5}$$

The following lemma is proven in [11, Section 2].

Lemma 2.1 *Let $(\lambda, r, t) \in \mathbb{R}^3$ and $r \neq 0$. Then we have*

$$K(-\lambda, r, t) = -K(\lambda, r, t), \tag{2.6}$$

$$K(\lambda, r, -t) = K(\lambda, r, t). \tag{2.7}$$

Moreover, suppose that $r > 0$, $t \geq 0$ and $|r - t| \leq \lambda \leq r + t$. Then we have

$$|\partial^\sigma \phi^m(\lambda, r, t)| \leq C r^{2m-|\sigma|} \quad \text{for } |\sigma| \leq 2m, \tag{2.8}$$

$$|\partial^\sigma K(\lambda, r, t)| \leq C(r^{-m-1}\lambda^{m+1-|\sigma|} + r^{-m-1-|\sigma|}\lambda^{m+1}) \quad \text{for } |\sigma| \leq 2, \tag{2.9}$$

where $\partial = (\partial_\lambda, \partial_r, \partial_t)$, and C is a constant depending only on m.

In order to state the decay estimates for solutions of (2.1)-(2.2), we introduce

$$Y_\mu(\varepsilon) = \{(f(r), g(r)) \in C^2([0,\infty)) \times C^1([0,\infty));\ \sup_{r>0}(1+r)^{1+\mu}\|(f,g)(r)\| \leq \varepsilon\} \tag{2.10}$$

for ε, $\mu > 0$, where $\|(f,g)(r)\|$ is defined by

$$\|(f,g)(r)\| = \sum_{j=0}^{2}\Big|\Big(\frac{d}{dr}\Big)^j f(r)\Big|(1+r)^{m-1+j} + \sum_{j=0}^{1}\Big|\Big(\frac{d}{dr}\Big)^j g(r)\Big|(1+r)^{m+j}. \tag{2.11}$$

In addition, for $(r,t) \in \Omega$ we set

$$w_+(r,t) = (1 + r + t), \quad w_c(r,t) = (1 + |r - ct|). \tag{2.12}$$

Then we have the following.

Theorem 2.1 *Let ε, μ be positive numbers such that $\mu \neq 1$. If $(f,g) \in Y_\mu(\varepsilon)$, then the Cauchy problem (2.1)–(2.2) admits uniquely a solution $u(r,t) := K_c[f,g](r,t) \in C^2(\Omega)$ satisfying*

$$|u(r,t)| \leq \begin{cases} C\varepsilon r^{1-m}(1+r)^{-1} w_+(r,|t|)^{-1} w_c(r,|t|)^{1-\mu}, & \text{if } \mu > 1, \\ C\varepsilon r^{1-m}(1+r)^{-1} w_+(r,|t|)^{-\mu}, & \text{if } 0 < \mu < 1, \end{cases} \tag{2.13}$$

$$|\partial u(r,t)| \leq C\varepsilon r^{-m}(1+r)^{-1} w_c(r,|t|)^{-\mu}, \tag{2.14}$$

$$|\partial^2 u(r,t)| \leq C\varepsilon r^{-m-1} w_c(r,|t|)^{-1-\mu} \tag{2.15}$$

for $(r,t) \in \Omega$, where $\partial = (\partial_r, \partial_t)$, C is a constant depending only on m, c and μ.

Proof When $\mu > 1$, the theorem follows immediately from [11, Theorem 1.1]. The case where $0 < \mu < 1$ can be also proven analogously, if we make use of the estimate

$$\int_{t-r}^{t+r} (1 + |\xi|)^{-\mu} d\xi \leq Cr(1 + r + |t|)^{-\mu} \quad \text{for } (r,t) \in \Omega,\ 0 < \mu < 1. \tag{2.16}$$

Hence we omit the details.

Next we consider the inhomogeneous wave equation

$$u_{tt} - c^2\Big(u_{rr} + \frac{n-1}{r}u_r\Big) = F(r,t) \tag{2.17}$$

in $\Omega = \{(r,t) \in \mathbb{R}^2; r > 0\}$ or $\Omega_+ = \{(r,t) \in \Omega; t \geq 0\}$, where c and n are as in (2.1) and $F(r,t)$ is a given function. A solution of the Cauchy problem for (2.17) in Ω_+ with the zero initial data at $t = 0$ is given by

$$L_c^+(F)(r,t) = \frac{1}{c}\int_0^t ds \int_{|r-c(t-s)|}^{r+c(t-s)} F(\lambda,s)K(\lambda,r,c(t-s))d\lambda \quad \text{for } (r,t) \in \Omega_+, \tag{2.18}$$

provided F satisfies certain conditions. Moreover, a solution of (2.17) in Ω having the asymptotic behavior $|u(r,t)| + |\partial_t u(r,t)| \to 0$ as $t \to -\infty$ is given by

$$L_c(F)(r,t) = \frac{1}{c}\int_{-\infty}^t ds \int_{|r-c(t-s)|}^{r+c(t-s)} F(\lambda,s)K(\lambda,r,c(t-s))d\lambda \quad \text{for } (r,t) \in \Omega, \tag{2.19}$$

provided F is chosen appropriately (see (2.72) below).

The aim of the present section is to prove the basic a priori estimates for $L_c^+(F)$ and $L_c(F)$ which will be stated in Theorem 2.2 below.

Let μ and a be positive numbers with $\mu \neq 1$. Let α, β, γ and δ be nonnegative real numbers satisfying

$$\alpha < m+1, \tag{2.20}$$

$$\gamma + \delta \geq \mu, \tag{2.21}$$

$$\alpha + \beta + \gamma - (m+1) \geq \mu, \tag{2.22}$$

$$\alpha + \beta + \gamma + \delta - (m+1) > 1 + \mu. \tag{2.23}$$

For a function $F(r,t) \in C(\Omega_+)$ with $\partial_r F(r,t) \in C(\Omega_+)$, we set

$$\begin{aligned} M^+(F,a) = \sup_{(\lambda,s)\in\Omega_+} &\{|F(\lambda,s)| + |\partial_\lambda F(\lambda,s)|\lambda(1+\lambda)^{-1}\} \\ &\times \lambda^\alpha (1+\lambda)^\beta (1+\lambda+s)^\gamma (1+|\lambda - as|)^\delta. \end{aligned} \tag{2.24}$$

For a function $F(r,t) \in C(\Omega)$ with $\partial_r F(r,t) \in C(\Omega)$, we also put

$$\begin{aligned} M(F,a) = \sup_{(\lambda,s)\in\Omega} &\{|F(\lambda,s)| + |\partial_\lambda F(\lambda,s)|\lambda(1+\lambda)^{-1}\} \\ &\times \lambda^\alpha (1+\lambda)^\beta (1+\lambda+|s|)^\gamma (1+|\lambda - a|s||)^\delta. \end{aligned} \tag{2.25}$$

Then the main result of this section is the following.

Theorem 2.2 *Let μ and a be positive numbers with $\mu \neq 1$. Assume that α, β, γ and δ fulfill* (2.20) *through* (2.23)*.*

(A) *Let $F(r,t) \in C(\Omega)$, $\partial_r F(r,t) \in C(\Omega)$ and $M(F,a) < \infty$. Suppose that either $\mu > 1$ or $a \neq c$. Then $L_c(F) \in C^2(\Omega)$ and we have*

$$|L_c(F)(r,t)| \leq CM(F,a)r^{1-m}(1+r)^{-1}\Phi_1(r,t;\mu,c), \tag{2.26}$$

$$|\partial^\sigma L_c(F)(r,t)| \leq CM(F,a)r^{1-m-|\sigma|}(1+r)^{-2+|\sigma|}\Phi_2(r,t;\mu,c) \tag{2.27}$$

for $(r,t)\in\Omega$ and $|\sigma|=1,2$, where $\partial=(\partial_r,\partial_t)$, and C is a constant depending only on μ, m, c, a, α, β, γ and δ. Moreover Φ_1 and Φ_2 are defined as follows:

$$\Phi_1(r,t;\mu,c)=\begin{cases} w_+(r,|t|)^{-1}w_c(r,t)^{1-\mu}, & \text{if } \mu>1,\\ w_+(r,|t|)^{-\mu}, & \text{if } 0<\mu<1,\end{cases} \tag{2.28}$$

$$\Phi_2(r,t;\mu,c)=\begin{cases} w_c(r,|t|)^{-1}w_c(r,t)^{1-\mu}, & \text{if } \mu>1,\\ w_c(r,t)^{-\mu}, & \text{if } 0<\mu<1.\end{cases} \tag{2.29}$$

Here $w_+(r,t)$ and $w_c(r,t)$ are given by (2.12).

(B) *Let $F(r,t)\in C(\Omega_+)$, $\partial_r F(r,t)\in C(\Omega_+)$ and $M^+(F,a)<\infty$. Then $L_c^+(F)\in C^2(\Omega_+)$ and we have*

$$|L_c^+(F)(r,t)|\le CM^+(F,a)r^{1-m}(1+r)^{-1}\Phi_1(r,t;\mu,c), \tag{2.30}$$

$$|\partial^\sigma L_c^+(F)(r,t)|\le CM^+(F,a)r^{1-m-|\sigma|}(1+r)^{-2+|\sigma|}\Phi_3(r,t;\mu,c) \tag{2.31}$$

for $(r,t)\in\Omega_+$ and $|\sigma|=1,2$, where we have set

$$\Phi_3(r,t;\mu,c)=\begin{cases} w_c(r,t)^{-\mu}, & \text{if } \mu>1,\\ w_c(r,t)^{-\mu}\Big(\dfrac{1+r+ct}{1+|r-ct|}\Big)^{(1-\mu)\chi}, & \text{if } 0<\mu<1.\end{cases} \tag{2.32}$$

Here $\chi=1$ if $a=c$ and $\alpha+\beta+\gamma-(m+1)\le 1$, while $\chi=0$ otherwise.

Remark 2.1 When $(r,t)\in\Omega$ with $t\le 0$, we see from (2.12) and (2.28) that $\Phi_1(r,t;\mu,c)$ is equivalent to $(1+r+|t|)^{-\mu}$ for any $\mu(\ne 1)$ and $c>0$.

Proof of Theorem 2.2 We begin with proving the part (A). It suffices to show the theorem for $c=1$, since $L_c(F)(r,t)=L_1(F_c)(r,ct)$ with $F_c(r,t)=\frac{F(r,\frac{t}{c})}{c^2}$. We set

$$w(s,r,t)=\int_{|\lambda_-|}^{\lambda_+}F(\lambda,s)K(\lambda,r,t-s)d\lambda \tag{2.33}$$

with $\lambda_\pm=t-s\pm r$, so that (2.19) yields

$$L_1(F)(r,t)=\int_{-\infty}^{t}w(s,r,t)ds \quad \text{for } (r,t)\in\Omega. \tag{2.34}$$

First we show that $L_1(F)\in C^2(\Omega)$. Let l be an arbitrary positive number and set

$$\Omega_l=\{(r,t)\in\Omega : r+|t|<l\}.$$

For $(r,t)\in\Omega_l$ and $s\le t$, we shall prove that there is a number $\theta>1$ such that

$$|w(s,r,t)|\le CM(F,a)(1+|s|)^{-\theta}r^{-m}, \tag{2.35}$$

$$|\partial_{r,t}w(s,r,t)|\le CM(F,a)(1+|s|)^{-\theta}(r^{-m}+r^{-m-1}), \tag{2.36}$$

$$|\partial_{r,t}^2 w(s,r,t)|\le CM(F,a)(1+|s|)^{-\theta}\{r^{-m}+r^{-m-2}+r^{-m-1}(1+\psi(|\lambda_-|))\} \tag{2.37}$$

hold, provided either $\mu > 1$ or $a \neq 1$ $(c = 1)$. Here $\psi(\lambda) = 0$ for $\lambda > 1$ and we have set for $0 < \lambda \leq 1$,

$$\psi(\lambda) = \begin{cases} 0, & \text{if } \alpha < m, \\ |\log \lambda|, & \text{if } \alpha = m, \\ \lambda^{m-\alpha}, & \text{if } \alpha > m. \end{cases} \tag{2.38}$$

Suppose that (2.35) through (2.37) are valid. Then we see from (2.34), (2.35) and (2.36) that $L_1(F) \in C^1(\Omega_l)$. If $\alpha < m$, then we have $L_1(F) \in C^2(\Omega_l)$, making use of (2.37). When $m \leq \alpha < m+1$, one can also show that $L_1(F) \in C^2(\Omega_l)$, analogously to [11, Proposition 4.5]. Hence in order to prove $L_1(F) \in C^2(\Omega)$, we have only to show (2.35) through (2.37).

We begin with proving them for $\theta = \mu > 0$. It follows from (2.33), (2.9) and (2.25) that

$$\begin{aligned} |w(s,r,t)| &\leq CM(F,a)r^{-m-1}\int_{|\lambda_-|}^{\lambda_+} \lambda^{m+1-\alpha}(1+\lambda)^{-\beta}(1+\lambda+|s|)^{-\gamma}(1+|\lambda-a|s||)^{-\delta}d\lambda \\ &= CM(F,a)r^{-m-1}\int_{|\lambda_-|}^{\lambda_+} \lambda(1+\lambda)^{-1}W(\lambda,s)d\lambda \quad \text{for } (r,t)\in\Omega,\ s\leq t, \end{aligned} \tag{2.39}$$

where C is a constant depending only on m and we have set

$$W(\lambda,s) = \lambda^{m-\alpha}(1+\lambda)^{1-\beta}(1+\lambda+|s|)^{-\gamma}(1+|\lambda-a|s||)^{-\delta}. \tag{2.40}$$

From (2.21) and (2.22) we get

$$W(\lambda,s) \leq C(1+\lambda+|s|)^{-\mu}\lambda^{m-\alpha} \quad \text{for } 0<\lambda\leq 1,\ s\in\mathbb{R}, \tag{2.41}$$

$$W(\lambda,s) \leq C(1+\lambda+|s|)^{-\mu}\{(1+\lambda)^{-1-\rho}+(1+|\lambda-a|s||)^{-1-\rho}\} \quad \text{for } \lambda\geq 1,\ s\in\mathbb{R}, \tag{2.42}$$

where we have set

$$\rho = \alpha+\beta+\gamma+\delta-(m+1)-(1+\mu), \tag{2.43}$$

and C is a constant depending only on α, β, γ, δ, μ and a. Note that

$$\rho > 0 \text{ and } \rho \geq \delta - 1, \tag{2.44}$$

according to (2.22) and (2.23). Therefore we see from (2.39) and (2.20) that (2.35) holds for $\theta = \mu$.

To derive (2.36), we use the following identity

$$\begin{aligned} \partial_{r,t}w(s,r,t) = &\int_{|\lambda_-|}^{\lambda_+} F(\lambda,s)\partial_{r,t}K(\lambda,r,t-s)d\lambda + F(\lambda_+,s)K(\lambda_+,r,t-s) \\ &- (\partial_{r,t}\lambda_-)F(|\lambda_-|,s)K(\lambda_-,r,t-s) \quad \text{for } s\leq t, \end{aligned} \tag{2.45}$$

which follows from (2.33) and (2.6). By (2.9) and (2.25) we get

$$\begin{aligned} |\partial_{r,t}w(s,r,t)| \leq CM(F,a)\Big[&r^{-m-1}\int_{|\lambda_-|}^{\lambda_+}(1+\lambda)^{-1}W(\lambda,s)d\lambda \\ &+ r^{-m-2}\int_{|\lambda_-|}^{\lambda_+}\lambda(1+\lambda)^{-1}W(\lambda,s)d\lambda + r^{-m-1}\lambda(1+\lambda)^{-1}W(\lambda,s)\big|_{\lambda=|\lambda_\pm|}\Big]. \end{aligned}$$

Using (2.41), (2.42) and (2.20), we obtain (2.36) for $\theta = \mu$. Similarly one can also prove (2.37), hence the detail is omitted (see also [11, Lemma 4.3]).

Next we show (2.35) through (2.37) for $\theta = 1 + \mu$, when $\mu > 0$ and $a \neq 1$. If $(r, t) \in \Omega_l$ and $t \geq s \geq -2l(1 + |a-1|^{-1}) - 2$, then $1 + |s|$ is bounded by some constant which depends on l and a. Therefore the previous argument shows that they are also valid for $\theta = 1 + \mu$.

On the contrary, suppose that

$$s \leq -2l(1 + |a-1|^{-1}) - 2. \tag{2.46}$$

Then we see that

$$\frac{|s|}{2} \geq l, \quad \frac{|a-1|}{2}|s| \geq l, \quad |s| \geq 2.$$

Hence we get

$$\lambda_- = t - s - r \geq |s| - l \geq \frac{|s|}{2} \geq 1, \tag{2.47}$$

$$|\lambda - a|s|| \geq \frac{|a-1|}{2}|s| \quad \text{for } |\lambda_-| \leq \lambda \leq \lambda_+,\ (r, t) \in \Omega_l. \tag{2.48}$$

Indeed, (2.48) follows from the fact that $|\lambda - |s|| \leq l$ for $(r, t) \in \Omega_l$, $s \leq 0$ and $|\lambda_-| \leq \lambda \leq \lambda_+$.

Now we see from (2.23), (2.39), (2.42), (2.47) and (2.48) that (2.35) holds for $\theta = 1 + \mu$. In a similar fashion, one can also prove (2.36) and (2.37) for $\theta = 1 + \mu$, making use of (2.45). Thus we have shown that $L_1(F) \in C^2(\Omega)$.

In order to derive the estimates (2.26) and (2.27), we will repeatedly make use of the following two lemmas.

Lemma 2.2 *Let μ and a be positive numbers with $\mu \neq 1$. Let α, β, γ and δ be nonnegative numbers satisfying* (2.20) *through* (2.23). *Suppose that either $\mu > 1$ or $a \neq 1$. Then for $(r, t) \in \Omega$ we have*

$$\int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} W(\lambda, s) d\lambda \leq Cr\Phi_1(r, t; \mu, 1), \tag{2.49}$$

$$\int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} \lambda^{-1} W(\lambda, s) d\lambda \leq C(1 + |r - t|)^{-\mu}, \tag{2.50}$$

where $\lambda_\pm = t - s \pm r$, $W(\lambda, s)$ and $\Phi_1(r, t; \mu, c)$ are defined by (2.40) *and* (2.28) *respectively, and C is a constant depending only on m, μ, a, α, β, γ and δ.*

Proof Fisrtly we shall prove (2.49). By (2.41) and (2.42) we see that the left-hand side of (2.49) is estimated by some constant times a sum of the following integrals:

$$\mathrm{I}_1 = \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + |s|)^{-\mu} (1 + \lambda)^{-1-\rho} d\lambda,$$

$$\mathrm{I}_2 = \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + |s|)^{-\mu} (1 + |\lambda - a|s||)^{-1-\rho} d\lambda,$$

$$\mathrm{I}_3 = \int_{t-r-1}^{(t-r+1)\wedge t} ds \int_{|\lambda_-|}^{1\wedge\lambda_+} (1 + |s|)^{-\mu} \lambda^{m-\alpha} d\lambda.$$

Here and in what follows we write

$$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\} \quad \text{for } a, b \in \mathbb{R}. \tag{2.51}$$

First we shall prove

$$\mathrm{I}_3 \leq Cr\Phi_1(r, t; \mu, 1) \quad \text{for } (r, t) \in \Omega. \tag{2.52}$$

Let $r \geq 1$. By (2.20) we easily have $\mathrm{I}_3 \leq C(1 + |t - r|)^{-\mu}$, which yields

$$\mathrm{I}_3 \leq \begin{cases} C(1 + |t| + r)^{-\mu}, & \text{if } t \geq 2r \text{ or } t \leq 0, \\ Cr(1 + r + |t|)^{-1}(1 + |t - r|)^{-\mu}, & \text{if } 0 \leq t \leq 2r. \end{cases}$$

Hence (2.52) follows when $r \geq 1$. While, it follows that

$$\mathrm{I}_3 \leq C(1 + |t - r|)^{-\mu} \int_{t-r-1}^{t-r+1} (1 + |\lambda_-|^{m-\alpha}) ds \int_{\lambda_-}^{\lambda_+} d\lambda \leq Cr(1 + |t - r|)^{-\mu}$$

by (2.20). Since the above estimate implies (2.52) if $0 < r \leq 1$, we have proved (2.52).

Next we shall prove

$$\mathrm{I}_1 + \mathrm{I}_2 \leq Cr\Phi_1(r, t; \mu, 1) \quad \text{for } (r, t) \in \Omega, \tag{2.53}$$

or equivalently

$$\int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + |s|)^{-\mu} (1 + |\lambda - b|s||)^{-1-\rho} d\lambda \leq Cr\Phi_1(r, t; \mu, 1) \tag{2.54}$$

for $(r, t) \in \Omega$, $b \geq 0$ and $\rho > 0$, provided either $\mu > 1$ or $0 < \mu < 1$ and $b \neq 1$.

If $\mu > 1$ and $b > 0$, then the estimate follows from [14, Propositions 2.4 and 2.5]. Besides, the proofs given there are still valid when $\mu > 1$ and $b = 0$. Therefore, we have only to consider the case where

$$b \neq 1 \quad \text{and} \quad 0 < \mu < 1. \tag{2.55}$$

When $t > 0$, we divide the integral in (2.54) at $s = 0$ and denote by $J_\pm$ the integrals over $\pm s \geq 0$, so that the left-hand side of (2.54) is estimated by $J_+ + J_-$. If $t \leq 0$, then we put $J_+ = 0$ and J_- stands for the integral in (2.54) itself. Introducing new variables ξ, η by

$$\xi = \lambda + s, \quad \eta = \lambda - s, \tag{2.56}$$

we have

$$J_+ = \frac{1}{2} \int_{|r-t|}^{r+t} (1 + \xi)^{-\mu} d\xi \int_{r-t}^{\xi} \Big(1 + \Big|\frac{1-b}{2}\xi + \frac{1+b}{2}\eta\Big|\Big)^{-1-\rho} d\eta \quad \text{for } t > 0, \tag{2.57}$$

$$J_- = \frac{1}{2} \int_{t-r}^{t+r} d\xi \int_{\xi \vee |r-t|}^{\infty} (1 + \eta)^{-\mu} \Big(1 + \Big|\frac{1+b}{2}\xi + \frac{1-b}{2}\eta\Big|\Big)^{-1-\rho} d\eta, \tag{2.58}$$

where we have used the notation in (2.51).

First we deal with J_+. Let $t > 0$. Since $\rho > 0$ and $1 + b \neq 0$, we have

$$J_+ \leq C \int_{|r-t|}^{r+t} (1+\xi)^{-\mu} d\xi,$$

which yields

$$J_+ \leq Cr(1 + r + |t|)^{-\mu} \quad \text{for } r > 0,\ t > 0,\ 0 < \mu < 1. \tag{2.59}$$

Next we consider J_-. Since $(1+\eta)^{-\mu} \leq (1+|\xi|)^{-\mu}$ for $\eta \geq \xi \vee |r-t|$ and $t - r \leq \xi \leq t + r$, we have

$$J_- \leq C \int_{t-r}^{t+r} (1+|\xi|)^{-\mu} d\xi$$

by (2.55). Therefore, for $(r,t) \in \Omega$ and $0 < \mu < 1$, we see from (2.16) that J_- has the same bounds as in (2.59). Thus the desired estimate (2.54) follows. Now we get (2.49) from (2.52) and (2.53).

Secondly we shall prove (2.50). From (2.21) and (2.22) we have

$$\begin{aligned} \lambda^{-1} W(\lambda, s) \leq C\{&(1 + \lambda + |s|)^{-\mu} (1+\lambda)^{-2-\rho} \\ &+ (1 + \lambda + |s|)^{-1-\mu} (1 + |\lambda - a|s||)^{-1-\rho}\} \quad \text{for } \lambda \geq 1,\ s \in \mathbb{R}, \end{aligned} \tag{2.60}$$

where $\rho (> 0)$ is the same number as in (2.43). By the above estimate together with (2.41), we see that the left-hand side of (2.50) is evaluated by some constant times a sum of the following integrals:

$$\begin{aligned} \mathrm{I}_4 &= \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + |s|)^{-\mu} (1+\lambda)^{-2-\rho} d\lambda, \\ \mathrm{I}_5 &= \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + |s|)^{-1-\mu} (1 + |\lambda - a|s||)^{-1-\rho} d\lambda, \\ \mathrm{I}_6 &= \int_{t-r-1}^{t-r+1} (1+|s|)^{-\mu} ds \int_{|\lambda_-|}^{1} \lambda^{m-\alpha-1} d\lambda. \end{aligned}$$

It follows that

$$\mathrm{I}_6 \leq C(1 + |r-t|)^{-\mu} \int_{t-r-1}^{t-r+1} (1 + \psi(|\lambda_-|)) ds \leq C(1 + |r-t|)^{-\mu}$$

by (2.20). Here $\psi(\lambda)$ is the function defined by (2.38). Applying (2.54) to I_5, we get

$$\mathrm{I}_5 \leq Cr\Phi_1(r, t; 1+\mu, 1) \leq C(1 + |r-t|)^{-\mu}$$

by (2.28). Besides, thanks to the following inequality

$$|\lambda_-| + |s| \geq |r - t| \quad \text{for } s \leq t, \tag{2.61}$$

we get

$$(1 + |r-t|)^{\mu} \mathrm{I}_4 \leq \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1+\lambda)^{-2-\rho} d\lambda \leq C.$$

Thus we obtain (2.50) and the proof is complete.

Lemma 2.3 *Suppose that the hypotheses of the preceding lemma are fulfilled. Then for* $(r,t)\in\Omega$ *we have*

$$\int_{-\infty}^{t} W(\lambda_+,s)ds \le C\Phi_2(r,t;\mu,1), \tag{2.62}$$

$$\int_{-\infty}^{t} W(|\lambda_-|,s)ds \le C(1+|r-t|)^{-\mu}, \tag{2.63}$$

where $\lambda_\pm = t-s\pm r$, $W(\lambda,s)$ *and* $\Phi_2(r,t;\mu,c)$ *are defined by* (2.40) *and* (2.29) *respectively, and* C *is a constant depending only on* m, μ, a, α, β, γ *and* δ.

Proof First we shall prove (2.62). By (2.41) and (2.42) we see that the left-hand side is estimated by some constant times a sum of the following integrals:

$$\mathrm{I}_1 = \int_{-\infty}^{t} (1+\lambda_+ +|s|)^{-\mu}(1+\lambda_+)^{-1-\rho}ds,$$

$$\mathrm{I}_2 = \int_{-\infty}^{t} (1+\lambda_+ +|s|)^{-\mu}(1+|\lambda_+ - a|s||)^{-1-\rho}ds,$$

$$\mathrm{I}_3 = 0 \quad \text{for } r>1,$$

$$\mathrm{I}_3 = \int_{t+r-1}^{t} (1+|s|)^{-\mu}(\lambda_+)^{m-\alpha}ds \quad \text{for } 0<r\le 1.$$

From (2.20) we easily have

$$\mathrm{I}_3 \le C(1+r+|t|)^{-\mu} \quad \text{for } (r,t)\in\Omega. \tag{2.64}$$

Next we shall prove

$$\mathrm{I}_1+\mathrm{I}_2 \le C\Phi_2(r,t;\mu,1) \quad \text{for } (r,t)\in\Omega, \tag{2.65}$$

which follows, as is easily seen, from

$$\int_{-\infty}^{t} (1+\lambda_+ +|s|)^{-\mu}(1+|\lambda_+ - bs|)^{-1-\rho}ds \le C\Phi_2(r,t;\mu,1) \tag{2.66}$$

for $(r,t)\in\Omega$, $b\in\mathbb{R}$ and $\rho>0$, provided either $\mu>1$ or $0<\mu<1$ and $b\neq -1$.

By I we denote the left-hand side of (2.66). Since

$$\lambda_+ +|s| \ge r+|t| \quad \text{for } s\le t, \tag{2.67}$$

it is easy to see that

$$\mathrm{I} \le (1+r+|t|)^{-\mu}\int_{-\infty}^{\infty}(1+|t+r-(1+b)s|)^{-1-\rho}ds,$$

which yields the desired estimate, if $b\neq -1$.

Therefore, it remains to consider the case where $b=-1$ and $\mu>1$. Then

$$\mathrm{I} = (1+|t+r|)^{-1-\rho}\Big[\int_{-\infty}^{t\wedge 0}(1+\lambda_+ - s)^{-\mu}ds + \int_{0}^{t\vee 0}(1+\lambda_+ + s)^{-\mu}ds\Big].$$

Since $\mu > 1$, the above integrals are both evaluated by $C(1+r+|t|)^{1-\mu}$, hence (2.66) follows. Now we get (2.62) from (2.64) and (2.65).

Next we shall prove (2.63). By (2.41) and (2.42) we see that the left-hand side is estimated by some constant times a sum of the following integrals:

$$\mathrm{I}_1 = \int_{-\infty}^{t} (1+|\lambda_-|+|s|)^{-\mu}(1+|\lambda_-|)^{-1-\rho} ds,$$
$$\mathrm{I}_2 = \int_{-\infty}^{t} (1+|\lambda_-|+|s|)^{-\mu}(1+||\lambda_-|-a|s||)^{-1-\rho} ds,$$
$$\mathrm{I}_3 = \int_{t-r-1}^{(t-r+1)\wedge t} (1+|s|)^{-\mu}|\lambda_-|^{m-\alpha} ds.$$

From (2.20) we easily have

$$\mathrm{I}_3 \leq C(1+|r-t|)^{-\mu} \quad \text{for } (r,t) \in \Omega. \tag{2.68}$$

Next we shall prove that I_1 and I_2 have the same bound as in (2.68), which is a consequence of

$$\int_{-\infty}^{t} (1+|\lambda_-|+|s|)^{-\mu}(1+|\lambda_- - bs|)^{-1-\rho} ds \leq C(1+|r-t|)^{-\mu} \tag{2.69}$$

for $(r,t) \in \Omega$, $b \in \mathbb{R}$ and $\rho > 0$, provided either $\mu > 1$ or $0 < \mu < 1$ and $b \neq -1$.

When $b \neq -1$, making use of (2.61), we see that (2.69) follows, as before.

If $b = -1$ and $\mu > 1$, then the left-hand side of (2.69) is estimated by

$$(1+|r-t|)^{-\rho+\varepsilon-\mu} \int_{-\infty}^{\infty} (1+|s|)^{-1-\varepsilon} ds \leq C(1+|r-t|)^{-\mu},$$

where we have used (2.61) and taken a positive number ε satisfying $\varepsilon \leq \mu - 1$ and $\varepsilon \leq \rho$. Thus we have proved (2.69), and hence (2.63) follows. The proof is complete.

We are now in a position to derive the estimates (2.26) and (2.27). First we deal with the case where $r \geq 1$. It follows from (2.19), (2.25), (2.9) with $|\sigma| = 0$ and (2.40) that

$$|L_1(F)(r,t)| \leq CM(F,a) r^{-m-1} \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} W(\lambda,s) d\lambda.$$

Making use of (2.49), we get (2.26) for $r \geq 1$.

Next we shall prove (2.27) for $|\sigma| = 1$. It follows from (2.34), (2.45), (2.9), (2.25) and (2.40) that

$$|\partial L_1(F)(r,t)| \leq CM(F,a) \sum_{k=1}^{4} \mathrm{I}_k \quad \text{for } (r,t) \in \Omega, \tag{2.70}$$

where

$$\mathrm{I}_1 = r^{-m-2} \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} W(\lambda,s) d\lambda,$$
$$\mathrm{I}_2 = r^{-m-1} \int_{-\infty}^{t} ds \int_{|\lambda_-|}^{\lambda_+} (1+\lambda)^{-1} W(\lambda,s) d\lambda,$$

$$\mathrm{I}_3 = r^{-m-1}\int_{-\infty}^{t} W(\lambda_+, s)ds,$$
$$\mathrm{I}_4 = r^{-m-1}\int_{-\infty}^{t} W(|\lambda_-|, s)ds.$$

Making use of Lemma 2.2 for I_1, I_2 and Lemma 2.3 for I_3, I_4, we get

$$\mathrm{I}_1 \le Cr^{-m-1}\Phi_1(r,t;\mu,1),$$
$$\mathrm{I}_3 \le Cr^{-m-1}\Phi_2(r,t;\mu,1),$$
$$\mathrm{I}_2 + \mathrm{I}_4 \le Cr^{-m-1}(1+|r-t|)^{-\mu}.$$

Thus (2.27) with $|\sigma| = 1$ for $r \ge 1$ immediately follows, since $\Phi_1(r,t;\mu,1)$ and $(1+|r-t|)^{-\mu}$ are dominated by $\Phi_2(r,t;\mu,1)$ (recall (2.28) and (2.29)).

Finally we shall prove (2.27) for $|\sigma| = 2$. Since if $r \ge 1$, then (2.9) implies

$$|\partial K(\lambda, r, t)| \le Cr^{-m-1}\lambda^m(1+\lambda),$$

analogously to (2.70) we have

$$|\partial_r\partial_{r,t}L_1(F)(r,t)| \le CM(F,a)\sum_{k=1}^{4}\mathrm{I}_k \quad \text{for } (r,t)\in\Omega \text{ with } r\ge 1, \tag{2.71}$$

where

$$\mathrm{I}_1 = r^{-m-3}\int_{-\infty}^{t} ds\int_{|\lambda_-|}^{\lambda_+} W(\lambda, s)d\lambda,$$
$$\mathrm{I}_2 = r^{-m-1}\int_{-\infty}^{t} ds\int_{|\lambda_-|}^{\lambda_+} \lambda^{-1}(1+\lambda)^{-1}W(\lambda, s)d\lambda,$$
$$\mathrm{I}_3 = r^{-m-1}\int_{-\infty}^{t} W(\lambda_+, s)ds,$$
$$\mathrm{I}_4 = r^{-m-1}\int_{-\infty}^{t} W(|\lambda_-|, s)ds.$$

As before, we see that (2.27) holds for $\partial_r\partial_{r,t}L_1(F)$, if $r \ge 1$.

To estimate $\partial_t^2 L_1(F)$ we note that $L_1(F)$ is a solution of (2.17) with $c = 1$, namely

$$\partial_t^2 L_1(F)(r,t) = \Big(\partial_r^2 + \frac{n-1}{r}\partial_r\Big)L_1(F)(r,t) + F(r,t) \quad \text{for } (r,t)\in\Omega. \tag{2.72}$$

This identity can be derived from (2.34) based on the estimtes (2.35) through (2.37).

We also claim that

$$|F(r,t)| \le CM(F,a)r^{-m-1}(1+r+|t|)^{-\mu} \quad \text{for } (r,t)\in\Omega. \tag{2.73}$$

In fact, it follows from (2.25) and (2.20) that

$$|F(r,t)| \le CM(F,a)r^{-m-1}(1+r)^{m+1-\alpha-\beta}(1+r+|t|)^{-\gamma}(1+|r-a|t||)^{-\delta} \quad \text{for } (r,t)\in\Omega.$$

Using (2.22) (resp. (2.21)) when $r \ge 1$ and $a|t| \le 2r$ (resp. $0 < r \le 1$ or $a|t| \ge 2r$), we get (2.73).

Since we have already shown that (2.27) holds for $r \geq 1$ except for $\partial_t^2 L_1(F)$, we can conclude that it is also valid for $\partial_t^2 L_1(F)$ from (2.72) and (2.73). Thus it remains to show (2.26) and (2.27) for $0 < r \leq 1$. This can be done if we show the following proposition.

Proposition 2.1 *Suppose that the hypotheses of Lemma* 2.2 *are fulfilled and that* $F(r,t) \in C(\Omega)$, $\partial_r F(r,t) \in C(\Omega)$ *and* $M(F,a) < \infty$. *Then for* $(r,t) \in \Omega$ *with* $0 < r \leq 1$ *and* $|\sigma| \leq 2$ *we have*

$$|\partial^\sigma L_1(F)(r,t)| \leq CM(F,a) r^{1-m-|\sigma|}(1+|t|)^{-\mu}. \tag{2.74}$$

Proof In what follows we suppose that $(r,t) \in \Omega$ with $0 < r \leq 1$. From (2.34) we have for $|\sigma| \leq 2$,

$$\begin{aligned}\partial_{r,t}^\sigma L_1(F)(r,t) &= \int_{-\infty}^{t-2r} \partial_{r,t}^\sigma w(s,r,t)ds + \int_{t-2r}^{t} \partial_{r,t}^\sigma w(s,r,t)ds + \chi_\sigma F(r,t) \\ &\equiv A_\sigma(r,t) + B_\sigma(r,t) + \chi_\sigma F(r,t),\end{aligned} \tag{2.75}$$

where $\chi_\sigma = 1$ if $\partial_{r,t}^\sigma = \partial_t^2$, while $\chi_\sigma = 0$ otherwise.

First we shall prove

$$|A_\sigma(r,t)| \leq CM(F,a) r^{1-m-|\sigma|}(1+|t|)^{-\mu} \quad \text{for } |\sigma| \leq 2, \tag{2.76}$$

which follows from

$$|\partial_{r,t}^\sigma w(s,r,t)| \leq CM(F,a) r^{1-m-|\sigma|}\Big(\frac{1}{r}\int_{\lambda_-}^{\lambda_+} W(\lambda,s)ds + W(\lambda_+,s) + W(\lambda_-,s)\Big) \quad \text{for } s \leq t-2r,\ |\sigma| \leq 2 \tag{2.77}$$

by Lemmas 2.2 and 2.3, since $0 < r \leq 1$.

It follows from (2.4) that $K(\lambda,r,t)$ is of the following form

$$K(\lambda,r,t) = r^{-2m-1}\sum_{j=0}^{m} C_j \lambda^{j+1}\partial_\lambda^j \phi^m(\lambda,r,t),$$

where C_j are constants. Hence, if we set

$$w_j(s,r,t) = \int_{\lambda_-}^{\lambda_+} \lambda^{j+1} F(\lambda,s)\partial_\lambda^j \phi^m(\lambda,r,t-s)d\lambda, \tag{2.78}$$

then (2.33) yields

$$w(s,r,t) = r^{-2m-1}\sum_{j=0}^{m} C_j w_j(s,r,t) \quad \text{for } s \leq t-2r. \tag{2.79}$$

Therefore, (2.77) follows from

$$|\partial_{r,t}^\sigma w_j(s,r,t)| \leq CM(F,a) r^{m+2-|\sigma|}\Big(\frac{1}{r}\int_{\lambda_-}^{\lambda_+} W(\lambda,s)ds + W(\lambda_+,s) + W(\lambda_-,s)\Big) \quad \text{for } 0 \leq j \leq m,\ s \leq t-2r,\ |\sigma| \leq 2. \tag{2.80}$$

Fisrt we consider the case $0 \le j \le m-1$. When $\lambda \ge \lambda_- \ge r$, we get from (2.8)

$$|\partial_{r,t}^{\sigma}\partial_{\lambda}^{j}\phi^m(\lambda,r,t-s)| \le Cr^{2m-j-|\sigma|} \le Cr^{m+1-|\sigma|}\lambda^{m-1-j} \quad \text{for } 0 \le j \le m-1,\ |\sigma| \le 2. \tag{2.81}$$

Therefore, we obtain (2.80) for $0 \le j \le m-1$ from (2.78) and (2.25). The last two terms in (2.80) appear from the estimate for the second derivatives of $w_{m-1}(s,r,t)$.

Next we consider the case $j = m$. Integrating by parts, we have

$$w_m(s,r,t) = \int_{\lambda_-}^{\lambda_+} F_m(\lambda,s)\partial_{\lambda}^{m-1}\phi^m(\lambda,r,t-s)d\lambda, \tag{2.82}$$

where $F_m(\lambda,s) = -\partial_\lambda(\lambda^{m+1}F(\lambda,s))$. Since (2.25) yields

$$|F_m(\lambda,s)| \le CM(F,a)\lambda^{m-\alpha}(1+\lambda)^{1-\beta}(1+\lambda+|s|)^{-\gamma}(1+|\lambda-a|s||)^{-\delta} \quad \text{for } (\lambda,s)\in\Omega,$$

we get (2.80) for $j = m$, by using (2.81) with $j = m-1$. Thus we have shown (2.76).

Next we shall prove

$$|B_\sigma(r,t)| \le CM(F,a)r^{1-m-|\sigma|}(1+|t|)^{-\mu} \quad \text{for } |\sigma| \le 2. \tag{2.83}$$

Since $0 < r \le 1$, we have

$$|\lambda_-| \le \lambda_+ \le 3r \le 3 \quad \text{for } t-2r \le s \le t. \tag{2.84}$$

From (2.40), (2.84), (2.21) and $0 < r \le 1$, we get

$$W(\lambda,s) \le C\lambda^{m-\alpha}(1+|t|)^{-\mu} \quad \text{for } |\lambda_-| \le \lambda \le \lambda_+,\ t-2r \le s \le t. \tag{2.85}$$

Note that (2.9) yields

$$|\partial^\sigma K(\lambda,s,t-s)| \le Cr^{-m-1}\lambda^{m+1-|\sigma|} \quad \text{for } \lambda \le 3r,\ |\sigma| \le 2, \tag{2.86}$$

where $\partial = (\partial_\lambda, \partial_r, \partial_t)$.

First we derive (2.83) for $\sigma = 0$. It follows from (2.39), (2.85), (2.84) and (2.20) that

$$\begin{aligned} |w(s,r,t)| &\le CM(F,a)r^{-m-1}(1+|t|)^{-\mu}\int_{|\lambda_-|}^{\lambda_+} d\lambda \\ &\le CM(F,a)r^{-m}(1+|t|)^{-\mu} \quad \text{for } t-2r \le s \le t. \end{aligned}$$

Therefore we get (2.83) for $\sigma = 0$.

Next we deal with the case $|\sigma| = 1$. It follows from (2.45), (2.25), (2.86) and (2.40) that

$$|\partial_{r,t}w(s,r,t)| \le CM(F,a)r^{-m-1}\Big[\int_{|\lambda_-|}^{\lambda_+} W(\lambda,s)d\lambda + \lambda_+W(\lambda_+,s) + |\lambda_-|W(|\lambda_-|,s)\Big].$$

By (2.85), (2.20) and (2.84), we see that (2.83) holds for $|\sigma| = 1$.

Finally we deal with the case $|\sigma| = 2$. As above we have

$$\sum_{|\sigma|=2} |B_\sigma(r,t)| \le CM(F,a)r^{-m-1}(1+|t|)^{-\mu}\int_{t-2r}^{t} \Big[(\lambda_+)^{m-\alpha} + |\lambda_-|^{m-\alpha} + \int_{|\lambda_-|}^{3} \lambda^{m-1-\alpha}d\lambda\Big]ds.$$

Since the above integral is bounded for $0 < r \le 1$, according to (2.20), we thus obtain (2.83) for $|\sigma| = 2$.

Now the desired estimate (2.74) follows from (2.75), (2.76), (2.83) and (2.73) when $(r,t) \in \Omega$ with $0 < r \le 1$. The proof is complete.

Since we have shown the part (A) in Theorem 2.2 so far, it remains to show the part (B). The procedure is analogous to the proof of the part (A) when either $\mu > 1$ or $a \ne c$. Hence we suppose in what follows that

$$a = c = 1 \quad \text{and} \quad 0 < \mu < 1. \tag{2.87}$$

Seeing the proof of (2.35) through (2.37) with $\theta = 1 + \mu$, we find that they are still valid for $0 \le s \le t$. Hence $L_1^+(F) \in C^2(\Omega_+)$. As for the estimates (2.30) and (2.31), it suffices to modify Lemmas 2.2 and 2.3 so that they include the case of $a = c$. In fact, the following estimates (2.88), (2.89), (2.91) and (2.92) enable us to derive the same conclusion of Proposition 2.1 under the assumption (2.87).

Fisrt we show that

$$\int_0^t ds \int_{|\lambda_-|}^{\lambda_+} W(\lambda, s) d\lambda \le C r \Phi_1(r,t;\mu,1), \tag{2.88}$$

$$\int_0^t ds \int_{|\lambda_-|}^{\lambda_+} \lambda^{-1} W(\lambda, s) d\lambda \le C(1 + |r-t|)^{-\mu} \tag{2.89}$$

hold for $(r,t) \in \Omega_+$, $a = 1$ and $0 < \mu < 1$. Since (2.59) holds for $b = 1$, we have

$$\int_0^t ds \int_{|\lambda_-|}^{\lambda_+} (1 + \lambda + s)^{-\mu} (1 + |\lambda - bs|)^{-1-\rho} d\lambda \le C r \Phi_1(r,t;\mu,1) \tag{2.90}$$

for $(r,t) \in \Omega_+$, $b \ge 0$, $\rho > 0$, and $0 < \mu < 1$. Therefore, repeating the argument in the proof of Lemma 2.2, we get (2.88) and (2.89).

Next we prove

$$\int_0^t W(\lambda_+, s) ds \le C \Phi_2(r,t;\mu,1), \tag{2.91}$$

$$\int_0^t W(|\lambda_-|, s) ds \le C \Phi_3(r,t;\mu,1) \tag{2.92}$$

for $(r,t) \in \Omega_+$, $a = 1$ and $0 < \mu < 1$. Here $\Phi_3(r,t;\mu,1)$ is defined by (2.32). Since (2.66) holds for $b = 1$, we get (2.91) analogously to (2.62).

To prove (2.92), we devide the argument into two cases. First suppose that

$$\alpha + \beta + \gamma - (m+1) \le 1. \tag{2.93}$$

Then $\chi = 1$ in (2.32). Seeing the proof of (2.63), we find that our task is reduced to the following estimate:

$$\int_0^t (1 + |\lambda_-| + s)^{-\mu} (1 + |\lambda_- - bs|)^{-1-\rho} ds \le C \Phi_3(r,t;\mu,1) \tag{2.94}$$

for $(r, t) \in \Omega_+$, $b \in \mathbb{R}$, $\rho > 0$, and $0 < \mu < 1$.

When $b \neq -1$, making use of (2.61), we see that the left-hand side of (2.94) is estimated by $C(1 + |r - t|)^{-\mu}$. Hence (2.94) holds. On the contrary, if $b = -1$, then the left-hand side of (2.94) is equal to

$$(1 + |r - t|)^{-1-\rho} \int_0^t (1 + |\lambda_-| + s)^{-\mu} ds \leq C(1 + |r - t|)^{-1-\rho}(1 + t)^{1-\mu}$$
$$= C(1 + |r - t|)^{-\mu-\rho}\Big(\frac{1 + t}{1 + |r - t|}\Big)^{1-\mu},$$

which implies (2.94).

Next suppose that (2.93) does not hold. Then (2.43) yields $\delta < \mu + \rho$. By (2.21) and (2.43), we have

$$W(\lambda, s) \leq C\{(1 + \lambda + s)^{-\mu}(1 + \lambda)^{-1-\rho} + (1 + \lambda + s)^{\delta-\mu-1-\rho}(1 + |\lambda - as|)^{-\delta}\}$$

for $\lambda \geq 1$ and $s \geq 0$. Employing this bound instead of (2.42), we see that it suffces to show

$$\int_0^t (1 + |\lambda_-| + s)^{\delta-\mu-1-\rho}(1 + |\lambda_- - bs|)^{-\delta} ds \leq C(1 + |r - t|)^{-\mu} \tag{2.95}$$

for $(r, t) \in \Omega_+$, $b \in \mathbb{R}$ and $0 < \mu < 1$, provided ρ satisfies (2.44).

When $b \neq -1$, it is easy to see from (2.44) and (2.61) that (2.95) holds. While, in the case where $b = -1$, we take a positive number ε so that $\varepsilon \leq \mu + \rho - \delta$ and $\varepsilon \leq \rho$. Then we see that the left-hand side of (2.95) is equal to

$$(1 + |r - t|)^{-\delta} \int_0^t (1 + |\lambda_-| + s)^{\delta-\mu-1-\rho} ds \leq C(1 + |r - t|)^{\varepsilon-\mu-\rho} \int_0^t (1 + s)^{-1-\varepsilon} ds.$$

Thus we get (2.95), hence (2.92). This completes the proof of the part (B) in Theorem 2.2.

Theorem 2.3 *Let μ and a be positive numbers with $\mu \neq 1$. Assume that α, β, γ and δ fulfill* (2.20) *through* (2.23).

(A) *Let $F(r, t) \in C(\Omega)$ and*

$$\widetilde{M}(F, a) := \sup_{(\lambda,s)\in\Omega} |F(\lambda, s)|\lambda^{\alpha+1}(1 + \lambda)^{\beta-1}(1 + \lambda + |s|)^{\gamma}(1 + |\lambda - a|s||)^{\delta} < \infty. \tag{2.96}$$

Suppose that either $\mu > 1$ or $a \neq c$. Then we have

$$|L_c(F)(r, t)| \leq C\widetilde{M}(F, a)r^{-m}\Phi_1(r, t; \mu, c) \quad \textit{for } (r, t) \in \Omega, \tag{2.97}$$

where C is a constant depending only on μ, m, c, a, α, β, γ and δ.

(B) *Let $F(r, t) \in C(\Omega_+)$ and*

$$\widetilde{M}^+(F, a) := \sup_{(\lambda,s)\in\Omega_+} |F(\lambda, s)|\lambda^{\alpha+1}(1 + \lambda)^{\beta-1}(1 + \lambda + s)^{\gamma}(1 + |\lambda - as|)^{\delta} < \infty. \tag{2.98}$$

Then we have

$$|L_c^+(F)(r, t)| \leq C\widetilde{M}^+(F, a)r^{-m}\Phi_1(r, t; \mu, c) \quad \textit{for } (r, t) \in \Omega_+. \tag{2.99}$$

Proof We may assume $c = 1$ without loss of generality. It follows from (2.33), (2.9), (2.96) and (2.40) that

$$|w(s,r,t)| \le C\widetilde{M}(F,a)r^{-m-1}\int_{|\lambda_-|}^{\lambda_+} W(\lambda,s)d\lambda.$$

Therefore (2.97) with $c = 1$ follows from (2.34) and (2.49). Moreover, we get (2.99) with $c = 1$, if we use (2.88) instead of (2.49). This completes the proof.

3 Asymptotic Behavior

In this section we study asymptotic behavior as $t \to \pm\infty$ of radially symmetric solutions to the system (1.1) in odd space dimensions. For simplicity, we take $F(v) = |v|^p$ and $G(u) = |u|^q$. We shall write the solutions as $u_1(x,t) = u(|x|,t)$ and $u_2(x,t) = v(|x|,t)$, so that $u(r,t)$ and $v(r,t)$ satisfy the following system:

$$\begin{cases} u_{tt} - c_1^2\Big(u_{rr} + \dfrac{n-1}{r}u_r\Big) = |v|^p, \\ v_{tt} - c_2^2\Big(v_{rr} + \dfrac{n-1}{r}v_r\Big) = |u|^q \end{cases} \tag{3.1}$$

in Ω, where n and Ω are as in (2.1), and $1 < p \le q$.

First we prepare two preliminary lemmas.

Lemma 3.1 *Let $n = 2m+3$ with m a positive integer and let $1 < p \le q$. Assume that conditions*

$$\Gamma = \Gamma(p,q) > 0, \tag{3.2}$$

$$q^* < q, \quad i.e., \quad q < \frac{n+1}{n-3} = 1 + \frac{2}{m} \tag{3.3}$$

hold, where Γ and q^* are defined by (1.3) *and* (1.2).

(A) *If $m \ge 3$, then we have*

$$p^* > 0, \quad i.e., \quad p > \frac{n+1}{n-1} = \frac{m+2}{m+1}. \tag{3.4}$$

(B) *If $m = 1$ or $m = 2$, then we have*

$$p^* > -\frac{1}{2}. \tag{3.5}$$

If we assume in addition that

$$q^* \le 1, \quad i.e., \quad q \le \frac{n+3}{n-1} = 1 + \frac{2}{m+1}, \tag{3.6}$$

then (3.4) *holds.*

Proof First we prove (A). Suppose contrary that

$$p^* \le 0. \tag{3.7}$$

It follows from (1.3) that

$$p^* = \frac{\Gamma - pq^* + 1 + p}{pq}.$$

Therefore, (3.7), (3.2) and (3.3) give $p > \frac{m}{2}$, which is inconsistent with (3.7) if $m \geq 3$. Thus (A) holds.

Next we prove the first part of (B). Suppose contrary that

$$p^* + \frac{1}{2} \leq 0 \text{ and either } m = 1 \text{ or } m = 2. \tag{3.8}$$

It follows from (1.3) that

$$p^* + \frac{1}{2} = \frac{2\Gamma - 2pq^* + 2 + 2p + pq}{2pq}.$$

Therefore, $p^* + \frac{1}{2} \leq 0$, (3.2) and (3.3) give

$$\Big(\frac{2}{m} - 1\Big)p > 2. \tag{3.9}$$

When $m = 2$, we have a contradiction immediately. While, if $m = 1$, then (3.9) yields $p > 2$. This is contradictory to $p^* + \frac{1}{2} \leq 0$ when $m = 1$.

The second part of (B) can be shown similarly to the proof of (A). This completes the proof.

Lemma 3.2 *Let $1 < p \leq q$. Assume that* (3.2) *is satisfied.*

(A) *Suppose that $p^* > 0$. Then there exist positive numbers κ_1, κ_2 satisfying the following four conditions:*

$$0 < \kappa_1 \leq p^*, \tag{3.10}$$

$$0 < \kappa_2 \leq q^*, \tag{3.11}$$

$$1 + \kappa_1 < p^* + p\kappa_2, \tag{3.12}$$

$$1 + \kappa_2 < q^* + q\kappa_1. \tag{3.13}$$

(B) *Suppose that $-\frac{1}{2} < p^* \leq 0$. Then there exist a negative number κ_1 and a positive number κ_2 satisfying* (3.11), (3.12), (3.13) *and the following condition:*

$$-\frac{1}{2} < \kappa_1 \leq p^* \quad \textit{and} \quad \kappa_1 < 0. \tag{3.14}$$

Proof Since (A) follows from [14, Lemma 3.1], we shall prove (B). Note that (3.2) yields

$$p^* > \frac{1}{q}\Big(-q^* + 1 + \frac{1}{p}\Big).$$

Since $-\frac{1}{2} < p^* \leq 0$, we can therefore take κ_1 satisfying (3.14) and

$$\kappa_1 > \frac{1}{q}\Big(-q^* + 1 + \frac{1}{p}\Big), \quad \text{i.e.,} \quad q^* + q\kappa_1 - 1 > \frac{1}{p}.$$

Hence, for such κ_1, there is κ_2 such that

$$q^* + q\kappa_1 - 1 > \kappa_2 > \frac{1}{p}.$$

In conclusion we find that (3.11), (3.12) and (3.13) hold for κ_1, κ_2 chosen in the above. The proof is complete.

In what follows we shall fix a pair of numbers κ_1 and κ_2 satisfying conditions (3.11) through (3.13) and either (3.10) or (3.14). Then we introduce two Banach spaces X_1 and X_2 by

$$X_j = \{u(r,t) \in C^1(\Omega);\ \|u\|_j < \infty\}, \quad j = 1, 2. \tag{3.15}$$

Here the norm is defined by

$$\|u\|_j = \sup_{(r,t)\in\Omega} (|u(r,t)|r^{m-1}(1+r) + |\partial_{r,t}u(r,t)|r^m)\{\Phi_1(r,|t|;1+\kappa_j,c_j)\}^{-1}, \tag{3.16}$$

where $\Phi_1(r,t;\mu,c)$ is given by (2.28).

Let $(f_j, g_j) \in Y_{1+\kappa_j}(\varepsilon)$ for $\varepsilon > 0$, $j = 1, 2$, where $Y_\mu(\varepsilon)$ is defined by (2.10). If we set

$$u^-(r,t) = K_{c_1}[f_1,g_1](r,t), \quad v^-(r,t) = K_{c_2}[f_2,g_2](r,t), \tag{3.17}$$

then we find from Theorem 2.1 that

$$u^- \in X_1 \cap C^2(\Omega), \quad v^- \in X_2 \cap C^2(\Omega), \tag{3.18}$$

$$\|u^-\|_1 \le C_0\varepsilon, \quad \|v^-\|_2 \le C_0\varepsilon, \tag{3.19}$$

where C_0 is a positive constant depending only on m, c_1, c_2, κ_1 and κ_2, because

$$(1+r)^{-1}w_c(r,|t|)^{-\mu} \le C\Phi_1(r,|t|;\mu,c).$$

We are now in a position to state the main results in this section.

Theorem 3.1 *Assume that conditions* $1 < p \le q$, (3.2) *and* (3.3) *are satisfied. Besides, when* $m = 1$ *or* $m = 2$, *we suppose that* (3.4) *holds. Let* κ_1 *and* κ_2 *be positive numbers satisfying* (3.10) *through* (3.13). *Suppose that* $(f_j, g_j) \in Y_{1+\kappa_j}(\varepsilon)$ *for* $\varepsilon > 0$, $j = 1, 2$. *Then there is a positive constant* ε_0 *(depending only on* $m, c_1, c_2, p, q, \kappa_1$ *and* κ_2*) such that for any* ε *with* $0 < \varepsilon \le \varepsilon_0$, *there exists uniquely a solution* (u, v) *of the system* (3.1) *satisfying*

$$u \in X_1 \cap C^2(\Omega), \quad v \in X_2 \cap C^2(\Omega), \tag{3.20}$$

$$\|u\|_1 + \|v\|_2 \le 2(\|u^-\|_1 + \|v^-\|_2), \tag{3.21}$$

$$E(u(t) - u^-(t); c_1) \le C\|v\|_2^p(1+|t|)^{-\theta_1} \quad \text{for } t \le 0, \tag{3.22}$$

$$E(v(t) - v^-(t); c_2) \le C\|u\|_1^q(1+|t|)^{-\theta_2} \quad \text{for } t \le 0, \tag{3.23}$$

where

$$\theta_1 = \min\Big\{p^*, p + p\kappa_2 - 1, \frac{1}{2} + \kappa_1\Big\}, \quad \theta_2 = \min\Big\{q^*, q + q\kappa_1 - 1, \frac{1}{2} + \kappa_2\Big\},$$

C *is a constant depending only on* $m, c_1, c_2, p, q, \kappa_1$ *and* κ_2, *and we have set*

$$E(w(t); c) = \Big\{\frac{1}{2}\int_{\mathbb{R}^n} (|\partial_t w(|x|,t)|^2 + c^2|\partial_x w(|x|,t)|^2)dx\Big\}^{\frac{1}{2}} \tag{3.24}$$

for a function $w(r,t) \in C^1(\Omega)$. *Moreover we have for* $(r,t) \in \Omega$,

$$|u(r,t) - u^-(r,t)| \le C\|v\|_2^p r^{1-m}(1+r)^{-1}\Phi_1(r,t;1+\kappa_1,c_1), \tag{3.25}$$

$$|\partial_{r,t}^\sigma(u(r,t) - u^-(r,t))| \le C\|v\|_2^p r^{1-m-|\sigma|}(1+r)^{|\sigma|-2}\Phi_2(r,t;1+\kappa_1,c_1) \quad \textit{for } |\sigma| = 1,2, \tag{3.26}$$

$$|v(r,t) - v^-(r,t)| \le C\|u\|_1^q r^{1-m}(1+r)^{-1}\Phi_1(r,t;1+\kappa_2,c_2), \tag{3.27}$$

$$|\partial_{r,t}^\sigma(v(r,t) - v^-(r,t))| \le C\|u\|_1^q r^{1-m-|\sigma|}(1+r)^{|\sigma|-2}\Phi_2(r,t;1+\kappa_2,c_2) \quad \textit{for } |\sigma| = 1,2, \tag{3.28}$$

where $\Phi_1(r,t;\mu,c)$ *and* $\Phi_2(r,t;\mu,c)$ *are given by* (2.28) *and* (2.29).

Furthermore there exist uniquely solutions $u^+(r,t) \in X_1 \cap C^2(\Omega)$ *and* $v^+(r,t) \in X_2 \cap C^2(\Omega)$ *of the homogeneous wave equation* (2.1) *with* $c = c_1$ *and* $c = c_2$ *respectively such that*

$$E(u(t) - u^+(t);c_1) \le C\|v\|_2^p(1+t)^{-\theta_1} \quad \textit{for } t \ge 0, \tag{3.29}$$

$$E(v(t) - v^+(t);c_2) \le C\|u\|_1^q(1+t)^{-\theta_2} \quad \textit{for } t \ge 0. \tag{3.30}$$

In addition, we have for $(r,t) \in \Omega$,

$$|u(r,t) - u^+(r,t)| \le C\|v\|_2^p r^{1-m}(1+r)^{-1}\Phi_1(r,-t;1+\kappa_1,c_1), \tag{3.31}$$

$$|\partial_{r,t}^\sigma(u(r,t) - u^+(r,t))| \le C\|v\|_2^p r^{1-m-|\sigma|}(1+r)^{|\sigma|-2}\Phi_2(r,-t;1+\kappa_1,c_1) \quad \textit{for } |\sigma| = 1,2, \tag{3.32}$$

$$|v(r,t) - v^+(r,t)| \le C\|u\|_1^q r^{1-m}(1+r)^{-1}\Phi_1(r,-t;1+\kappa_2,c_2), \tag{3.33}$$

$$|\partial_{r,t}^\sigma(v(r,t) - v^+(r,t))| \le C\|u\|_1^q r^{1-m-|\sigma|}(1+r)^{|\sigma|-2}\Phi_2(r,-t;1+\kappa_2,c_2) \quad \textit{for } |\sigma| = 1,2. \tag{3.34}$$

Remark 3.1 (1) The existence of positive numbers κ_1 and κ_2 satisfying (3.10) through (3.13) is guaranteed by Lemmas 3.1 and 3.2.

(2) The extra assumption (3.4) for $m = 1$ and $m = 2$ can be removed when the propagation speeds c_1 and c_2 are different from each other. More precisely we have the following

Theorem 3.2 *Let* $m = 1$ *or* $m = 2$, *i.e,* $n = 5$ *or* $n = 7$. *Assume that conditions* $1 < p \le q$, (3.2), (3.3) *and* (3.7) *hold. Moreover suppose that* $c_1 \ne c_2$. *Let* κ_1 *and* κ_2 *be real numbers satisfying* (3.11) *through* (3.14). *Suppose that* $(f_j, g_j) \in Y_{1+\kappa_j}(\varepsilon)$ *for* $\varepsilon > 0$, $j = 1,2$. *Then the conclusions of the preceding theorem are still valid, if we replace* θ_1 *and* θ_2 *by* θ_3 *and* θ_4 *respectively, where*

$$\theta_3 = \frac{1}{2} + \kappa_1, \quad \theta_4 = \min\Big\{q + q\kappa_1 - 1, \frac{1}{2} + \kappa_2\Big\}. \tag{3.35}$$

Remark 3.2 Since (3.3) and (3.13) imply $q + q\kappa_1 - 1 > \kappa_2$, we see that θ_4 is positive.

The rest of the present section will be devoted to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1 First we shall look for a solution $(u,v) \in X_1 \times X_2$ of the following system of integral equations:

$$u(r,t) = u^-(r,t) + L_{c_1}(|v|^p)(r,t) \quad \text{in } \Omega, \tag{3.36}$$

$$v(r,t) = v^-(r,t) + L_{c_2}(|u|^q)(r,t) \quad \text{in } \Omega, \tag{3.37}$$

where $L_c(F)$ is the linear operator defined by (2.19). The a priori estimates which will be given in Lemmas 3.3 and 3.4 below are crucial in solving the system.

Lemma 3.3 *Suppose that p, q, κ_1 and κ_2 satisfy the hypotheses of Theorem 3.1. Let $u \in X_1$ and $v \in X_2$. Then $L_{c_1}(|v|^p) \in X_1 \cap C^2(\Omega)$, $L_{c_2}(|u|^q) \in X_2 \cap C^2(\Omega)$, and we have*

$$\|L_{c_1}(|v|^p)\|_1 \leq K_0 \|v\|_2^p, \tag{3.38}$$

$$\|L_{c_2}(|u|^q)\|_2 \leq K_0 \|u\|_1^q, \tag{3.39}$$

where K_0 is a constant depending only on m, c_1, c_2, p, q, κ_1 and κ_2.

Proof We shall prove only the assertions concerning $L_{c_1}(|v|^p)$, since the others can be handled analogously. To this end we want to apply the part (A) of Theorem 2.2 by taking $F(r,t) = |v(r,t)|^p$, $c = c_1$, $a = c_2$, $\mu = 1 + \kappa_1$, $\alpha = (m-1)p$, $\beta = \gamma = p$ and $\delta = p\kappa_2$.

First we examine conditions (2.20) through (2.23). Since (3.3) with $p \leq q$ implies

$$(m-1)p < m+1, \quad (m-1)q < m+1, \tag{3.40}$$

the condition (2.20) is satisfied. Moreover, (1.2) with $n = 2m+3$ yields

$$(m+1)p = p^* + m + 2, \quad (m+1)q = q^* + m + 2. \tag{3.41}$$

Hence (2.22) and (2.23) follows from (3.10) and (3.12) respectively. Furthermore we get (2.21) by (3.12), since $p \geq p^*$ according to (3.3) with $p \leq q$. Thus all hypotheses of Theorem 2.2 are fulfilled.

For $v \in X_2$ we have $|v(r,t)|^p \in C^1(\Omega)$ and

$$M(|v|^p, c_2) \leq (p+1)\|v\|_2^p, \tag{3.42}$$

where $M(F,a)$ is defined by (2.25). Indeed, since $\kappa_2 > 0$, we have from (2.28)

$$\Phi_1(r,t; 1+\kappa_2, c_2) = (1 + r + |t|)^{-1}(1 + |r - c_2 t|)^{-\kappa_2}. \tag{3.43}$$

Hence (3.16) with $j = 2$ implies (3.42).

Since $\kappa_1 > 0$, we see from the part (A) of Theorem 2.2 that $L_{c_1}(|v|^p) \in C^2(\Omega)$ and that

$$|L_{c_1}(|v|^p)(r,t)| \leq CM(|v|^p, c_2) r^{1-m}(1+r)^{-1}\Phi_1(r,t; 1+\kappa_1, c_1),$$
$$|\partial_{r,t} L_{c_1}(|v|^p)(r,t)| \leq CM(|v|^p, c_2) r^{-m}(1+r)^{-1}\Phi_2(r,t; 1+\kappa_1, c_1)$$

hold for $(r,t) \in \Omega$. Since $\Phi_1(r,t;\mu,c) \leq \Phi_1(r,|t|;\mu,c)$ and

$$(1+r)^{-1}\Phi_2(r,t;\mu,c) \leq C\Phi_1(r,t;\mu,c) \quad \text{for } (r,t) \in \Omega,\ \mu > 0, \tag{3.44}$$

we obtain (3.38) by (3.42) and (3.16) with $j = 1$. The proof is complete.

To show the existence of solutions to the system (3.36)-(3.37), we also need a Lipschitz continuity of $L_{c_1}(|\cdot|^p)$ and $L_{c_2}(|\cdot|^q)$. To state this we introduce auxiliary norms by

$$|u|_j = \sup_{(r,t)\in\Omega} |u(r,t)| r^m \{\Phi_1(r,|t|; 1+\kappa_j, c_j)\}^{-1} \quad \text{for } u \in X_j,\ j = 1,2. \tag{3.45}$$

Then we have the following

Lemma 3.4 *Let p, q, κ_1 and κ_2 be as in the preceding lemma. Let u, $\bar{u} \in X_1$ and v, $\bar{v} \in X_2$. Then we have*

$$|L_{c_1}(|v|^p) - L_{c_1}(|\bar{v}|^p)|_1 \le K_1 |v-\bar{v}|_2 (\|v\|_2 + \|\bar{v}\|_2)^{p-1}, \tag{3.46}$$

$$|L_{c_2}(|u|^q) - L_{c_2}(|\bar{u}|^q)|_2 \le K_1 |u-\bar{u}|_1 (\|u\|_1 + \|\bar{u}\|_1)^{q-1}, \tag{3.47}$$

$$\|L_{c_1}(|v|^p) - L_{c_1}(|\bar{v}|^p)\|_1 \le K_2 \|v-\bar{v}\|_2 (\|v\|_2 + \|\bar{v}|_2)^{p-1} + K_3 |v-\bar{v}|_2^{p-1} (\|v\|_2 + \|\bar{v}\|_2), \tag{3.48}$$

$$\|L_{c_2}(|u|^q) - L_{c_2}(|\bar{u}|^q)\|_2 \le K_2 \|u-\bar{u}\|_1 (\|u\|_1 + \|\bar{u}\|_1)^{q-1} + K_4 |u-\bar{u}|_1^{q-1} (\|u\|_1 + \|\bar{u}\|_1), \tag{3.49}$$

where K_j $(j = 1, 2, 3, 4)$ are constants depending only on m, c_1, c_2, p, q, κ_1 and κ_2 such that $K_3 = 0$ if $p > 2$, and $K_4 = 0$ if $q > 2$.

Proof We shall prove only (3.46) and (3.48), since the others can be treated analogously.

First we prove (3.46). Let $\alpha = (m-1)p$, $\beta = \gamma = p$ and $\delta = p\kappa_2$. We see from the proof of the preceding lemma that such α, β, γ and δ satisfy conditions (2.20) through (2.23). For convenience we set

$$F(r,t) = |v(r,t)|^p - |\bar{v}(r,t)|^p, \tag{3.50}$$

so that

$$L_{c_1}(|v|^p)(r,t) - L_{c_1}(|\bar{v}|^p)(r,t) = L_{c_1}(F)(r,t). \tag{3.51}$$

Besides, we have

$$\widetilde{M}(F, c_2) \le p|v-\bar{v}|_2 (\|v\|_2 + \|\bar{v}\|_2)^{p-1}, \tag{3.52}$$

where $\widetilde{M}(F,a)$ is defined by (2.96). Indeed, we have

$$|F(\lambda,s)| \le p|v(\lambda,s) - \bar{v}(\lambda,s)|(|v(\lambda,s)| + |\bar{v}(\lambda,s)|)^{p-1} \quad \text{for } (\lambda,s) \in \Omega. \tag{3.53}$$

Therefore it is easy to see from (3.16), (3.45) with $j = 2$ and (3.43) that (3.52) holds.

Applying the part (A) of Theorem 2.3 as $c = c_1$, $a = c_2$, $\mu = 1 + \kappa_1$ to (3.51), we get (3.46) by (3.52), (3.45) with $j = 1$.

Next we consider (3.48). The procedure is similar to the proof of (3.38). We let $p \le 2$, since one can more easily prove the estimate for $p > 2$. We keep the notation (3.50). Let $\alpha = mp-1$, $\beta = 1$, $\gamma = p$ and $\delta = p\kappa_2$. Then (3.3) with $p \le q$ implies (2.20) for $\alpha = mp-1$. In addition, we see from the proof of the preceding lemma that such α, β, γ and δ satisfy conditions (2.21) through (2.23). Therefore it follows from the part (A) of Theorem 2.2 with $c = c_1$, $a = c_2$, $\mu = 1 + \kappa_1$, (3.44) and (3.16) with $j = 1$ that

$$\|L_{c_1}(F)\|_1 \le CM(F, c_2). \tag{3.54}$$

Thus, by (3.51), it suffices to show

$$M(F, c_2) \le 2p\{\|v-\bar{v}\|_2 (\|v\|_2 + \|\bar{v}\|_2)^{p-1} + |v-\bar{v}|_2^{p-1} (\|v\|_2 + \|\bar{v}\|_2)\}. \tag{3.55}$$

Since $1 < p \le 2$, we have from (3.50)

$$|\partial_\lambda F(\lambda,s)| \le p|\partial_\lambda(v(\lambda,s) - \bar{v}(\lambda,s))||v(\lambda,s)|^{p-1} + 2p|v(\lambda,s) - \bar{v}(\lambda,s)|^{p-1}|\partial_\lambda \bar{v}(\lambda,s)|.$$

Hence, recalling (3.53), we see from (3.16), (3.45) with $j=2$ and (3.43) that (3.55) holds for $\alpha=mp-1$, $\beta=1$, $\gamma=p$ and $\delta=p\kappa_2$. Thus we get (3.48) by (3.54). The proof is complete.

We are now in a position to solve (3.36)–(3.37).

Proposition 3.1 *Suppose that the hypotheses of Theorem* 3.1 *are fulfilled. Then there is a positive constant* ε_0 *(depending only on* $m, c_1, c_2, p, q, \kappa_1$ *and* κ_2*) such that for any* ε *with* $0<\varepsilon\le\varepsilon_0$*, there exists uniquely a solution* (u,v) *of the system* (3.36)–(3.37) *satisfying* (3.20) *and* (3.21)*. Moreover we have* (3.22) *through* (3.28) *and* (3.1)*.*

Proof Using (3.18), (3.19) and Lemmas 3.3 and 3.4, one can find a positive number ε_0, depending only on C_0, K_0, K_1 and K_2, such that there exists a unique solution (u,v) of the system (3.36)–(3.37) satisfying (3.20) and (3.21) for $0<\varepsilon\le\varepsilon_0$ (for the detail, see e.g. [13, Section 7]. Applying the part (A) of Theorem 2.2 to $L_{c_1}(|v|^p)$, we see from (3.42) that (3.25) and (3.26) hold. Analogously we have (3.27) and (3.28). Moreover we find from (2.72) that (u,v) is a solution of (3.1).

Therefore it remains to show the estimates (3.22) and (3.23). First we deal with (3.22). We see from [13, Proposition 8.1] that

$$E(u(t)-u^-(t);c_1)\le C\int_{-\infty}^t\Big(\int_0^\infty |v(r,s)|^{2p}r^{2m+2}dr\Big)^{\frac12}ds, \tag{3.56}$$

where C is a constant depending only on m and c_1. It follows from (3.16) with $j=2$, (3.43), (3.40) and (3.41) that the inner integral in the above is estimated by $\|v\|_2^{2p}$ times

$$\begin{aligned}
&\int_0^\infty (1+r)^{2p-2p^*-2}(1+r+|s|)^{-2p}(1+|r-c_2|s||)^{-2p\kappa_2}dr\\
&\le C(1+|s|)^{-2p-2p\kappa_2}\int_0^{\frac{c_2|s|}{2}}(1+r)^{2p-2p^*-2}dr\\
&\quad+C(1+|s|)^{-2p^*-2}\int_{\frac{c_2|s|}{2}}^{2c_2|s|}(1+|r-c_2|s||)^{-2p\kappa_2}dr+C\int_{2c_2|s|}^\infty(1+r)^{-2p^*-2p\kappa_2-2}dr\\
&\le C(1+|s|)^{-2p-2p\kappa_2}(1+|s|)^{[2p-2p^*-1]_+}+C(1+|s|)^{-2p^*-2}(1+|s|)^{[1-2p\kappa_2]_+}.
\end{aligned}$$

Here we have used the follwing notation:

$$[a]_+=\max\{a,0\},\quad A^{[0]_+}=1+\log A \tag{3.57}$$

for $a\in\mathbb{R}$ with $a\ne 0$ and $A\ge 1$. Therefore we see from (3.56) that (3.22) follows from the following two inequalities:

$$\begin{gathered}
\int_{-\infty}^t(1+|s|)^{-p-p\kappa_2}((1+|s|)^{[2p-2p^*-1]_+})^{\frac12}ds\le C(1+|t|)^{-\min\{p+p\kappa_2-1,\frac12+\kappa_1\}},\\
\int_{-\infty}^t(1+|s|)^{-p^*-1}((1+|s|)^{[1-2p\kappa_2]_+})^{\frac12}ds\le C(1+|t|)^{-\min\{p^*,\frac12+\kappa_1\}}
\end{gathered}$$

for $t\le 0$. By virtue of (3.12), those inequalities are the consequence of the following elementary lemma.

Lemma 3.5 *Let $\alpha > 1$ if $\beta < 0$ and $\alpha > 1 + \frac{\beta}{2}$ if $\beta \geq 0$, and let $a \geq 0$. Then we have*

$$\int_a^\infty (1+s)^{-\alpha}((1+s)^{[\beta]_+})^{\frac{1}{2}} ds \leq C(1+a)^{-\alpha+1}((1+a)^{[\beta]_+})^{\frac{1}{2}}. \tag{3.58}$$

Thus we get (3.22). Since we can prove (3.23) analogously to (3.22), we omit the details. This completes the proof.

End of Proof of Theorem 3.1 The assertions concerning (u, v) follow from Proposition 3.1 except for the uniqueness. Besides, we find from [13, the proof of Theorem 5.1] that a solution (u, v) of (3.1) satisfying (3.20) through (3.23) is unique.

Next we define $u^+(r,t)$ and $v^+(r,t)$ by

$$u^+(r,t) = u(r,t) - L_{c_1}(F)(r,-t), \quad v^+(r,t) = v(r,t) - L_{c_2}(G)(r,-t) \quad \text{for } (r,t) \in \Omega, \tag{3.59}$$

where $F(r,t) = |v(r,-t)|^p$, $G(r,t) = |u(r,-t)|^q$ and L_c is the operator defined by (2.19). Note that $v(r,-t) \in X_2$ if $v(r,t) \in X_2$ and that $M(|v(r,-t)|^p, c_2) = M(|v(r,t)|^p, c_2)$. Besides, the analogue for $u(r,t)$ is also valid. Therefore, by repeating exactly the procedure in the proof of Proposition 3.1, we obatin the assertions for (u^+, v^+) except for the uniqueness. In addition, we see from (3.29) and (3.30) that such solutions u^+ and v^+ of (2.1) are unique. Thus we finish the proof of the theorem.

Proof of Theorem 3.2 The procedure is analogous to the proof of preceding theorem. Hence we shall point out only the difference.

First we derive (3.38) and (3.39) under the hypotheses of Theorem 3.2. Since $\frac{1}{2} < 1+\kappa_1 < 1$ according to (3.14), it follows from (2.28) that

$$\Phi_1(r,t; 1+\kappa_1, c_1) = (1+r+|t|)^{-1-\kappa_1}. \tag{3.60}$$

Thanks to the assumption that $c_1 \neq c_2$, it is possible to apply the part (A) of Theorem 2.2 to $L_{c_1}(|v|^p)$. Since (3.43) is still valid, we see that (3.38) can be shown as before.

To prove (3.39), we want to apply the part (A) of Theorem 2.2 to $L_{c_2}(|u|^q)$, by taking $a = c_1$, $\mu = 1+\kappa_2 > 1$, $\alpha = (m-1)q$, $\beta = q$, $\gamma = q + q\kappa_1$ and $\delta = 0$. Using (3.40), (3.3), (3.41) and (3.13), one can show that (2.20) through (2.23) are fulfilled. Moreover, it follows from (3.60) and (3.16) with $j = 1$ that

$$M(|u|^q, c_1) \leq (1+q)\|u\|_1^q.$$

Therefore, by (3.44), we obtain (3.39).

Next we prove (3.46) through (3.49) under the hypotheses of Theorem 3.2. We have (3.46) and (3.48) as before. Applying the part (A) of Theorem 2.3 with the same numbers α, β, γ and δ as above, we get (3.47). The proof of (3.49) is similar to that of (3.39), if we take $\alpha = mq-1$, $\beta = 1$, $\gamma = q + q\kappa_1$ and $\delta = 0$.

Finally we show that (3.22) and (3.23) with θ_1 and θ_2 replaced by θ_3 and θ_4 respectively. Fisrt we deal with the former. Since $\frac{1}{2} < 1+\kappa_1 < 1$, it follows from (2.29) that

$$\Phi_2(r,t; 1+\kappa_1, c_1) = (1+r+c_1|t|)^{-1-\kappa_1} \quad \text{for } t \leq 0. \tag{3.61}$$

Therefore we see from (3.24) and (3.26) with $|\sigma| = 1$ that

$$E(u(t) - u^-(t); c_1) \le C\|v\|_2^p \Big(\int_0^\infty r^2(1+r)^{-2}(1+r+|t|)^{-2-2\kappa_1} dr \Big)^{\frac{1}{2}} \quad \text{for } t \le 0.$$

Since $-2 - 2\kappa_1 < -1$ according to (3.14), we get the desired estimate due to (3.35).

Next we handle the latter. Analogously to (3.56) we have

$$E(v(t) - v^-(t); c_2) \le C \int_{-\infty}^t \Big(\int_0^\infty |u(r,s)|^{2q} r^{2m+2} dr \Big)^{\frac{1}{2}} ds, \tag{3.62}$$

where C is a constant depending only on m and c_2. It follows from (3.16) with $j = 1$, (3.60), (3.40) and (3.41) that the inner integral in the above is estimated by $\|u\|_1^{2q}$ times

$$\begin{aligned} &\int_0^\infty (1+r)^{2q-2q^*-2}(1+r+|s|)^{-2q(1+\kappa_1)} dr \\ &\le C(1+|s|)^{-2q(1+\kappa_1)} \int_0^{|s|} (1+r)^{2q-2q^*-2} dr + C \int_{|s|}^\infty (1+r+|s|)^{-2q^*-2q\kappa_1-2} dr \\ &\le C(1+|s|)^{-2q(1+\kappa_1)}(1+|s|)^{[2q-2q^*-1]_+}, \end{aligned}$$

since $-q^* - q\kappa_1 < -1$ according to (3.13). Here we have used the notation (3.57). By (3.13) and (3.3), we can apply Lemma 3.5. Hence (3.62) yields

$$E(v(t) - v^-(t); c_2) \le C\|u\|_1^q (1+|t|)^{-q(1+\kappa_1)+1}((1+|t|)^{[2q-2q^*-1]_+})^{\frac{1}{2}} \quad \text{for } t \le 0. \tag{3.63}$$

In view of (3.13) and (3.35), we see that (3.23) with θ_2 replaced by θ_4 holds. This completes the proof of the theorem.

4 Initial Value Problems

This section is concerned with the initial value problems in $\Omega_+ = \{(r,t) \in \Omega; t \ge 0\}$ for the system (3.1) with initial conditions

$$\begin{cases} u(r,0) = f_1(r), \quad u_t(r,0) = g_1(r), \\ v(r,0) = f_2(r), \quad v_t(r,0) = g_2(r) \end{cases} \tag{4.1}$$

for $r > 0$. Here $(f_j, g_j) \in Y_{1+\kappa_j}(\varepsilon)$, $Y_\mu(\varepsilon)$ is defined by (2.10), and κ_j will be specified later.

To state the main results in the present section, we shall modify the Banach spaces X_1 and X_2 defined by (3.15) as follows. We set

$$X_j^+ = \{u(r,t) \in C^1(\Omega_+) \; ; \; \|u\|_j^+ < \infty\}, \quad j = 1, 2,$$

where

$$\|u\|_j^+ = \sup_{(r,t)\in\Omega_+} (|u(r,t)|r^{m-1}(1+r) + |\partial_{r,t} u(r,t)|r^m)\{\Phi_1(r,t;1+\kappa_j,c_j)\}^{-1}. \tag{4.2}$$

First we consider the case where (3.4) holds. Analogously to Theorem 3.1 we have the following

Theorem 4.1 *Let the hypotheses of Theorem* 3.1 *be fulfilled. Then there is a positive constant* ε_0 *such that for any* ε *with* $0 < \varepsilon \le \varepsilon_0$*, there exists uniquely a solution* (u, v) *of the Cauchy problem* (3.1)-(4.1) *in* Ω_+ *satisfying*

$$u \in X_1^+ \cap C^2(\Omega_+), \quad v \in X_2^+ \cap C^2(\Omega_+), \tag{4.3}$$

$$\|u\|_1^+ + \|v\|_2^+ \le 2(\|u^-\|_1^+ + \|v^-\|_2^+), \tag{4.4}$$

where u^- *and* v^- *are defined by* (3.17).

Remark 4.1 For the solutions $u(r, t)$ and $v(r, t)$ obtained in the above theorem, we can show the existence of free profiles of them. Indeed, if we define $u^+(r, t)$ and $v^+(r, t)$ as in (3.59) for $(r, t) \in \Omega_+$, then we see that $u^+ \in X_1^+ \cap C^2(\Omega_+)$, $v^+ \in X_2^+ \cap C^2(\Omega_+)$ and that u^+ (resp. v^+) is the solution of the homogeneous wave equation (2.1) with $c = c_1$ (resp. $c = c_2$). Moreover, they satisfy (3.29) through (3.34) with $\|u\|_1$, $\|v\|_2$ and Ω replaced by $\|u\|_1^+$, $\|v\|_2^+$ and Ω_+ respectively. These assertions can be also proven analogously to the proof of Theorem 3.1.

Next we consider the case where (3.4) does not hold. Then we have the following

Theorem 4.2 *Let* $m = 1$ *or* $m = 2$*, i.e,* $n = 5$ *or* $n = 7$*. Assume that conditions* $1 < p \le q$*,* (3.2)*,* (3.3) *and* (3.7) *hold. Let* κ_1 *and* κ_2 *be real numbers satisfying* (3.11) *through* (3.14)*. Suppose that* $(f_j, g_j) \in Y_{1+\kappa_j}(\varepsilon)$ *for* $\varepsilon > 0$*,* $j = 1, 2$*. Then the conclusions of the preceding theorem are still valid.*

Proof The following lemma is essential for the proof of the theorem.

Lemma 4.1 *Suppose that* p*,* q*,* κ_1 *and* κ_2 *satisfy the hypotheses of Theorem* 4.2*. Let* $u \in X_1^+$ *and* $v \in X_2^+$*. Then we have*

$$\|L_{c_1}^+(|v|^p)\|_1^+ \le K_0(\|v\|_2^+)^p, \quad \|L_{c_2}^+(|u|^q)\|_2^+ \le K_0(\|u\|_1^+)^q, \tag{4.5}$$

where K_0 *is a constant depending only on* m*,* c_1*,* c_2*,* p*,* q*,* κ_1 *and* κ_2*, and* L_c^+ *is the operator defined by* (2.18).

Proof Note that $\Phi_3(r, t; \mu, c) = \Phi_2(r, t; \mu, c)$ for $\mu > 1$ and $t \ge 0$. In view of Theorem 2.2 and the proof of Theorem 3.2, one can show the second estimate of (4.5).

Next we consider the first one. When $c_1 \ne c_2$, one can prove it analogously to (3.38). Hence we suppose from now on that $c_1 = c_2$.

By (3.14) we have (3.60) for $(r, t) \in \Omega_+$, hence it suffices to show that

$$|L_{c_1}^+(|v|^p)(r, t)| r^{m-1}(1 + r)(1 + r + t)^{1+\kappa_1} \le C(\|v\|_2^+)^p, \tag{4.6}$$

$$|\partial_{r,t} L_{c_1}^+(|v|^p)(r, t)| r^m (1 + r + t)^{1+\kappa_1} \le C(\|v\|_2^+)^p \tag{4.7}$$

for $(r, t) \in \Omega_+$. Applying the part (B) of Theorem 2.2 with the same choice of the parameters as in the proof of (3.38), we get (4.6) and

$$|\partial_{r,t} L_{c_1}^+(|v|^p)(r, t)| \le C(\|v\|_2^+)^p r^{-m}(1 + r)^{-1}\Phi_3(r, t; 1 + \kappa_1, c_1) \quad \text{for } (r, t) \in \Omega_+, \tag{4.8}$$

because analogously to (3.42) we have

$$M^+(|v|^p, c_2) \le (p+1)(\|v\|_2^+)^p.$$

Note that (3.7) implies (2.93), for $\alpha+\beta+\gamma=(m+1)p$. Since $\frac{1}{2} < 1+\kappa_1 < 1$ and $c_1 = c_2$, we have from (2.32)

$$\begin{aligned}(1+r)^{-1}\Phi_3(r,t;1+\kappa_1,c_1) &= (1+r)^{-1}(1+|r-c_1t|)^{-1}(1+r+c_1t)^{-\kappa_1}\\ &\le C(1+r+t)^{-1-\kappa_1} \quad \text{for } (r,t)\in\Omega_+. \end{aligned} \tag{4.9}$$

Thus we obtain (4.7). The proof is complete.

End of Proof of Theorem 4.2 Seeing the proof of Lemma 4.1, one can establish a priori estimates analogous to Lemma 3.4. Repeating a part of the proof of Theorem 3.1, we complete the proof.

5 Three Space Dimensional Case

In this section we deal with the following initial value problem:

$$\begin{cases}\partial_t^2 u_1 - c_1^2\Delta u_1 = F(u_2) & \text{in } \mathbb{R}^3\times(0,\infty),\\ \partial_t^2 u_2 - c_2^2\Delta u_2 = G(u_1) & \text{in } \mathbb{R}^3\times(0,\infty),\end{cases} \tag{5.1}$$

$$u_j(x,0) = f_j(x), \quad \partial_t u_j(x,0) = g_j(x) \quad \text{for } x\in\mathbb{R}^3,\ j=1,2, \tag{5.2}$$

where c_1, c_2, F and G are as in (1.1). The aim here is to show the global existence of a small solution to the Cauchy problem by assuming only the condition (3.2), i.e., $\Gamma > 0$, where Γ is defined by (1.3) with $p^* = p-2$ and $q^* = q-2$. This condition is optimal for the global existence, since if $\Gamma \le 0$, then solutions of the Cauchy problem generically blow up in finite time even though the initial data are small (see [1, 3–6, 15] for the case $c_1 = c_2$ and [16] for the case $c_1 \ne c_2$).

Since one can not expect that $u_j(x,t)\in C^2(\mathbb{R}^3\times[0,\infty))$ in the case $p^*\le 0$, we say in what follows that (u_1,u_2) is a solution of the Cauchy problem (5.1)–(5.2), if $u_j(x,t)\in C^1(\mathbb{R}^3\times[0,\infty))$ for $j=1,2$, (5.2) holds, and (u_1,u_2) satisfies (5.1) in the sense of distributions on $\mathbb{R}^3\times(0,\infty)$. (See e.g. [12, Lemma 5.1]). As for the initial data, we suppose that $f_j\in C^3(\mathbb{R}^3)$ and $g_j\in C^2(\mathbb{R}^3)$ for $j=1,2$. Then we have the following

Theorem 5.1 *Assume that the condition* (3.2) *as well as* $1<p\le q$ *holds. Then there exists a small solution* (u_1,u_2) *of the Cauchy problem* (5.1)–(5.2)*, provided the initial data are sufficiently small and decay rapidly as* $|x|\to\infty$*.*

The theorem was proven in [3, 4] when $c_1 = c_2$ and the initial data are of compact support. As for the case of general speeds of propagation c_1, c_2, we proved it in [14] when $p^*>0$, i.e., (3.4) holds. More precisely, we showed in [14, Theorem 3.1] the following: Let κ_1 and κ_2 be positive numbers satisfying (3.10) through (3.13). Suppose that the initial data f_j and g_j,

$j = 1, 2$ satisfy the following condition

$$\sup_{x\in\mathbb{R}^3}\Big[(1+|x|)^{1+\kappa_j}|f_j(x)| + \sum_{1\le|\alpha|\le 3}(1+|x|)^{2+\kappa_j}|\partial_x^\alpha f_j(x)| \\ + \sum_{0\le|\alpha|\le 2}(1+|x|)^{2+\kappa_j}|\partial_x^\alpha g_j(x)|\Big] \le \varepsilon \quad \text{for } \varepsilon > 0,\ j = 1, 2. \tag{5.3}$$

Then there is a positive constant ε_0 such that for any ε with $0 < \varepsilon \le \varepsilon_0$, there exists uniquely a solution (u_1, u_2) of (5.1)–(5.2) satisfying $u_j(x,t) \in C^2(\mathbb{R}^3 \times [0,\infty)) \cap Y_j$ for $j = 1, 2$, and $[u_1]_1 + [u_2]_2 \le 2([u_1^-]_1 + [u_2^-]_2)$, where Y_j, $[u]_j$ and u_j^- are defined for $j = 1, 2$ as follows:

$$[u]_j = \sup_{(x,t)\in\mathbb{R}^3\times[0,\infty)} |u(x,t)|\{\Phi_1(|x|, t; 1+\kappa_j, c_j)\}^{-1}, \tag{5.4}$$

where $\Phi_1(r, t; \mu, c)$ is given by (2.28),

$$Y_j = \{u(x,t) \in C^1(\mathbb{R}^3 \times [0,\infty));\ [\partial_{x,t}^\alpha u]_j < \infty \text{ for } |\alpha| \le 1\}. \tag{5.5}$$

Besides, we set

$$u_j^-(x,t) = K_{c_j}[f_j, g_j](x,t) \quad \text{for } (x,t) \in \mathbb{R}^3 \times [0,\infty), \tag{5.6}$$

where

$$K_c[f,g](x,t) = \frac{t}{4\pi}\int_{|\omega|=1} g(x+ct\omega)dS_\omega + \frac{\partial}{\partial t}\Big(\frac{t}{4\pi}\int_{|\omega|=1} f(x+ct\omega)dS_\omega\Big). \tag{5.7}$$

Note that (5.7) coincides with (2.3) if $f(x)$ and $g(x)$ are radially symmetric, since $K(\lambda, r, t) = \frac{\lambda}{2r}$ for $m = 0$. (For the proof see e.g. [18, Lemma 1] or [14, Lemma 2.1]). Thus it remains to handle the case where $p^* \le 0$, i.e., (3.7) holds.

Proof of Theorem 5.1 In what follows we suppose that conditions (3.2) and (3.7) as well as $1 < p \le q$ hold. Then, since $-1 < p^* \le 0$, one can take a negative number κ_1 and a positive number κ_2 satisfying (3.11) through (3.13) together with

$$-1 < \kappa_1 \le p^* \quad \text{and} \quad \kappa_1 < 0, \tag{5.8}$$

analogously to the proof of the part (B) in Lemma 3.2. We also keep using the notations (5.4) and (5.5) for such κ_1 and κ_2. Note that the functions given by (5.6) are classical solutions of the homogeneous wave equations (1.4) in $\mathbb{R}^3 \times [0,\infty)$ satisfying the initial conditions (5.2). Moreover analogously to [18, Lemma 2] we see from (5.3) that

$$[\partial_{x,t}^\alpha u_j^-]_j \le C_0\varepsilon \quad \text{for } \varepsilon > 0,\ |\alpha| \le 2,\ j = 1, 2, \tag{5.9}$$

where C_0 is a constant depending only on c_j and κ_j.

To show the existence of solutions to the Cauchy probelm (5.1)–(5.2), we shall look for a solution $(u_1, u_2) \in Y_1 \times Y_2$ to the following system of integral equations:

$$\begin{cases} u_1(x,t) = u_1^-(x,t) + L_{c_1}^+(F(u_2))(x,t) & \text{in } \mathbb{R}^3 \times [0,\infty), \\ u_2(x,t) = u_2^-(x,t) + L_{c_2}^+(G(u_1))(x,t) & \text{in } \mathbb{R}^3 \times [0,\infty), \end{cases} \tag{5.10}$$

where we have set

$$L_c^+(F)(x,t) = \frac{1}{4\pi}\int_0^t (t-s)ds \int_{|\omega|=1} F(x+c(t-s)\omega, s)dS_\omega \tag{5.11}$$

for $(x,t) \in \mathbb{R}^3 \times [0,\infty)$ and $F(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$. Then we have the following

Proposition 5.1 *Assume that conditions* $1 < p \le q$, (3.2) *and* (3.7) *are fulfilled. Let* κ_1 *and* κ_2 *be real numbers satisfying* (3.11) *through* (3.13) *together with* (5.8). *Suppose that* (5.3) *holds for* $\varepsilon > 0$, $j = 1, 2$. *Then there is a positive constant* ε_0 *such that for any* ε *with* $0 < \varepsilon \le \varepsilon_0$, *there exists uniquely a solution* (u_1, u_2) *of the system* (5.10) *satisfying*

$$u_j \in Y_j \quad \text{for } j = 1, 2, \tag{5.12}$$

$$[u_1]_1 + [u_2]_2 \le 2([u_1^-]_1 + [u_2^-]_2). \tag{5.13}$$

This proposition can be proven as usual (see for instance [13, the proof of Theorem 5.1]), if we make use of Lemma 5.1 below. Hence we omit the details.

Lemma 5.1 *Suppose that* p, q, κ_1 *and* κ_2 *satisfy the assumptions of Proposition* 5.1. *Then we have*

$$[L_{c_1}^+(|u_2|^p)]_1 \le K_0[u_2]_2^p, \quad [L_{c_2}^+(|u_1|^q)]_2 \le K_0[u_1]_1^q \tag{5.14}$$

for $u_j(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$ *with* $[u_j]_j < \infty$, $j = 1, 2$, *where* K_0 *is a constant depending only on* c_1, c_2, p, q, κ_1 *and* κ_2.

To prove this we shall extend [14, Theorem 1.1] as follows:

Proposition 5.2 *Let* μ *and* a *be positive numbers with* $\mu \ne 1$. *Let* $F(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$ *and*

$$N(F,a,\mu,\rho) := \sup_{(y,s)\in\mathbb{R}^3\times[0,\infty)} |y|(1+|y|+s)^\mu(1+||y|-as|)^{1+\rho}|F(y,s)| < \infty \tag{5.15}$$

for some $\rho > 0$. *Then we have*

$$|L_c^+(F)(x,t)| \le CN(F,a,\mu,\rho)\Phi_1(|x|,t;\mu,c) \quad \text{for } (x,t) \in \mathbb{R}^3 \times [0,\infty), \tag{5.16}$$

where C *is a constant depending only on* μ, ρ, c, a.

Proof When $\mu > 1$, the estimate (5.16) coincides with the case $n = 3$ of [14, Theorem 1.1]. The proof of the estimate for $0 < \mu < 1$ is analogous to the case $\mu > 1$, hence we omit the details.

Proof of Lemma 5.1 By (3.11) and (5.8) we have $\kappa_2 > 0$ and $-1 < \kappa_1 < 0$. Hence (5.4) yields

$$\begin{cases} |u_2(y,s)| \le [u_2]_2(1+|y|+s)^{-1}(1+||y|-c_2 s|)^{-\kappa_2}, \\ |u_1(y,s)| \le [u_1]_1(1+|y|+s)^{-1-\kappa_1} \end{cases} \tag{5.17}$$

for $(y,s) \in \mathbb{R}^3 \times [0,\infty)$. Taking

$$\rho_1 = p^* + p\kappa_2 - 1 - \kappa_1, \quad \rho_2 = q^* + q\kappa_1 - 1 - \kappa_2,$$

we see from (3.12) and (3.13) that $\rho_j > 0$ for $j = 1, 2$. Moreover, by (5.15) and (5.17) we get

$$\begin{cases} N(|u_2|^p, c_2, 1+\kappa_1, \rho_1) \le C[u_2]_2^p, \\ N(|u_1|^q, c_1, 1+\kappa_2, \rho_2) \le C[u_1]_1^q, \end{cases}$$

where C is a constant depending only on c_1, c_2, p, q, κ_1 and κ_2, because (5.8) and (3.13) imply

$$\begin{cases} (1+|y|+s)^{\kappa_1 - p^*} \le C(1+||y|-c_2 s|)^{\kappa_1 - p^*}, \\ (1+|y|+s)^{\kappa_2 - q^* - q\kappa_1} \le C(1+||y|-c_1 s|)^{\kappa_2 - q^* - q\kappa_1} \end{cases}$$

for $(y, s) \in \mathbb{R}^3 \times [0, \infty)$. Thus we obtain (5.14) from (5.16). The proof is complete.

End of Proof of Theorem 5.1 Let ε_0 and (u_1, u_2) be as in Proposition 5.1, and let $0 < \varepsilon \le \varepsilon_0$. Then we see that (u_1, u_2) is a solution of the Cauchy problem (5.1)–(5.2) by means of the following fact: If $F(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$, then $L_1^+(F)(x,t) \in C(\mathbb{R}^3 \times [0,\infty))$ and we have

$$\int_0^\infty dt \int_{\mathbb{R}^3} L_1^+(F)(x,t)(\partial_t^2 - \Delta)\phi(x,t)dx = \int_0^\infty dt \int_{\mathbb{R}^3} F(x,t)\phi(x,t)dx$$

for any $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,\infty))$. Thus we complete the proof.

Remark 5.1 One can show that a solution of the Cauchy problem (5.1)–(5.2) which satisfies (5.12) and (5.13) is unique, provided $\varepsilon > 0$ is sufficiently small. To see this, it suffices to prove that such a solution (u_1, u_2) satisfies the system (5.10) in view of Proposition 5.1. If we set

$$\begin{cases} v_1(x,t) = u_1(x,t) - u_1^-(x,t) - L_{c_1}^+(F(u_2))(x,t), \\ v_2(x,t) = u_2(x,t) - u_2^-(x,t) - L_{c_2}^+(G(u_1))(x,t) \end{cases}$$

for $(x,t) \in \mathbb{R}^3 \times [0,\infty)$, then we see from the proof of Theorem 5.1 that $v_j(x,t)$ belongs to $C^1(\mathbb{R}^3 \times [0,\infty))$ and satisfies the homogeneous wave equations (1.4) in $\mathbb{R}^3 \times (0,\infty)$. Moreover, we have $v_j(x,0) = \partial_t v_j(x,0) = 0$ for $x \in \mathbb{R}^3$, $j = 1, 2$. These informations imply $v_j(x,t) = 0$ for $(x,t) \in \mathbb{R}^3 \times [0,\infty)$ (see e.g. [12, Lemma 5.1]), namely, (u_1, u_2) satisfy (5.10).

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