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Jensen's Inequality for Backward Stochastic Differential Equations**

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Abstract Under the Lipschitz assumption and square integrable assumption on g, the author proves that Jensen's inequality holds for backward stochastic differential equations with generator g if and only if g is independent of y, $g(t,0) \equiv 0$ and g is super homogeneous with respect to z. This result generalizes the known results on Jensen's inequality for g-expectation in [4, 7–9].

Keywords Backward stochastic differential equation, g-Expectation, Jensen's inequality for g-expectation, Jensen's inequality for BSDEs
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1 Introduction

It is well known that there exists a unique adapted and square integrable solution to a Backward Stochastic Differential Equation (BSDE in short) of type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dB_s, \quad 0 \le t \le T, \tag{1.1}$$

provided that the generator g is Lipschitz in both variables y and z, and that ξ and the process $g(\,\cdot\,, 0, 0)$ are square integrable. We denote the unique adapted and square integrable solution of the BSDE (1.1) by $(Y_t(g,T,\xi), Z_t(g,T,\xi))_{t\in[0,T]}$. When g also satisfies $g(t,y,0) \equiv 0$ for any (t,y), then, $Y_0(g,T,\xi)$, denoted by $\mathcal{E}_g[\xi]$, is called g-expectation of ξ; $Y_t(g,T,\xi)$, denoted by $\mathcal{E}_g[\xi|\mathcal{F}_t]$, is called conditional g-expectation of ξ (see [1] for details).

The notion of g-expectation can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying g-expectation comes from the theory of expected utility, which is the foundation of modern mathematical economics. Chen and Epstein [2] gave an application of g-expectation to recursive utility, Rosazza [3] showed how g-expectations and conditional g-expectations provide examples of static and dynamic risk measures. Since the notion of g-expectation was introduced, many properties of g-expectation have been studied in [1, 4–10]. [6] obtained an important result, where the authors proved that if a filtration consistent (nonlinear) expectation $\mathcal{E}$ can be dominated by a kind of g-expectation, then $\mathcal{E}$ must be a g-expectation.

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Some papers, such as [4, 7–9] have been devoted to Jensen's inequality for g-expectation. Roughly speaking, the problem of Jensen's inequality for g-expectation is: For convex function $\varphi : \mathbb{R} \to \mathbb{R}$, what conditions should be given to the generator g such that the following inequality

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \geq \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)] \tag{1.2}$$

will hold in general?

[4] gave a counterexample to indicate that even for a linear function φ, Jensen's inequality for g-expectation usually does not hold. [7, 8] obtained a necessary and sufficient condition for Jensen's inequality for g-expectation under three additional assumptions that g is independent of y, is continuous respect to t and is convex in z. [9] obtained a necessary and sufficient condition for Jensen's inequality for g-expectation under two additional assumptions that g is independent of y and g is continuous respect to t. These results yield a natural question:

Without these additional assumptions on g, can we solve the problem on Jensen's inequality for g-expectation generally?

More generally, we want to investigate the following problem of Jensen's inequality for BSDEs:

For each $0 \leq t \leq r \leq T$, $\xi \in L^2(\Omega, \mathcal{F}_r, P)$ and convex function $\varphi : \mathbb{R} \to \mathbb{R}$. Suppose $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_r, P)$. Let $(Y_t(g, r, \xi), Z_t(g, r, \xi))_{t\in[0,r]}$ and $(Y_t(g, r, \varphi(\xi)), Z_t(g, r, \varphi(\xi)))_{t\in[0,r]}$ denote the solutions of the following BSDE (1.3) and BSDE (1.4) respectively:

$$y_t^1 = \xi + \int_t^r g(s, y_s^1, z_s^1)ds - \int_t^r z_s^1 \cdot dB_s, \qquad 0 \leq t \leq r, \tag{1.3}$$

$$y_t^2 = \varphi(\xi) + \int_t^r g(s, y_s^2, z_s^2)ds - \int_t^r z_s^2 \cdot dB_s, \quad 0 \leq t \leq r. \tag{1.4}$$

Then what conditions should be given to the generator g such that the following inequality

$$Y_t(g, r, \varphi(\xi)) \geq \varphi(Y_t(g, r, \xi)) \tag{1.5}$$

will hold in general?

The objective of this paper is to investigate these problems. The remainder of this paper is organized as follows: In Section 2, we introduce some notations, assumptions, definitions and lemmas which will be useful in this paper; in Section 3, we introduce the main result of this paper, we prove that Jensen's inequality holds for BSDEs with generator g if and only if g is independent of y, $g(t, 0) \equiv 0$ and g is super homogeneous in z. This result generalizes the known results on Jensen's inequality for g-expectation in [4, 7–9].

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $(B_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$; let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by this Brownian motion, i.e., $\mathcal{F}_t := \sigma\{B_s, s \in [0, t]\} \vee \mathcal{N}$, $t \in [0, T]$, where $\mathcal{N}$ is the set of all P-null subsets.

Let $T > 0$ be a given real number. In this paper, we always consider processes indexed by $t \in [0, T]$. For any positive integer n and $z \in \mathbb{R}^n$, $|z|$ denotes its Euclidean norm.

We define the following usual spaces of processes:

$$\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) := \Big\{\psi \text{ continuous and progressively measurable; } \mathbf{E}\Big[\sup_{0\le t\le T}|\psi_t|^2\Big] < \infty\Big\};$$

$$\mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^n) := \Big\{\psi \text{ progressively measurable; } ||\psi||_2^2 = \mathbf{E}\Big[\int_0^T |\psi_t|^2 dt\Big] < \infty\Big\}.$$

The generator g of a BSDE is a function

$$g : \Omega\times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

such that the process $(g(t,y,z))_{t\in[0,T]}$ is progressively measurable for each pair (y,z) in $\mathbb{R}\times\mathbb{R}^d$, and furthermore, g satisfies the following assumptions (A1) and (A2):

(A1) There exists a constant $K \geq 0$, such that $dP \times dt$-a.s., we have

$$\forall\, y_1, y_2 \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d, \quad |g(t,y_1,z_1) - g(t,y_2,z_2)| \le K(|y_1-y_2| + |z_1-z_2|).$$

(A2) The process $(g(t,0,0))_{t\in[0,T]} \in \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R})$.

(A3) $dP \times dt$-a.s., $\forall\, y \in \mathbb{R},\ g(t,y,0) \equiv 0$.

Let g satisfy (A1) and (A2). Then for each $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, there exists a unique pair of adapted processes in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, denoted by $(Y_t(g,T,\xi),\ Z_t(g,T,\xi))_{t\in[0,T]}$, solving the BSDE (1.1) (see [11]).

For the convenience of the reader, we recall the notion of g-expectation and conditional g-expectation defined in [1]. We also list some basic properties of BSDEs and g-expectation. In the following Definitions 2.1 and 2.2, we always assume that g satisfies (A1) and (A3).

Definition 2.1 *The g-expectation $\mathcal{E}_g[\,\cdot\,] : L^2(\Omega,\mathcal{F}_T,P) \to \mathbb{R}$ is defined by*

$$\mathcal{E}_g[\xi] := Y_0(g,T,\xi).$$

Definition 2.2 *The conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[\xi|\mathcal{F}_t] := Y_t(g,T,\xi).$$

Lemmas 2.1–2.4 come from [1], where g is assumed to satisfy (A1) and (A3).

Lemma 2.1 (1) (Preserving of Constants) *For each constant c, $\mathcal{E}_g[c] = c$;*
(2) (Monotonicity) *If $X_1 \geq X_2$, a.s., then $\mathcal{E}_g[X_1] \ge \mathcal{E}_g[X_2]$;*
(3) (Strict Monotonicity) *If $X_1 \ge X_2$, a.s., and $P(X_1 > X_2) > 0$, then $\mathcal{E}_g[X_1] > \mathcal{E}_g[X_2]$.*

Lemma 2.2 (1) *If X is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[X|\mathcal{F}_t]=X$;*
(2) *For each $t \in [0,T]$, $\mathcal{E}_g[\mathcal{E}_g[X|\mathcal{F}_t]] = \mathcal{E}_g[X]$.*

Lemma 2.3 *$\mathcal{E}_g[X|\mathcal{F}_t]$ is the unique random variable η in $L^2(\Omega,\mathcal{F}_t,P)$, such that*

$$\mathcal{E}_g[X1_A] = \mathcal{E}_g[\eta 1_A] \quad \textit{for all } A \in \mathcal{F}_t.$$

A generator g of a BSDE is said to be independent of y if g is defined on $\Omega\times[0,T]\times\mathbb{R}^d$. We often denote this kind of generator g by $g(t,z)$.

Lemma 2.4 *Let g satisfy* (A1) *and* (A3)*. If g is independent of y, then $\mathcal{E}_g[X+\eta|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t] + \eta$, $\forall\, \eta \in L^2(\Omega,\mathcal{F}_t,P)$, $X \in L^2(\Omega,\mathcal{F}_T,P)$.*

The following Lemma 2.5 is [12, Theorem 1.3.5]. It will play a key role in this paper.

Lemma 2.5 (Representation Lemma) *Let* (A1) *and* (A2) *hold for* g; *let* $1 \leq p < 2$. *Then for each pair* $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, *the following equality*

$$g(t, y, z) = L^p\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)) - y]$$

holds for almost every $t \in [0, T[$.

Definition 2.3 *Let* g *be independent of* y. g *is said to be super homogeneous with respect to* z *if* g *also satisfies*

$$dP \times dt\text{-a.s.}, \quad \forall\, a \in \mathbb{R},\ z \in \mathbb{R}^d, \quad g(t, az) \geq a g(t, z).$$

Lemma 2.6 (See [9]) *Let* g *be independent of* y *and satisfy* (A1), (A3) *and the following assumption*

(A4) : P*-a.s.*, $\forall\, (y, z) \in \mathbb{R} \times \mathbb{R}^d$, $t \to g(t, y, z)$ *is continuous.*

Then the following two conditions are equivalent:

(i) g *is super homogeneous in* z;

(ii) *Jensen's inequality for* g*-expectation holds in general, i.e., for each* $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ *and convex function* $\varphi : \mathbb{R} \to \mathbb{R}$, *if* $\varphi(\xi) \in L^2(\Omega, \mathcal{F}_T, P)$, *then for each* $t \in [0, T]$, P*-a.s.,*

$$\mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t] \geq \varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Remark 2.1 When the authors of [9] proved that (i) implies (ii), they did not need (A4). (See the proof of Theorem 3.1 and Remark 3.1 of [9]).

3 Jensen's Inequality for BSDEs

In this Section, we study Jensen's inequality for BSDEs. Let us first introduce two notations. Let (A1) and (A2) hold for g; let $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. We set

$$S_y^z(g) := \left\{ t \,\middle|\, t \in [0, T[\text{ and } g(t, y, z) = L^1\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t)) - y] \right\}.$$

If g is independent of y, then g is often denoted by $g(t, z)$ for simplicity of notation. For this kind generator g, we set

$$S^z(g) := \left\{ t \,\middle|\, t \in [0, T[\text{ and } g(t, z) = L^1\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} Y_t(g, t+\varepsilon, z \cdot (B_{t+\varepsilon} - B_t)) \right\}.$$

We often denote $S_y^z(g)$ by S_y^z, denote $S^z(g)$ by S^z for simplicity of notations.

By Lemma 2.5 we understand that $\lambda\left([0, T] \setminus S_y^z\right) = 0$ (or $\lambda\left([0, T] \setminus S^z\right) = 0$, respectively), where λ denotes the Lebesgue measure.

For any $a, b \in \mathbb{R}$, we define a corresponding linear function $\varphi_a^b : \mathbb{R} \to \mathbb{R}$, such that $\varphi_a^b(x) = ax + b$, $\forall\, x \in \mathbb{R}$. Now let us state and prove our main result on Jensen's inequality for BSDEs.

Theorem 3.1 *Let* (A1) *and* (A2) *hold for* g.

(i) *If for any* $0 \leq t < T, 0 < \varepsilon \leq T - t$, *and any* $a, b, y \in \mathbb{R}, z \in \mathbb{R}^d$, *we have* P*-a.s.,*

$$Y_t(g, t+\varepsilon, \varphi_a^b(y + z \cdot (B_{t+\varepsilon} - B_t))) \geq \varphi_a^b(Y_t(g, t+\varepsilon, y + z \cdot (B_{t+\varepsilon} - B_t))). \tag{3.1}$$

Then g is independent of y, is super homogeneous in z and $g(t,0)\equiv 0$, $dP\times dt$-a.s..

Suppose furthermore that (A3) *also holds for g. Then the following three statements are equivalent:*

(ii) *g is independent of y and g is super homogeneous in z;*

(iii) *For each $a,b\in\mathbb{R}$, $\xi\in L^2(\Omega,\mathcal{F}_T,P)$, we have $\mathcal{E}_g[a\xi+b]\geq a\mathcal{E}_g[\xi]+b$;*

(iv) *Jensen's inequality for g-expectation holds in general, i.e., for each $\xi\in L^2(\Omega,\mathcal{F}_T,P)$ and convex function $\varphi:\mathbb{R}\to\mathbb{R}$, if $\varphi(\xi)\in L^2(\Omega,\mathcal{F}_T,P)$, then*

$$P\text{-a.s.},\quad \forall\, t\in[0,T],\quad \mathcal{E}_g[\varphi(\xi)|\mathcal{F}_t]\geq\varphi[\mathcal{E}_g(\xi|\mathcal{F}_t)].$$

Proof (i) For any $t\in[0,T[\,,\ y\in\mathbb{R},z\in\mathbb{R}^d$, since (3.1) holds, for any large enough positive integer n such that $n\geq\frac{1}{T-t}$, we have, P-a.s.,

$$Y_t\Big(g,t+\frac{1}{n},\varphi_1^y(z\cdot(B_{t+\frac{1}{n}}-B_t))\Big)\geq\varphi_1^y\Big(Y_t\Big(g,t+\frac{1}{n},z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)\Big).$$

Therefore we have

$$Y_t\Big(g,t+\frac{1}{n},y+(z\cdot(B_{t+\frac{1}{n}}-B_t))\Big)-y\geq Y_t\Big(g,t+\frac{1}{n},z\cdot(B_{t+\frac{1}{n}}-B_t)\Big).\tag{3.2}$$

Suppose $t\in S_y^z\cap S_0^z$. Then

$$L^1\text{-}\lim_{n\to\infty}n\Big[Y_t\Big(g,t+\frac{1}{n},y+z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)-y\Big]=g(t,y,z),\tag{3.3}$$

$$L^1\text{-}\lim_{n\to\infty}nY_t\Big(g,t+\frac{1}{n},z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)=g(t,0,z).\tag{3.4}$$

Then there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$P\text{-a.s.},\quad\lim_{k\to\infty}n_k\Big[Y_t\Big(g,t+\frac{1}{n_k},y+z\cdot(B_{t+\frac{1}{n_k}}-B_t)\Big)-y\Big]=g(t,y,z),\tag{3.5}$$

$$P\text{-a.s.},\quad\lim_{k\to\infty}n_kY_t\Big(g,t+\frac{1}{n_k},z\cdot(B_{t+\frac{1}{n_k}}-B_t)\Big)=g(t,0,z).\tag{3.6}$$

Thus for each $t\in S_y^z\cap S_0^z$, it follows from (3.5)–(3.6) and (3.2) that

$$P\text{-a.s.},\quad g(t,y,z)\geq g(t,0,z).$$

By Lemma 2.5 we know that $\lambda([0,T]\setminus S_y^z\cap S_0^z)=0$. Thus for each $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, we have

$$dP\times dt\text{-a.s.},\quad g(t,y,z)\geq g(t,0,z).\tag{3.7}$$

By (i) we also know that

$$Y_t\Big(g,t+\frac{1}{n},\varphi_1^{-y}(y+z\cdot(B_{t+\frac{1}{n}}-B_t))\Big)\geq\varphi_1^{-y}\Big(Y_t\Big(g,t+\frac{1}{n},y+z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)\Big).$$

Thus we have

$$Y_t\Big(g,t+\frac{1}{n},z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)\geq Y_t\Big(g,t+\frac{1}{n},y+z\cdot(B_{t+\frac{1}{n}}-B_t)\Big)-y.$$

Then for any y,z, in a similar manner, we can prove that

$$dP\times dt\text{-a.s.},\quad g(t,0,z)\geq g(t,y,z).\tag{3.8}$$

Thus for any $z \in \mathbb{R}^d,\ y \in \mathbb{R}$,

$$dP \times dt\text{-a.s.,} \quad g(t,0,z) = g(t,y,z). \tag{3.9}$$

In view of that g is Lipschitz with respect to y, we know that for each $z \in \mathbb{R}^d$,

$$dP \times dt\text{-a.s.,} \quad \forall\, y \in \mathbb{R}, \quad g(t,0,z) = g(t,y,z).$$

Therefore g is independent of y indeed.

For each $a \in \mathbb{R}, z \in \mathbb{R}^d,\ t \in [0,T[$, we choose a large enough positive integer n such that $n \geq \frac{1}{T-t}$. By (3.1) we have, P-a.s.,

$$Y_t\Big(g, t+\frac{1}{n}, \varphi_a^0(z\cdot(B_{t+\frac{1}{n}} - B_t))\Big) \geq \varphi_a^0\Big(Y_t\Big(g, t+\frac{1}{n}, z\cdot(B_{t+\frac{1}{n}} - B_t)\Big)\Big);$$

that is, P-a.s.,

$$Y_t\Big(g, t+\frac{1}{n}, az\cdot(B_{t+\frac{1}{n}} - B_t)\Big) \geq a\Big[Y_t\Big(g, t+\frac{1}{n}, z\cdot(B_{t+\frac{1}{n}} - B_t)\Big)\Big]. \tag{3.10}$$

Since g is independent of y, for each $t \in S^z \cap S^{az}$ we have

$$L^1\text{-}\lim_{n\to\infty} nY_t\Big(g, t+\frac{1}{n}, z\cdot(B_{t+\frac{1}{n}} - B_t)\Big) = g(t,z),$$
$$L^1\text{-}\lim_{n\to\infty} nY_t\Big(g, t+\frac{1}{n}, az\cdot(B_{t+\frac{1}{n}} - B_t)\Big) = g(t,az).$$

Therefore there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$P\text{-a.s.,} \quad \lim_{k\to\infty} an_k\Big[Y_t\Big(g, t+\frac{1}{n_k}, z\cdot(B_{t+\frac{1}{n_k}} - B_t)\Big)\Big] = ag(t,z), \tag{3.11}$$

$$P\text{-a.s.,} \quad \lim_{k\to\infty} n_k\Big[Y_t\Big(g, t+\frac{1}{n_k}, az\cdot(B_{t+\frac{1}{n_k}} - B_t)\Big)\Big] = g(t,az). \tag{3.12}$$

Thus for each given $a \in \mathbb{R}, z \in \mathbb{R}^d$, if $t \in S^z \cap S^{az}$, then it follows from (3.11)–(3.12) and (3.10) that

$$P\text{-a.s.,} \quad g(t,az) \geq ag(t,z). \tag{3.13}$$

By Lemma 2.5 we know that $\lambda([0,T] \setminus S^z \cap S^{az}) = 0$. Thus

$$dP \times dt\text{-a.s.,} \quad g(t,az) \geq ag(t,z).$$

Thus for each $a \in \mathbb{R}$, it follows from the Lipschitz assumption (A1) and the above inequality that

$$dP \times dt\text{-a.s.,} \quad \forall\, z \in \mathbb{R}^d, \quad g(t,az) \geq ag(t,z). \tag{3.14}$$

Also by (A1) we understand that

$$dP \times dt\text{-a.s.,} \quad \forall\, z \in \mathbb{R}^d, \quad a \to g(t,az) \ \text{ is continuous.}$$

Obviously ag is continuous with respect to a. Thus

$$dP \times dt\text{-a.s.,} \quad \forall\, a \in \mathbb{R},\ z \in \mathbb{R}^d, \quad g(t,az) \geq ag(t,z). \tag{3.15}$$

Therefore g is super-homogeneous in z.

By (3.14) we understand that

$$dP \times dt\text{-a.s.,} \quad g(t,0) \geq -g(t,0); \tag{3.16}$$

$$dP \times dt\text{-a.s.,} \quad g(t,0) \geq 2g(t,0). \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$dP \times dt\text{-a.s.,} \quad g(t,0) = 0.$$

The proof of (i) is complete.

Now let us prove the second part of Theorem 3.1.

Suppose (A3) also holds for g. By Lemma 2.6 and Remark 2.1 we know that (ii) implies (iv). It is obvious that (iv) implies (iii), so for completing the proof of Theorem 3.1, we only need to prove that (iii) implies (ii). Suppose (iii) holds. Then we have

$$\mathcal{E}_g[X+c] \geq \mathcal{E}_g[X] + c, \quad \forall\, c \in \mathbb{R},\ X \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.18}$$

Thus we have

$$\mathcal{E}_g[X] = \mathcal{E}_g[X+c-c] \geq \mathcal{E}_g[X+c] - c, \quad \forall\, c \in \mathbb{R},\ X \in L^2(\Omega, \mathcal{F}_T, P).$$

Therefore

$$\mathcal{E}_g[X] + c \geq \mathcal{E}_g[X+c], \quad \forall\, c \in \mathbb{R},\ X \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\mathcal{E}_g[X+c] = \mathcal{E}_g[X] + c, \quad \forall\, c \in \mathbb{R},\ X \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.20}$$

Then we have

Proposition 3.1 *Suppose $\mathcal{E}_g$ satisfies* (3.20). *Then g is independent of y.*

Proof For any $c \in \mathbb{R}$, we define a new generator

$$g^c(t,y,z) := g(t, y-c, z), \quad \forall\, t \in [0,T],\ y \in \mathbb{R},\ z \in \mathbb{R}^d.$$

Then g^c satisfies (A1), (A2) and (A3).

For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, let $(y_t, z_t)_{t\in[0,T]}$ denote the solution of the following BSDE (3.21):

$$y_t = X + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dB_s, \quad 0 \leq t \leq T. \tag{3.21}$$

Then we have

$$y_t + c = X + c + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dB_s, \quad 0 \leq t \leq T.$$

We set $\bar{y}_t = y_t + c,\ \bar{z}_t = z_t,\ \forall\, t \in [0,T]$. Then we have

$$\bar{y}_t = X + c + \int_t^T g^c(s, \bar{y}_s, \bar{z}_s)ds - \int_t^T \bar{z}_s dB_s, \quad 0 \leq t \leq T. \tag{3.22}$$

By the uniqueness of the solution of the BSDE (3.22) we have

$$\mathcal{E}_{g^c}[X+c|\mathcal{F}_t] = \bar{y}_t = y_t + c = \mathcal{E}_g[X|\mathcal{F}_t] + c, \quad \forall\, t \in [0,T]. \tag{3.23}$$

Combining (3.23) with (3.20) we have

$$\mathcal{E}_{g^c}[X+c] = \mathcal{E}_g[X+c], \quad \forall\, X \in L^2(\Omega, \mathcal{F}_T, P).$$

Thus $\forall\, c \in \mathbb{R}$, we have

$$\mathcal{E}_{g^c}[\xi] = \mathcal{E}_g[\xi], \quad \forall\, \xi \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.24}$$

Hence for any $c \in \mathbb{R}$, $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, it follows from (3.24) and [10, Proposition 3.1] that

$$P\text{-a.s.}, \quad \mathcal{E}_{g^c}[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t], \quad \forall\, t \in [0,T]. \tag{3.25}$$

For any $c \in \mathbb{R}, z \in \mathbb{R}^d, t \in [0,T[$, for any large enough positive integer n such that $n \geq \frac{1}{T-t}$, by (3.25) we have P-a.s.,

$$\mathcal{E}_{g^c}[z \cdot (B_{t+\frac{1}{n}} - B_t)|\mathcal{F}_t] = \mathcal{E}_g[z \cdot (B_{t+\frac{1}{n}} - B_t)|\mathcal{F}_t]; \tag{3.26}$$

that is,

$$Y_t\Big(g^c, t+\frac{1}{n}, z \cdot (B_{t+\frac{1}{n}} - B_t)\Big) = Y_t\Big(g, t+\frac{1}{n}, z \cdot (B_{t+\frac{1}{n}} - B_t)\Big). \tag{3.27}$$

For any $t \in S_0^z(g^c) \cap S_0^z(g)$, it is obvious that

$$L^1\text{-}\lim_{n\to\infty} n\Big[Y_t\Big(g^c, t+\frac{1}{n}, z \cdot (B_{t+\frac{1}{n}} - B_t)\Big)\Big] = g^c(t,0,z), \tag{3.28}$$

$$L^1\text{-}\lim_{n\to\infty} nY_t\Big(g, t+\frac{1}{n}, z \cdot (B_{t+\frac{1}{n}} - B_t)\Big) = g(t,0,z). \tag{3.29}$$

Thus there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$P\text{-a.s.}, \quad \lim_{k\to\infty} n_k Y_t\Big(g^c, t+\frac{1}{n_k}, z \cdot (B_{t+\frac{1}{n_k}} - B_t)\Big) = g^c(t,0,z), \tag{3.30}$$

$$P\text{-a.s.}, \quad \lim_{k\to\infty} n_k Y_t\Big(g, t+\frac{1}{n_k}, z \cdot (B_{t+\frac{1}{n_k}} - B_t)\Big) = g(t,0,z). \tag{3.31}$$

Thus for each given $t \in S_y^z(g^c) \cap S_0^z(g)$, it follows from (3.30)–(3.31) and (3.27) that

$$P\text{-a.s.}, \quad g^c(t,0,z) = g(t,0,z),$$

that is,

$$P\text{-a.s.}, \quad g(t,-c,z) = g(t,0,z).$$

Hence for each $c \in \mathbb{R}, z \in \mathbb{R}^d$, by Lemma 2.5 we have

$$dP \times dt\text{-a.s.}, \quad g(t,-c,z) = g(t,0,z). \tag{3.32}$$

For each $z \in \mathbb{R}^d$, since g is Lipschitz with respect to y, it follows that

$$dP \times dt\text{-a.s.}, \quad \forall\, y \in \mathbb{R}, \quad g(t,y,z) = g(t,0,z). \tag{3.33}$$

Therefore g is independent of y. The proof of Proposition 3.1 is complete.

Let us return to the proof of Theorem 3.1. We now prove that g is super homogeneous with respect to z. For each $a \in \mathbb{R},\ a \neq 0$, by (iii) we know

$$\mathcal{E}_g[aX] \geq a\mathcal{E}_g[X], \quad \forall\, X \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.34}$$

We define a new generator

$$\tilde{g}^a(t,z) := ag\Big(t, \frac{z}{a}\Big), \quad \forall\, t \in [0,T],\ z \in \mathbb{R}^d.$$

Then $\tilde{g}^a$ satisfies (A1), (A2) and (A3).

For any $X \in L^2(\Omega, \mathcal{F}_T, P)$, let $(y_t, z_t)_{t\in[0,T]}$ denote the solution of the following BSDE (3.35):

$$y_t = X + \int_t^T g(s, z_s)ds - \int_t^T z_s \cdot dB_s, \quad 0 \leq t \leq T. \tag{3.35}$$

Then we have

$$ay_t = aX + \int_t^T ag(s, z_s)ds - \int_t^T az_s \cdot dB_s, \quad 0 \leq t \leq T.$$

We set $\tilde{y}_t = ay_t,\ \tilde{z}_t = az_t,\ \forall\, t \in [0,T]$. Then we have

$$\tilde{y}_t = aX + \int_t^T \tilde{g}^a(s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s \cdot dB_s, \quad 0 \leq t \leq T. \tag{3.36}$$

Then by the uniqueness of solution of BSDE we have

$$\mathcal{E}_{\tilde{g}^a}[aX|\mathcal{F}_t] = \tilde{y}_t = ay_t = a\mathcal{E}_g[X|\mathcal{F}_t], \quad \forall\, t \in [0,T]. \tag{3.37}$$

Thus we have

$$\mathcal{E}_{\tilde{g}^a}[aX] = a\mathcal{E}_g[X].$$

Combining this with (3.34) we have

$$\mathcal{E}_g[\xi] \geq \mathcal{E}_{\tilde{g}^a}[\xi], \quad \forall\, \xi \in L^2(\Omega, \mathcal{F}_T, P). \tag{3.38}$$

Now let us prove that $\forall\, \xi \in L^2(\Omega, \mathcal{F}_T, P),\ t \in [0,T]$,

$$\mathcal{E}_g[\xi|\mathcal{F}_t] \geq \mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]. \tag{3.39}$$

The argument of (3.39) is derived from [1, 4].

We set $A := \{\mathcal{E}_g[\xi|\mathcal{F}_t] < \mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]\}$. Then A is $\mathcal{F}_t$-measurable. Since g and $\tilde{g}^a$ satisfy (A3), we can deduce that

$$P\text{-a.s.}, \quad \mathcal{E}_g[\mathbf{1_A}\xi|\mathcal{F}_s] = \mathbf{1_A}\mathcal{E}_g[\xi|\mathcal{F}_s], \quad \mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\xi|\mathcal{F}_s] = \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_s], \quad \forall\, s \in [t,T].$$

If $P(A) > 0$, then

$$\mathbf{1_A}\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t] \leq 0 \quad \text{and} \quad P(\{\mathbf{1_A}\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t] < 0\}) > 0. \tag{3.40}$$

In view of that g and $\tilde{g}^a$ are both independent of y, by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\xi - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] \\
&= \mathcal{E}_{\tilde{g}^a}[\mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\xi - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]|\mathcal{F}_t]] = \mathcal{E}_{\tilde{g}^a}[\mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] \\
&= \mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] = 0.
\end{aligned}$$

On the other hand, by Lemma 2.2, Lemma 2.4, (3.40) and Lemma 2.1 we can get that

$$\begin{aligned}
&\mathcal{E}_g[\mathbf{1_A}\xi - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] \\
&= \mathcal{E}_g[\mathcal{E}_g[\mathbf{1_A}\xi - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]|\mathcal{F}_t]] = \mathcal{E}_g[\mathcal{E}_g[\mathbf{1_A}\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] \\
&= \mathcal{E}_g[\mathbf{1_A}\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]] < 0 = \mathcal{E}_{\tilde{g}^a}[\mathbf{1_A}\xi - \mathbf{1_A}\mathcal{E}_{\tilde{g}^a}[\xi|\mathcal{F}_t]],
\end{aligned}$$

which is a contradiction to (3.38). Therefore $P(A) = 0$. Thus (3.39) does hold.

For each $z \in \mathbb{R}^d$, if $t \in S^z(g) \cap S^z(\tilde{g}^a)$, we have

$$\begin{aligned}
L^1\text{-}\lim_{n\to\infty} n\mathcal{E}_g[z\cdot(B_{t+\frac{1}{n}} - B_t)|\mathcal{F}_t] &= g(t,z), \\
L^1\text{-}\lim_{n\to\infty} n\mathcal{E}_{\tilde{g}^a}[z\cdot(B_{t+\frac{1}{n}} - B_t)|\mathcal{F}_t] &= \tilde{g}^a(t,z).
\end{aligned}$$

Therefore there exists a subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ such that

$$P\text{-a.s.},\quad \lim_{k\to\infty} n_k\mathcal{E}_g[z\cdot(B_{t+\frac{1}{n_k}} - B_t)|\mathcal{F}_t] = g(t,z), \tag{3.41}$$

$$P\text{-a.s.},\quad \lim_{k\to\infty} n_k\mathcal{E}_{\tilde{g}^a}[z\cdot(B_{t+\frac{1}{n_k}} - B_t)|\mathcal{F}_t] = \tilde{g}^a(t,z). \tag{3.42}$$

Thus for any $z \in \mathbb{R}^d$, $t \in S^z(g) \cap S^z(\tilde{g}^a)$, it follows from (3.39) and (3.41)–(3.42) that

$$P\text{-a.s.},\quad g(t,z) \geq \tilde{g}^a(t,z).$$

Since for any $z \in \mathbb{R}^d$, by Lemma 2.5 we know that $\lambda([0,T] \setminus S^z(g) \cap S^z(\tilde{g}^a)) = 0$. Thus

$$dP \times dt\text{-a.s.},\quad g(t,z) \geq \tilde{g}^a(t,z).$$

Since g and $\tilde{g}^a$ are both Lipschitz with respect to z, we have

$$dP \times dt\text{-a.s.},\quad \forall\, z \in \mathbb{R}^d,\quad g(t,z) \geq \tilde{g}^a(t,z);$$

that is,

$$dP \times dt\text{-a.s.},\quad \forall\, z \in \mathbb{R}^d,\quad g(t,z) \geq ag\Big(t, \frac{z}{a}\Big).$$

In view of $g(t,0) = 0$ we understand that for each $a \in \mathbb{R}$, we have

$$dP \times dt\text{-a.s.},\quad \forall\, z \in \mathbb{R}^d,\quad g(t,az) \geq ag(t,z).$$

Since $g(t,az)$ and ag are both continuous with respect to a, we have

$$dP \times dt\text{-a.s.},\quad \forall\, a \in \mathbb{R},\ z \in \mathbb{R}^d,\quad g(t,az) \geq ag(t,z).$$

Therefore g is super homogeneous in z. The proof of Theorem 3.1 is complete.

Remark 3.1 Suppose a generator g is independent of y. Then the super homogeneity of g is equivalent to the following condition: $dP \times dt$-a.s.,

$$g(t, az) = ag(t, z) \quad \text{and} \quad g(t, z) + g(t, -z) \geq 0, \quad \forall\, 0 \leq a \in \mathbb{R},\ z \in \mathbb{R}^d.$$

Corollary 3.1 *Let* (A1) *and* (A3) *hold for* g. *Let* $d = 1$, *i.e., the dimension of the Brownian motion* (B_t) *is one. Then the following three statements are equivalent:*

(i) *g is independent of y and g is super homogeneous in z.*

(ii) *There exist two bounded and progressively measurable processes* $(\alpha(t))_{t\in[0,T]}$ *and* $(\beta(t))_{t\in[0,T]}$ *such that* $\alpha(\cdot) \geq 0$ *and*

$$g(t, z) = \alpha(t)|z| + \beta(t)z, \quad \forall\, t \in [0, T],\ z \in \mathbb{R}.$$

In particular, g is convex with respect to z.

(iii) *Jensen's inequality for g-expectation holds in general.*

Proof It suffices to prove that (i) is equivalent to (ii). It is obvious that (ii) implies (i), so we only need to prove that (i) implies (ii). Suppose (i) holds, by Remark 3.1 we know that g is positively homogeneous with respect to z. Hence for any $z \in \mathbb{R}$, we have

$$g(t, z) = \Big(\frac{g(t, 1) + g(t, -1)}{2}\Big)|z| + \Big(\frac{g(t, 1) - g(t, -1)}{2}\Big)z = \alpha(t)|z| + \beta(t)z,$$

where $\alpha(t) := \frac{g(t,1)+g(t,-1)}{2}$, $\beta(t) := \frac{g(t,1)-g(t,-1)}{2}$, $t \in [0, T]$. Due to (A1) and (A3) we conclude that $\alpha(\cdot)$ and $\beta(\cdot)$ are both progressively measurable and bounded by the Lipschitz constant K.

It follows from the super homogeneity of g that

$$\alpha(t)|-z| + \beta(t)(-z) = g(t, -z) \geq -g(t, z) = -\alpha(t)|z| - \beta(t)z.$$

Hence

$$\alpha(t) \geq 0, \quad dP \times dt\text{-a.s.}$$

The proof of Corollary 3.1 is complete.

Remark 3.2 It is worth pointing out that if $d \geq 2$, a super homogeneous generator is not necessarily convex with respect to z. For more results on Jensen's inequality for g-expectation and on Jensen's inequality for g-(sub)martingale, we refer the reader to [12].

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