Chin. Ann. Math.
27B(5), 2006, 565–580
DOI: 10.1007/s11401-005-0041-z

Chinese Annals of Mathematics, Series B

Global Existence of Classical Solutions for Some Oldroyd-B Model via the Incompressible Limit**

Zhen LEI*

Abstract In this paper, we prove local and global existence of classical solutions for a system of equations concerning an incompressible viscoelastic fluid of Oldroyd-B type via the incompressible limit when the initial data are sufficiently small.

Keywords Incompressible limit, Global existence, Oldroyd model
2000 MR Subject Classification 11R70, 11R11, 11R27

1 Introduction

In the context of hydrodynamics, the motion of the fluid flow is classically described by the following system of equations

$$\begin{cases}\partial_t\rho + u\cdot\nabla\rho + \rho\nabla\cdot u = 0,\\ \rho(\partial_t u + u\cdot\nabla u) = \nabla\cdot\sigma,\end{cases}$$

where ρ is the density, u ($u(t,x)\in R^n$) the velocity and σ (an $n\times n$ symmetric matrix) the stress tensor. Moreover, σ can be decomposed as $\sigma = \tau_1 - pI$, where τ_1 is the tangential part of the stress tensor and $-pI$ is the normal part. For a Newtonian fluid, τ_1 depends linearly on ∇u. More precisely, $\tau_1 = 2\mu\Gamma(u)$, where $\Gamma(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$.

It turns out that many fluids do not satisfy the Newtonian law. For example, for some fluids with shear dependent viscosity, the Newtonian law $\tau_1 = 2\mu\Gamma(u)$ is replaced by $\tau_1 = 2\mu(|\Gamma(u)|^2)\Gamma(u)$. This model has been extensively studied, see [5, 6, 12, 14, 15, 18], for example. But all these models involve only instantaneous constitutive relation between the stress and the strain. Indeed, for many fluids, it is not possible to determine at some time t the value of τ_1 knowing only $\Gamma(u)$ at the same time t, and one also has to know the whole history of $\Gamma(u)$. In these cases, we say that the fluid has a “memory”.

In this paper, we are going to study a simpler viscoelastic model, namely the Oldroyd-B model. Fluids of this type has a memory and one has to write a differential equation for τ_1.

Manuscript received February 2, 2005.
*School of Mathematical Sciences, Fudan University, Shanghai 200433, China; School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China. E-mail: leizhn@yahoo.com
**Project supported by the National Natural Science Foundation of China (No.10225102), the 973 Project of the Ministry of Science and Technology of China and the Doctoral Program Foundation of the Ministry of Education of China.

The equations of incompressible fluid of Oldroyd-B type are subject to the following form

$$\begin{cases} \nabla\cdot u = 0, \\ \partial_t u + u\cdot\nabla u + \nabla q = \nu\Delta u + \mu_1\nabla\cdot\tau, \\ \partial_t\tau + u\cdot\nabla\tau + a\tau + Q(\tau,\nabla u) = \mu_2\Gamma(u), \end{cases} \tag{1.1}$$

where the density in the undeformed reference configuration has been set to be equal to one, q (a scalar function) is the pressure, τ is the elastic part of τ_1, $a,\ \nu,\ \mu_1,\ \mu_2$ are positive constants and $Q(\tau,\nabla u)$ is a quadratic form

$$Q(\tau,\nabla u) = \tau W(u) - W(u)\tau - b(\Gamma(u)\tau + \tau\Gamma(u)), \tag{1.2}$$

where $b\in[-1,1]$ is a constant and $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ is the vorticity tensor. We point out here that in this paper, we will use the notation $(\nabla u)_{ij} = \frac{\partial u^i}{\partial x_j}$, $(\nabla u)^T_{ij} = \frac{\partial u^j}{\partial x_i}$, $(\nabla\cdot\tau)_i = \partial_j\tau_{ij}$ and $u\cdot\nabla\tau = u_j\partial_j\tau$. Summation over repeated indices will always be understood.

On the other hand, the equations of compressible fluid of Oldroyd-B type with a large parameter λ are a distinctly different system taking the following form

$$\begin{cases} \partial_t\rho^\lambda + u^\lambda\cdot\nabla\rho^\lambda + \rho^\lambda\nabla\cdot u^\lambda = 0, \\ \partial_t u^\lambda + u^\lambda\cdot\nabla u^\lambda + \dfrac{1}{\rho^\lambda}\nabla p^\lambda = \dfrac{\nu}{\rho^\lambda}(\Delta u^\lambda + \nabla(\nabla\cdot u^\lambda)) + \dfrac{\mu_1}{\rho^\lambda}\nabla\cdot\tau^\lambda, \\ \partial_t\tau^\lambda + u^\lambda\cdot\nabla\tau^\lambda + a\tau^\lambda + \dfrac{1}{\rho^\lambda}Q(\tau^\lambda,\nabla u^\lambda) = \dfrac{\mu_2}{\rho^\lambda}\Gamma(u^\lambda). \end{cases} \tag{1.3}$$

We will concentrate below on a specific family of fluids which is given by the varying equations of state $p^\lambda(\rho) = \lambda^2 p(\rho)$ with fixed $p(\rho),\ p'(\rho) > 0,\ \rho > 0$, since we are ultimately interested in the incompressible limit. Such a family arises in a natural fashion from a fixed fluid with an equation of state $p(\rho) = A\rho^\gamma,\ \gamma > 1$. Indeed, denoting by $M = \frac{1}{\lambda} = \frac{|v_m|}{(p'(\rho_m))^{\frac{1}{2}}}$ the Mach number of the fluid defined as the ratio of typical fluid velocities v_m to typical sound speeds $(p'(\rho_m))^{\frac{1}{2}}$ and writing the corresponding compressible Oldroyd-B equations in non-dimensional form one obtains a varying family of equations of state of this type.

One expects, under appropriate conditions on the initial data, that the solutions ρ^λ, $\lambda^2\nabla p^\lambda$, u^λ, τ^λ of the compressible system (1.3) converge to 1, $\nabla q,\ u,\ \tau$, respectively, as $\lambda\to\infty$, where $\nabla q,\ u,\ \tau$ are the solutions of the incompressible Oldroyd-B system (1.1). It is well known that long time behavior of solutions to the viscoelastic equations depends on strong dispersive estimates (see [8, 9, 16]). For the wave equations, the generalized energy method, based on the Lorentz invariance and global Sobolev inequalities, provides an elegant and efficient means of combining energy and decay estimates (see [8, 9], for example). Recently Sideris and Thomases [17] studied an elastodynamic system which is not Lorentz invariant in three space dimensions via the incompressible limit through the use of weighted Sobolev inequalities involving the smaller number of generators. But the Oldroyd-B model we study in this paper is neither Lorentz invariant nor scaling invariant since the presence of damping mechanism of u.

In this paper, we first prove that classical local solutions of the equations of motion exist for sufficiently small disturbances from the general incompressible initial data. This result depends on a modified method of [10], where Klainerman and Majda developed a general theory to study the incompressible limit of compressible fluids in general framework of quasi-linear hyperbolic systems depending on a large parameter. Their method can also be extended to cover the viscous equations. But since the stress tensor τ in system (1.3) is not "fast scale" (in the sense of [10]), our system does not satisfy their structural conditions.

We also prove the uniform stability of the local existence family which yields a lifespan of the compressible system (1.3) and allows for convergence to a global solution of the limiting incompressible equations by means of compactness arguments. The strength of this convergence improves the degree of incompressibility satisfied by the initial data.

The existence and uniqueness of local strong solutions of incompressible fluid satisfying the Oldroyd constitutive law in Hilbert spaces H^s have been established by Guillopé and Saut in [4]. The existence and uniqueness for local and global solutions in some limit spaces which is invariant under the Navior-Stokes scaling was announced by Chemin and Masmoudi in [2]. For $b = 0$, the global weak solutions was also established in [12]. Ezquerra and Zonzaléz [3] proved the global existence of weak solutions in a unique cylinder. Recently, when the damping mechanism on the deformation tensor F is lost, Lin, Liu and Zhang [11] proved the global existence of classical solutions for two dimensional incompressible Oldroyd model by introducing the induced stress to find the dissipation of the system.

The paper is organized as follows. In Section 2, we state the main results of this paper. In Section 3, we establish the uniform stability estimates and the local existence of the solutions of the compressible system (1.3). In Section 4, we establish the dispersive energy estimates which allows us to take the limit as $\lambda \to \infty$ to obtain a global solution to the incompressible Oldroyd system.

2 Statements of Main Results

To avoid complications at the boundary, we concentrate below on the periodic case where $x \in \mathbf{T}^n$, the n-dimensional torus. In fact the whole space problem and the Dirichlet problem of smooth bounded domain can also be treated at the expense of complicating the proofs below. In the following, $\|\cdot\|$, $\|\cdot\|_s$ and $\|\cdot\|_\infty$ will denote the norms in $L^2(\mathbf{T}^n)$, $H^s(\mathbf{T}^n)$ and $L^\infty(\mathbf{T}^n)$, respectively.

Define

$$\begin{cases} E_s(U(t)) = \dfrac{1}{2}\displaystyle\sum_{|\alpha|\le s}\int_{\mathbf{T}^n}(\lambda^2|\nabla^\alpha(\rho-1)|^2 + |\nabla^\alpha u|^2 + |\nabla^\alpha \tau|^2)dx, \\ \widetilde{E}_s(U(t)) = \dfrac{1}{2}\displaystyle\sum_{|\alpha|\le s}\int_{\mathbf{T}^n}\Big(\mu_2\lambda^2\frac{p'(\rho)}{\rho}|\nabla^\alpha(\rho-1)|^2 + \mu_2\rho|\nabla^\alpha u|^2 + \mu_1\rho|\nabla^\alpha\tau|^2\Big)dx, \end{cases} \tag{2.1}$$

where $U(t) = (\rho,\ u,\ \tau)$. It is obvious that

$$E_s(U(t)) \sim \widetilde{E}_s(U(t)) \tag{2.2}$$

provided $|\rho - 1|$ is sufficiently small.

Theorem 2.1 *Consider the compressible Oldroyd-B model* (1.3) *with the following initial data in* $H^{s+1}(\mathbf{T}^n)$ ($n = 2$ *or* 3, *integer* $s \ge [\frac{n}{2}] + 3$)

$$\rho^\lambda(0,x) = 1 + \tilde{\rho}_0^\lambda(x), \quad u^\lambda(0,x) = u_0(x) + \tilde{u}_0^\lambda(x), \quad \tau^\lambda(0,x) = \tau_0(x) + \tilde{\tau}_0^\lambda(x), \tag{2.3}$$

where $\tau_0(x)$ *is symmetric,* $u_0(x)$ *satisfies the incompressible constraints*

$$\nabla \cdot u_0 = 0 \tag{2.4}$$

and $\tilde{\rho}_0^\lambda(x)$, $\tilde{u}_0^\lambda(x)$, $\tilde{\tau}_0^\lambda(x)$ *are assumed to satisfy*

$$\|\tilde{\rho}_0^\lambda(x)\|_s \le \frac{\delta_0}{\lambda^2}, \quad \|\tilde{u}_0^\lambda(x)\|_{s+1} \le \frac{\delta_0}{\lambda}, \quad \|\tilde{\tau}_0^\lambda(x)\|_s \le \frac{\delta_0}{\lambda}, \quad \delta_0 \textit{ small}. \tag{2.5}$$

Then the following statements hold.

Uniform stability: There exist fixed constants T_0, κ *independent of* λ *such that a unique classical* C^2 *solution* $(\rho^\lambda, u^\lambda, \tau^\lambda)$ *of system* (1.3) *exists for all large* λ *on the time interval* $[0, T_0]$. *Furthermore, the solution family satisfy*

$$\begin{cases} E_s(U^\lambda(t)) + \displaystyle\int_0^t (\|\nabla u^\lambda\|_s^2 + \|\tau^\lambda\|_s^2)\ dt \le 4(\|u_0\|_s^2 + \|\tau_0\|_s^2), \\ E_{s-1}(\partial_t U^\lambda(t)) + \displaystyle\int_0^t (\|\nabla \partial_t u^\lambda\|_{s-1}^2 + \|\partial_t \tau^\lambda\|_{s-1}^2)\ dt \le \kappa, \end{cases} \tag{2.6}$$

for all $t \in [0, T_0]$ *provided* λ *is sufficiently large.*

Local existence of solutions for incompressible system: There exist functions u, τ *with* $\|u\|_s^2 + \|\tau\|_s^2 \le 4(\|u_0\|_s^2 + \|\tau_0\|_s^2)$, $t \in [0, T_0]$, *such that*

$$\begin{cases} \rho^\lambda \to 1 & \textit{in} \quad L^\infty(0, T_0; H^s) \cap \mathrm{Lip}([0, T_0], H^{s-1}), \\ (u^\lambda, \tau^\lambda) \to (u, \tau) & \textit{weakly}^* \textit{ in } L^\infty(0, T_0; H^s) \cap \mathrm{Lip}([0, T_0], H^{s-1}), \\ (u^\lambda, \tau^\lambda) \to (u, \tau) & \textit{in } C([0, T_0], H^{s-\delta}), \end{cases} \tag{2.7}$$

where δ *is a small positive constant. The function* (u, τ) *is a* C^2 *solution of the incompressible system of Oldroyd-B type*

$$\begin{cases} \nabla \cdot u = 0, \\ P(\partial_t u + u \cdot \nabla u - \nu \Delta u - \mu_1 \nabla \cdot \tau) = 0, \\ \partial_t \tau + u \cdot \nabla \tau + a\tau + Q(\tau, \nabla u) = \mu_2 \Gamma(u) \end{cases} \tag{2.8}$$

with the initial data

$$u(0, x) = u_0(x), \quad \tau(0, x) = \tau_0(x), \tag{2.9}$$

where P *is the* L^2*-projection on the divergence free vector fields.*

Remark 2.1 If we denote

$$\partial_t u + u \cdot \nabla u - \nu \Delta u - \nabla \cdot \tau = \nabla q,$$

then we have

$$\frac{1}{\rho^\lambda} \lambda^2 \nabla p(\rho^\lambda) \longrightarrow \nabla q \quad \text{weakly}^* \text{ in } L^\infty(0, T_0; H^{s-2}) \cap L^2(0, T_0; H^{s-1}),$$

which means that $\lambda^2(\|\nabla \rho^\lambda\|_{s-2} + \int_0^t \|\nabla \rho^\lambda\|_{s-1}^2\ dt)$ is uniformly bounded in $t \in [0, T_0]$.

Theorem 2.2 *Consider the solutions $(\rho^\lambda, u^\lambda, \tau^\lambda)$ of the compressible Oldroyd model obtained in Theorem 2.1. Suppose that the initial data additionally satisfy*

$$\|u_0\|_s + \|\tau_0\|_s < \varepsilon, \tag{2.10}$$

where ε is a positive constant, and the displacements from $\rho_0 = 1, u_0, \tau_0$ satisfy (2.5). If ε is sufficiently small, then for every fixed $\overline{T} > 0$, the solution $(\rho^\lambda, u^\lambda, \tau^\lambda)$ satisfies the following estimates

$$\begin{cases} E_s(U^\lambda(t)) + \displaystyle\int_0^t (\|\nabla u^\lambda\|_s^2 + \|\tau\|_s^2)\ dt \le C\Big(\varepsilon + \frac{\delta_0^2}{\lambda^2}\Big), & t \in [0, T^\lambda), \\ E_{s-1}(\partial_t U^\lambda(t)) + \displaystyle\int_0^t (\|\nabla \partial_t u^\lambda\|_{s-1}^2 + \|\partial_t \tau\|_{s-1}^2)\ dt \le C \exp Ct, & 0 \le t \le \overline{T}, \end{cases} \tag{2.11}$$

where $T^\lambda > \overline{T}$ and $T^\lambda \to \infty$ as $\lambda \to \infty$.

We point out that the uniform bounds for the initial energy in (2.5) implies that, in the limit as $\lambda \to \infty$, the initial deformation is driven toward incompressibility. Since the bounds on the energy from Theorem 2.2 are uniform in λ, we will be able to take the limit as λ goes to infinity to obtain a global solution to the equations of incompressible system of Oldroyd-B type (1.1).

Theorem 2.3 *Consider the incompressible system of Oldroyd-B type* (1.1) *with the initial data* (2.9) *which satisfy the constraints* (2.4) *and* (2.10)*. Then there exists a unique classical solution (u, τ) which satisfies*

$$\|u\|_s + \|\tau\|_s \le C\varepsilon \tag{2.12}$$

provided ε is sufficiently small.

3 Local Existence and Uniform Stability

In this section we will derive the main estimates for our results and prove Theorem 2.1. We emphasize that solutions will depend on the value of the parameter λ, but this dependence will not always be displayed for reasons of notational convenience. Our object is to apply a modified classical method of existence originally for quasi-linear symmetric hyperbolic systems in weighted λ-norms.

Consider the set of functions $B_{T_0}^\lambda(U_0)$ contained in $C([0, T_0], H^s) \cap C^1([0, T_0], H^{s-1})$ with $s \ge [\frac{n}{2}] + 3$ and defined by

$$\begin{cases} \lambda|\rho - 1| + |u - u_0| + |\tau - \tau_0| < \delta, \\ E_s(U(t)) + \displaystyle\int_0^t (\|\nabla u\|_s^2 + \|\tau\|_s^2)\ dt \le 4(\|u_0\|_s^2 + \|\tau_0\|_s^2), \\ E_s(\partial_t U(t)) + \displaystyle\int_0^t (\|\nabla \partial_t u\|_{s-1}^2 + \|\partial_t \tau\|_{s-1}^2)\ dt \le \kappa \end{cases} \tag{3.1}$$

for $0 \le t \le T_0$, where $U = (\rho, u, \tau)$ and $U_0 = (1 + \tilde{\rho}_0^\lambda, u_0 + \tilde{u}_0^\lambda, \tau_0 + \tilde{\tau}_0^\lambda)$.

For $V=(\xi,v,\sigma)\in B^{\lambda}_{T_0}(U_0)$, define $U=(\rho,u,\tau)=\Lambda(V)$ as the unique solution of the linear problem

$$\begin{cases}\partial_t\rho+v\cdot\nabla\rho+\xi\nabla\cdot u=0,\\ \partial_t u+v\cdot\nabla u+\dfrac{p'(\xi)}{\xi}\lambda^2\nabla\rho=\dfrac{\nu}{\xi}(\Delta u+\nabla(\nabla\cdot u))+\dfrac{\mu_1}{\xi}\nabla\cdot\tau,\\ \partial_t\tau+v\cdot\nabla\tau+a\tau+\dfrac{1}{\xi}Q(\sigma,\nabla u)=\dfrac{\mu_2}{\xi}\Gamma(\nabla u)\end{cases}\tag{3.2}$$

with the initial condition

$$U(0,x)=U_0=(1+\tilde\rho_0^{\lambda},u_0+\tilde u_0^{\lambda},\tau_0+\tilde\tau_0^{\lambda}).$$

We plan to show that for appropriate choice of T_0, δ and κ independent of λ, Λ maps $B^{\lambda}_{T_0}(U_0)$ into itself and moreover it is a contraction in the norm $E(\,\cdot\,)^{\frac12}$.

Before proceeding any further, we apply D^α to (3.2) to get

$$\begin{cases}\partial_t D^\alpha\rho+v\cdot\nabla D^\alpha\rho+\xi\nabla\cdot D^\alpha u=\Pi_1,\\ \partial_t D^\alpha u+v\cdot\nabla D^\alpha u+\lambda^2\dfrac{p'(\xi)}{\xi}\nabla D^\alpha\rho=\dfrac{\nu}{\xi}(\Delta D^\alpha u+\nabla(D^\alpha\nabla\cdot u))+\dfrac{\mu_1}{\xi}\nabla\cdot D^\alpha\tau+\Pi_2,\\ \partial_t D^\alpha\tau+v\cdot\nabla D^\alpha\tau+aD^\alpha\tau=\Pi_3,\end{cases}\tag{3.3}$$

where

$$\begin{aligned}\Pi_1&=-[D^\alpha(v\cdot\nabla\rho)-v\cdot\nabla D^\alpha\rho]-[D^\alpha(\xi\nabla\cdot u)-\xi\nabla\cdot D^\alpha u],\\ \Pi_2&=-[D^\alpha(v\cdot\nabla u)-v\cdot\nabla D^\alpha u]-\lambda^2\Big[D^\alpha\Big(\frac{p'(\xi)}{\xi}\nabla\rho\Big)-\frac{p'(\xi)}{\xi}\nabla D^\alpha\rho\Big]\\ &\quad+\nu\Big[D^\alpha\Big(\frac{1}{\xi}(\Delta u+\nabla(\nabla\cdot u))\Big)-\frac{1}{\xi}(\Delta D^\alpha u+\nabla(D^\alpha\nabla\cdot u))\Big]\\ &\quad+\mu_1\Big[D^\alpha\Big(\frac{1}{\xi}\nabla\cdot\tau\Big)-\frac{1}{\xi}\nabla\cdot D^\alpha\tau\Big],\\ \Pi_3&=-[D^\alpha(v\cdot\nabla\tau)-v\cdot\nabla D^\alpha\tau]+\mu_2\Big[D^\alpha\Big(\frac{1}{\xi}\Gamma(\nabla u)\Big)-\frac{1}{\xi}\Gamma(D^\alpha\nabla u)\Big]-D^\alpha\Big\{\frac{1}{\xi}Q(\sigma,\nabla u)\Big\}.\end{aligned}$$

Now taking the L^2 inner product of the three equations of (3.3) with $\mu_2\lambda^2\frac{p'(\xi)}{\xi}D^\alpha(\rho-1)$, $\mu_2\xi D^\alpha u$ and $\mu_1\xi D^\alpha\tau$, respectively, we can use integration by parts to get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2\frac{p'(\xi)}{\xi}|D^\alpha(\rho-1)|^2+\xi(\mu_2|D^\alpha u|^2+\mu_1|D^\alpha\tau|^2)\Big]dx\\ &\quad+\ a\mu_1\|D^\alpha\tau\|^2+\nu\mu_2(\|\nabla D^\alpha u\|^2+\|\nabla\cdot D^\alpha u\|^2)\\ &=\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4+\mathrm{I}_5+\mathrm{I}_6+\mathrm{I}_7+\mathrm{I}_8+\mathrm{I}_9+\mathrm{I}_{10}+\mathrm{I}_{11},\end{aligned}\tag{3.4}$$

where

$$\mathrm{I}_1=\frac12\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2|D^\alpha(\rho-1)|^2\partial_t\frac{p'(\xi)}{\xi}+(\mu_2|D^\alpha u|^2+\mu_1|D^\alpha\tau|^2)\partial_t\xi\Big]dx,\tag{3.5}$$

$$\mathrm{I}_2=\frac12\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2|D^\alpha(\rho-1)|^2\nabla\cdot\Big(\frac{p'(\xi)}{\xi}v\Big)+(\mu_2|D^\alpha u|^2+\mu_1|D^\alpha\tau|^2)\nabla\cdot(\xi v)\Big]dx,\tag{3.6}$$

$$\mathrm{I}_3 = \mu_2\lambda^2 \int_{\mathbf{T}^n} p''(\xi) D^\alpha \rho D^\alpha u \cdot \nabla \xi dx, \tag{3.7}$$

$$\mathrm{I}_4 = \mu_1\mu_2 \int_{\mathbf{T}^n} [D^\alpha u_i \partial_j D^\alpha \tau_{ij} + D^\alpha \tau_{ij} D^\alpha \Gamma_{ij}(\nabla u)] dx, \tag{3.8}$$

$$\mathrm{I}_5 = -\mu_2\lambda^2 \int_{\mathbf{T}^n} \frac{p'(\xi)}{\xi} \{[D^\alpha(v\cdot\nabla\rho) - v\cdot\nabla D^\alpha\rho] + [D^\alpha(\xi\nabla\cdot u) - \xi\nabla\cdot D^\alpha u]\} D^\alpha(\rho-1) dx, \tag{3.9}$$

$$\mathrm{I}_6 = -\int_{\mathbf{T}^n} \xi\{\mu_2[D^\alpha(v\cdot\nabla u) - v\cdot\nabla D^\alpha u]\cdot D^\alpha u + \mu_1[D^\alpha(v\cdot\nabla\tau_{ij}) - v\cdot\nabla D^\alpha\tau_{ij}] D^\alpha\tau_{ij}\} dx, \tag{3.10}$$

$$\mathrm{I}_7 = -\mu_2\lambda^2 \int_{\mathbf{T}^n} \Big[D^\alpha\Big(\frac{p'(\xi)}{\xi}\nabla\rho\Big) - \frac{p'(\xi)}{\xi}\nabla D^\alpha \rho\Big] \cdot D^\alpha u dx, \tag{3.11}$$

$$\mathrm{I}_8 = \nu\mu_2 \int_{\mathbf{T}^n} \xi\Big[D^\alpha\Big(\frac{1}{\xi}(\Delta u + \nabla(\nabla\cdot u))\Big) - \frac{1}{\xi}(\Delta D^\alpha u + \nabla(D^\alpha \nabla\cdot u))\Big]\cdot D^\alpha u dx, \tag{3.12}$$

$$\mathrm{I}_9 = \mu_1\mu_2 \int_{\mathbf{T}^n} \xi\Big[D^\alpha\Big(\frac{1}{\xi}\nabla\cdot\tau\Big) - \frac{1}{\xi}\nabla\cdot D^\alpha\tau\Big]\cdot D^\alpha u dx, \tag{3.13}$$

$$\mathrm{I}_{10} = \mu_1\mu_2 \int_{\mathbf{T}^n} \xi\Big[D^\alpha\Big(\frac{1}{\xi}\Gamma(\partial_j u^i)\Big) - \frac{1}{\xi}\Gamma(D^\alpha\partial_j u^i)\Big] D^\alpha \tau_{ij} dx, \tag{3.14}$$

$$\mathrm{I}_{11} = -\mu_1 \int_{\mathbf{T}^n} \xi\Big[D^\alpha\Big\{\frac{1}{\xi}Q(\sigma,\nabla u)\Big\} - \frac{1}{\xi}Q(\sigma, D^\alpha\nabla u)\Big]_{ij} D^\alpha\tau_{ij} dx. \tag{3.15}$$

We point out here that the notations I_j (similarly, J_j in Section 4) will depend on the multi-index α, but this dependence will not be displayed either for reasons of notional convenience.

To estimate the quantities $\mathrm{I}_j,\ 1\le j\le 11$, we need

Lemma 3.1 *Assume $f,\ g\in H^k(\mathbf{T}^n)$, and k is an arbitrary positive integer. Then for any multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$, $|\alpha|\le k$, we have*

$$\begin{cases} \|\nabla^\alpha(fg)\| \le C(\|f\|_\infty\|\nabla^k g\| + \|g\|_\infty\|\nabla^k f\|), \\ \|\nabla^\alpha(fg) - f\nabla^\alpha g\| \le C(\|\nabla f\|_\infty\|\nabla^{k-1}g\| + \|g\|_\infty\|\nabla^{k-1}f\|), \end{cases}$$

where C depends only on k and the space dimension n.

Proof It is an immediate consequence of the well-known Gegliardo-Nirenberg calculus inequality

$$\|\nabla^\alpha f\|_{L^{\frac{2k}{|\alpha|}}} \le C\|f\|_\infty^{1-\frac{|\alpha|}{k}}\|\nabla^r\|^{\frac{|\alpha|}{k}}.$$

For more details, see [10], for example.

We also need the following lemma (see Proposition 1 of Chapter two in [1])

Lemma 3.2 *If $f: R^n\to R$ is a smooth function with $f(0)=0$, then, for any positive integer k, we have*

$$f(u)\in L^\infty\cap H^k \quad \text{and} \quad \|f(u)\|_k \le C\|u\|_k$$

provided $u\in L^\infty\cap H^k$, where C depends only on $f,\ k$ and $\|u\|_\infty$.

Corollary 3.1 *Assume $f,\ k$ are given as in Lemma 3.2, then we have*

$$\|\nabla f(u)\|_{k-1} \le C\|\nabla u\|_{k-1},$$

where C depends only on $f,\ k$ and $\|u\|_\infty$.

Now let us estimate the right-hand side of the above equality (3.4) term by term as follows. By the Sobolev imbedding

$$\begin{aligned}|\mathrm{I}_1| &\le \frac{1}{2}C\|\partial_t\xi\|_\infty \int_{\mathbf{T}^n} \mu_2[\lambda^2|D^\alpha(\rho-1)|^2 + \mu_2|D^\alpha u|^2 + \mu_1|D^\alpha\tau|^2]dx \\ &\le C\lambda^{-1}\|\lambda\partial_t\xi\|_{s-2} \int_{\mathbf{T}^n} \mu_2[\lambda^2|D^\alpha(\rho-1)|^2 + \mu_2|D^\alpha u|^2 + \mu_1|D^\alpha\tau|^2]dx \\ &\le C\lambda^{-1}(\|\lambda D^\alpha(\rho-1)\|^2 + \|D^\alpha u\|^2 + \|D^\alpha\tau\|^2). \end{aligned} \tag{3.16}$$

Similarly, we estimate $\mathrm{I}_2,\ \mathrm{I}_3$ as follows.

$$\begin{aligned}|\mathrm{I}_2| &\le C\lambda^{-1}(\|\lambda\nabla\xi\|_{s-2} + \|\lambda\nabla\cdot v\|_{s-2})(\|\lambda D^\alpha(\rho-1)\|^2 + \|D^\alpha u\|^2 + \|D^\alpha\tau\|^2) \\ &\le C\lambda^{-1}(\|\lambda D^\alpha(\rho-1)\|^2 + \|D^\alpha u\|^2 + \|D^\alpha\tau\|^2), \end{aligned} \tag{3.17}$$

$$\begin{aligned}|\mathrm{I}_3| &\le \mu_2 C\lambda^{-1}\|\lambda^2\nabla\xi\|_{s-1} \int_{\mathbf{T}^n} \lambda^2|D^\alpha(\rho-2)|^2 + |D^\alpha u|^2 dx \\ &\le\ C\lambda^{-1}(\|\lambda D^\alpha(\rho-1)\|^2 + \|D^\alpha u\|^2). \end{aligned} \tag{3.18}$$

To estimate I_4, noting that $\Gamma(u) = \frac{1}{2}(\nabla u +^t \nabla u)$, we can use integration by parts to get

$$\begin{aligned}\mathrm{I}_4 &= \mu_1\mu_2 \int_{\mathbf{T}^n} \Big[D^\alpha u_i \partial_j D^\alpha\tau_{ij} + \frac{1}{2}D^\alpha\tau_{ij}(\partial_j D^\alpha u^i + \partial_i D^\alpha u^j)\Big]dx \\ &= \mu_1\mu_2 \int_{\mathbf{T}^n} [-\partial_j D^\alpha u_i D^\alpha\tau_{ij} + D^\alpha\tau_{ij}\partial_j D^\alpha u^i]dx = 0, \end{aligned} \tag{3.19}$$

where in the second inequality we used the identity that the elastic stress tensor τ is symmetric.

Now we divide the rest estimates into four steps.

Step 1 Estimates when $|\alpha| = 0$ or $D^\alpha = \partial_t$.

When $|\alpha| = 0$, the quantities I_j are all zero except for $\mathrm{I}_1,\ \mathrm{I}_2,\ \mathrm{I}_3$ and I_{11}. Thus we need only to estimate I_{11}. In this case, noting the expression of $Q(\sigma, \nabla u)$ (see (1.2)), we have

$$|\mathrm{I}_{11}| = \mu_1|(Q(\sigma,\nabla u),\tau)| \le C\|\sigma\|_\infty\|\nabla u\|\|\tau\| \le \frac{\nu\mu_2}{8}\|\nabla u\|^2 + C\|\sigma\|_s\|\tau\|^2. \tag{3.20}$$

Now let $D^\alpha = \partial_t$. In this case, it is rather easy to get

$$|\mathrm{I}_5| \le C\|\lambda\rho_t\|(\|\lambda\nabla\rho\|\|v_t\|_\infty + \|\lambda\xi_t\|_\infty\|\nabla u\|) \le C(\|\lambda\rho_t\|^2 + \|\lambda\nabla\rho\|^2 + \|\nabla u\|^2) \tag{3.21}$$

provided λ is appropriately large. By a similar argument, we have

$$\begin{aligned}|\mathrm{I}_6| &\le C\mu_2\|v_t\|_\infty\|u_t\|\|\nabla u\| + C\mu_1\|v_t\|_\infty\|\nabla\tau\|\|\tau_t\| \\ &\le C(\|u_t\|^2 + \|\nabla u\|^2 + \|\tau_t\|^2 + \|\nabla\tau\|^2), \end{aligned} \tag{3.22}$$

$$|\mathrm{I}_7| \le C\mu_2\|\lambda\xi_t\|_\infty\|\lambda\nabla\rho\|\|u_t\| \le C(\|\lambda\nabla\rho\|^2 + \|u_t\|^2), \tag{3.23}$$

$$|\mathrm{I}_8| \le C\nu\mu_2\|\xi_t\|_\infty\|\Delta u\|\|u_t\| \le \frac{\nu\mu_2}{8}\|\Delta u\|^2 + C\|u_t\|^2, \tag{3.24}$$

$$|\mathrm{I}_9| \le C\mu_1\mu_2\|\xi_t\|_\infty\|\nabla\tau\|\|u_t\| \le C(\|\nabla\tau\|^2 + \|u_t\|^2), \tag{3.25}$$

$$|\mathrm{I}_{10}| \le C\mu_1\mu_2\|\xi_t\|\|\nabla u\|\|\tau_t\| \le C(\|\nabla u\|^2 + \|\tau_t\|^2). \tag{3.26}$$

Finally, let us estimate the last term I_{11} as follows:

$$|\mathrm{I}_{11}| \le C\mu_1\|\tau_t\|\Big(\|\nabla u\|\Big\|\xi\partial_t\frac{\sigma}{\xi}\Big\|_\infty + \|\nabla\partial_t u\|\|\sigma\|_\infty\Big) \le C(\|\tau_t\|^2 + \|\nabla u\|^2) + \frac{\nu\mu_2}{8}\|\nabla\partial_t u\|_{s-1}^2. \tag{3.27}$$

Step 2 Estimates when $D^\alpha = \nabla^\alpha,\ |\alpha| = s$.

By using Lemma 3.1, we obtain

$$\begin{aligned}
|\mathrm{I}_5| &\le C\|\lambda\nabla^\alpha(\rho-1)\|[(\|\nabla v\|_\infty\|\lambda\nabla\rho\|_{s-1} + \|\nabla v\|_{s-1}\|\lambda\nabla\rho\|_\infty) \\
&\quad + (\|\lambda\nabla\xi\|_\infty\|\|\nabla u\|_{s-1} + \|\lambda\nabla\xi\|_{s-1}\|\|\nabla u\|_\infty)] \\
&\le C\|\lambda\nabla^\alpha(\rho-1)\|(\|\nabla v\|_{s-1}\|\lambda\nabla\rho\|_{s-1} + \|\lambda\nabla\xi\|_{s-1}\|\|\nabla u\|_{s-1}) \\
&\le C(\|\lambda\nabla^\alpha(\rho-1)\|^2 + \|u\|_{s-1}^s),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
|\mathrm{I}_6| &\le C\mu_2\|\nabla^\alpha u\|(\|\nabla v\|_\infty\|\nabla u\|_{s-1} + \|\nabla u\|_\infty\|\nabla v\|_{s-1}) \\
&\quad + C\mu_1\|\nabla^\alpha\tau\|(\|\nabla v\|_\infty\|\nabla\tau\|_{s-1} + \|\nabla\tau\|_\infty\|\nabla v\|_{s-1}) \\
&\le C(\|u\|_s^2 + \|\tau\|_s^2).
\end{aligned} \tag{3.29}$$

Next we use Corollary 3.1 to estimate I_7–I_{10}.

$$\begin{aligned}
|\mathrm{I}_7| &\le C\mu_2\|\nabla^\alpha u\|(\|\lambda\nabla\xi\|_\infty\|\lambda\nabla\rho\|_{s-1} + \|\lambda\nabla\rho\|_\infty\|\lambda\nabla\xi\|_{s-1}) \\
&\le C(\|\nabla^\alpha u\|^2 + \|\lambda(\nabla-1)\rho\|_s^2),
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
|\mathrm{I}_8| &\le C\nu\mu_2\|\nabla^\alpha u\|[\|\nabla\xi\|_\infty(\|\Delta u\|_{s-1} + \|\nabla\nabla\cdot u\|_{s-1}) + (\|\Delta u\|_\infty + \|\nabla\nabla\cdot u\|_\infty)\|\nabla\xi\|_{s-1}] \\
&\le C\|u\|_s^2 + \frac{\nu\mu_2}{8}(\|\nabla u\|_s^2 + \|\nabla\cdot u\|_s^2),
\end{aligned} \tag{3.31}$$

$$|\mathrm{I}_9| \le C\mu_1\mu_2\|\nabla^\alpha u\|(\|\nabla\xi\|_\infty\|\nabla\tau\|_{s-1} + \|\nabla\tau\|_\infty\|\nabla\xi\|_{s-1}) \le C(\|u\|_s^2 + \|\tau\|_s^2), \tag{3.32}$$

$$|\mathrm{I}_{10}| \le C\mu_1\mu_2\|\nabla^\alpha\tau\|(\|\nabla\xi\|_\infty\|\nabla u\|_{s-1} + \|\nabla u\|_\infty\|\nabla\xi\|_{s-1}) \le C(\|u\|_s^2 + \|\tau\|_s^2). \tag{3.33}$$

Now let us estimate the last term I_{11}. Recalling the definition of $Q(\sigma,\nabla u)$ in (1.2), we see that

$$\begin{aligned}
|\mathrm{I}_{11}| &\le C\|\nabla^\alpha\tau\|\Big\|\xi\nabla^\alpha\Big\{\frac{1}{\xi}Q(\sigma,\nabla u)\Big\}\Big\| \le C\|\nabla^\alpha\tau\|\Big(\Big\|\frac{\sigma}{\xi}\Big\|_\infty\|\nabla u)\|_s + \|\nabla u\|_\infty\Big\|\frac{\sigma}{\xi}\Big\|_s\Big) \\
&\le C(\|u\|_s^2 + \|\tau\|_s^2) + \frac{\nu\mu_2}{8}\|\nabla u\|_s^2.
\end{aligned} \tag{3.34}$$

Step 3 Estimates when $D^\alpha = \nabla^\beta\partial_t$, $|\beta| = s-1$.

With the aid of Lemma 3.1 and Corollary 3.1, we have

$$
\begin{aligned}
|\mathrm{I}_5| \le{}& C\lambda\|\lambda\nabla^\beta\rho_t\|\{[\|\nabla^\beta(v\cdot\nabla\rho_t)-v\cdot\nabla\nabla^\beta\rho_t\|]+\|\nabla^\beta(v_t\cdot\nabla\rho)\|]\\
&+[\|\nabla^\beta(\xi\nabla\cdot u_t)-\xi\nabla\cdot\nabla^\beta u_t\|+\|\nabla^\beta(\xi_t\cdot\nabla u)\|]\}\\
\le{}& C\|\lambda\nabla^\beta\rho_t\|[(\|\nabla v\|_\infty\|\lambda\nabla\rho_t\|_{s-2}+\|\nabla v\|_{s-2}\|\lambda\nabla\rho_t\|_\infty)+(\|v_t\|_\infty\|\lambda\nabla\rho\|_{s-1}\\
&+\|v_t\|_{s-1}\|\lambda\nabla\rho\|_\infty)+(\|\nabla u_t\|_\infty\|\lambda\nabla\xi\|_{s-2}+\|\nabla u_t\|_{s-2}\|\lambda\nabla\xi\|_\infty)\\
&+(\|\nabla u\|_\infty\|\lambda\xi_t\|_{s-1}+\|\nabla u\|_{s-1}\|\lambda\xi_t\|_\infty)\\
\le{}& C(\|\lambda\rho_t\|_{s-1}^2+\|\lambda(\rho-1)\|_s^2+\|u\|_{s-1}^2+\|u_t\|_s^2).
\end{aligned}\tag{3.35}
$$

Following the same procedure, we can get

$$
\begin{aligned}
|\mathrm{I}_6| \le{}& C\|\nabla^\beta u_t\|(\|\nabla v\|_\infty\|\nabla u_t\|_{s-2}+\|\nabla u_t\|_\infty\|\nabla v\|_{s-2}+\|v_t\|_\infty\|\nabla u\|_{s-1}+\|\nabla u\|_\infty\|v_t\|_{s-1})\\
&+\|\nabla^\beta\tau_t\|(\|\nabla v\|_\infty\|\nabla\tau_t\|_{s-2}+\|\nabla\tau_t\|_\infty\|\nabla v\|_{s-2}+\|\nabla\tau\|_\infty\|\nabla v_t\|_{s-1}+\|\nabla v_t\|_\infty\|\nabla\tau\|_{s-1})\\
\le{}& C(\|\tau_t\|_{s-1}^2+\|\tau\|_s^2+\|u_t\|_{s-1}^2+\|u\|_s^2),
\end{aligned}\tag{3.36}
$$

$$
\begin{aligned}
|\mathrm{I}_7| \le{}& C\lambda^2\|\nabla^\beta u_t\|(\|\nabla\xi\|_\infty\|\nabla\rho_t\|_{s-2}+\|\nabla\rho_t\|_\infty\|\nabla\xi\|_{s-2}\|\xi_t\|_\infty\|\nabla\rho\|_{s-1}+\|\nabla\rho\|_\infty\|\xi_t\|_{s-1})\\
\le{}& C(\|\lambda\rho_t\|_{s-1}^2+\|\lambda(\rho-1)\|_s^2+\|u_t\|_{s-1}^2),
\end{aligned}\tag{3.37}
$$

$$
\begin{aligned}
|\mathrm{I}_8| \le{}& C\|\nabla^\beta u_t\|[\|\nabla\xi\|_\infty(\|\Delta u_t\|_{s-2}+\|\nabla\nabla\cdot u_t\|_{s-2})+\|\nabla\xi\|_{s-2}(\|\Delta u_t\|_\infty+\|\nabla\nabla\cdot u_t\|_\infty)\\
&+\|\xi_t\|_\infty(\|\Delta u\|_{s-1}+\|\nabla\nabla\cdot u\|_{s-1})+\|\xi_t\|_{s-1}(\|\Delta u\|_\infty+\|\nabla\nabla\cdot u\|_\infty)]\\
\le{}& C(\|u_t\|_{s-1}^2+\|u\|_s^2)+\frac{\nu\mu_2}{8}(\|\nabla\partial_t u\|_{s-1}^2+\|\nabla u\|_s^2),
\end{aligned}\tag{3.38}
$$

$$
\begin{aligned}
|\mathrm{I}_9| \le{}& C\|\nabla^\beta u_t\|(\|\nabla\xi\|_\infty\|\nabla\tau_t\|_{s-2}+\|\nabla\xi\|_{s-2}\|\nabla\tau_t\|_\infty+\|\xi_t\|_\infty\|\nabla\tau\|_{s-1}+\|\xi_t\|_{s-1}\|\nabla\tau\|_\infty)\\
\le{}& C(\|u_t\|_{s-1}^2+\|\tau\|_s^2+\|\tau_t\|_{s-1}^2),
\end{aligned}\tag{3.39}
$$

$$
\begin{aligned}
|\mathrm{I}_{10}| \le{}& C\|\nabla^\beta\tau_t\|(\|\nabla\xi\|_\infty\|\nabla u_t\|_{s-2}+\|\nabla\xi\|_{s-2}\|\nabla u_t\|_\infty+\|\xi_t\|_\infty\|\nabla u\|_{s-1}+\|\xi_t\|_{s-1}\|\nabla u\|_\infty)\\
\le{}& C(\|u\|_s^2+\|u_t\|_{s-1}^2+\|\tau_t\|_{s-1}^2),
\end{aligned}\tag{3.40}
$$

$$
\begin{aligned}
|\mathrm{I}_{11}| \le{}& C\|\nabla^\beta\tau_t\|\Big(\Big\|\partial_t\frac{\sigma}{\xi}\Big\|_\infty\|\nabla u\|_{s-1}+\Big\|\partial_t\frac{\sigma}{\xi}\Big\|_{s-1}\|\nabla u\|_\infty+\Big\|\frac{\sigma}{\xi}\Big\|_\infty\|\nabla\partial_t u\|_{s-1}\Big)+\Big\|\frac{\sigma}{\xi}\Big\|_{s-1}\|\nabla\partial_t u\|_\infty\\
\le{}& C(\|\tau_t\|_{s-1}^2+\|u\|_s^2)+\frac{\nu\mu_2}{8}\|\nabla u_t\|_{s-1}^2.
\end{aligned}\tag{3.41}
$$

Step 4 Energy estimates.

We now tie everything together. Substituting the estimates (3.16)–(3.20) and (3.28)–(3.34) into (3.4), we find that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|=0,s}\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2\frac{p'(\xi)}{\xi}|\nabla^\alpha(\rho-1)|^2+\xi(\mu_2|\nabla^\alpha u|^2+\mu_1|\nabla^\alpha\tau|^2)\Big]dx\\
&\quad+a\mu_1\|\nabla^\alpha\tau\|^2+\nu\mu_2\|\nabla\nabla^\alpha u\|^2\\
&\le C(\|\lambda(\rho-1)\|_s^2+\|\tau\|_s^2+\|u\|_s^2).
\end{aligned}\tag{3.42}
$$

Recalling the constraints on the initial data (2.5), we have

$$
E_s(U(0))=\|\lambda\tilde\rho_0^\lambda\|_s^2+\|u_0+\tilde u_0^\lambda\|_s^2+\|\tau_0+\tilde\tau_0^\lambda\|^2\le C\Big(\frac{\delta_0^2}{\lambda^2}+\|u_0\|_s^2+\|\tau_0\|_s^2\Big).\tag{3.43}
$$

Noting (2.2), we can use the Gronwall's inequality to obtain

$$E_s(U(t)) + \int_0^t (\|\tau\|_s^2 + \|\nabla u\|_s^2)\ dt \leq 4(\|u_0\|_s^2 + \|\tau_0\|_s^2) \quad \text{for } 0 \leq t \leq T_0 \tag{3.44}$$

provided T_0 is sufficiently small and λ is sufficiently large.

Substituting these estimates (3.16)–(3.19), (3.21)–(3.27), (3.35)–(3.41) and (3.44) into (3.4), we have

$$\frac{d}{dt}[E(\partial_t U(t)) + E_{s-1}(\partial_t U(t))] + (\|\nabla \partial_t u\|_{s-1}^2 + \|\partial_t \tau\|_{s-1}^2) \leq C + CE_{s-1}(\partial_t U(t)). \tag{3.45}$$

Similarly to (3.43), we have

$$\begin{aligned} E_{s-1}(\partial_t U(0)) \leq C \|&\lambda(u_0 + \tilde{u}_0^\lambda)\nabla\tilde{\rho}_0^\lambda\|_{s-1}^2 + \|\lambda(\tilde{\rho}_0^\lambda + 1)\nabla \cdot \tilde{u}_0^\lambda\|_{s-1}^2 \\ &+ \|\lambda^2 \nabla \tilde{\rho}_0^\lambda\|_{s-1}^2 + \|(u_0 + \tilde{u}_0^\lambda) \cdot \nabla(u_0 + \tilde{u}_0^\lambda)\|_{s-1}^2 \\ &+ \|(\Delta(u_0 + \tilde{u}_0^\lambda) + \nabla(\nabla \cdot \tilde{u}_0^\lambda)) + \nabla \cdot (\tau_0 + \tilde{\tau}_0^\lambda)\|_{s-1}^2 \\ &+ \|(u_0 + \tilde{u}_0^\lambda) \cdot \nabla(\tau_0 + \tilde{\tau}_0^\lambda) + a(\tau_0 + \tilde{\tau}_0^\lambda) \\ &+ Q(\tau_0 + \tilde{\tau}_0^\lambda, \nabla u) - \Gamma(\nabla u_0 + \tilde{u}_0^\lambda)\|_{s-1}^2 \\ \leq C&(\delta_0^2 + \|u_0\|_{s+1}^2 + \|\tau_0\|_s^2). \end{aligned} \tag{3.46}$$

Gronwall's inequality together with (3.45) and (3.46) yields

$$E_{s-1}(\partial_t U(t)) + \int_0^t (\|\nabla \partial_t u\|_{s-1}^2 + \|\partial_t \tau\|_{s-1}^2)\ dt \leq C \exp Ct \quad \text{for } 0 \leq t \leq T_0 \tag{3.47}$$

provided T_0 and δ_0 are sufficiently small and λ is sufficiently large.

Summing up, we have

Lemma 3.3 *Assume that $B_{T_0}^\lambda(U_0)$ is defined by (3.1) and $\Lambda : V \to U$ is defined by the linear partial differential equation (3.2). Then there exist uniform constants T_0, δ and κ independent of λ such that Λ maps $B_{T_0}^\lambda(U_0)$ into itself. Moreover it is a contraction in the L^2-norm.*

Proof It remains to show the first line of (3.1) and the L^2-contraction. To show the first line of (3.1), it is suffices to show $\|\lambda(\rho-1)\|_s + \|u-u_0\|_s + \|\tau-\tau_0\|_s \leq \delta$ by Sobolev's inequality. Let $\tilde{\rho} = \rho - 1$, $\tilde{u} = u - u_0$, $\tilde{\tau} = \tau - \tau_0$, we proceed as before. Similarly, to obtain (3.44), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbf{T}^n} \mu_2\Big[\lambda^2 \frac{p'(\xi)}{\xi}|\tilde{\rho}|_s^2 + \xi(\mu_2|\nabla^\alpha \tilde{u}|_s^2 + \mu_1|\tilde{\tau}|_s^2)\Big]dx + a\mu_1\|\tilde{\tau}\|_s^2 + \nu\mu_2(\|\nabla\tilde{u}\|_s^2 + \|\nabla \cdot \tilde{u}\|_s^2) \\ &\leq C(\|\lambda\tilde{\rho}\|_s^2 + \|\tilde{\tau}\|_s^2 + \|\tilde{u}\|_s^2) + C\Big(\|v \cdot \nabla u_0\|_s^2 + \|\nabla u_0\|_s^2 + \Big\|\nabla^s\Big(\frac{1}{\xi}\Delta u_0\Big) - \frac{1}{\xi}\nabla^s \Delta u_0\Big\|^2 \\ &\quad + a\|\tau_0\|_s^2 + \|Q(\sigma, \nabla u_0)\|_s^2 + \|\Gamma(\nabla u_0)\|_s^2\Big) \\ &\leq C(\|\lambda\tilde{\rho}\|_s^2 + \|\tilde{\tau}\|_s^2 + \|\tilde{u}\|_s^2 + 1). \end{aligned} \tag{3.48}$$

Since

$$E(\widetilde{U}(0)) = \|\lambda\tilde{\rho}_0^\lambda\|_s^2 + \|\tilde{u}_0^\lambda\|_s^2 + \|\tilde{\tau}_0^\lambda\|^2 \leq C\frac{\delta_0^2}{\lambda^2},$$

we conclude, by Gronwall's inequality, that

$$E(\widetilde{U}(t)) \leq (E(\widetilde{U}(0)) + CT_0) \exp CT_0.$$

Thus $U = \Lambda(V) \in B^\lambda_{T_0}(U_0)$ if λ is sufficiently large and T_0 is sufficiently small.

Next we show that Λ is a contraction in $B^\lambda_{T_0}(U_0)$ with respect to the L^2-norm. Let $U = \Lambda(V),\ \overline{U} = \Lambda(\overline{V}),\ V,\ \overline{V} \in B^\lambda_{T_0}(U_0)$, we have $U,\ \overline{U} \in B^\lambda_{T_0}(U_0)$. Moreover, by the definition of Λ, we have

$$\begin{cases} \partial_t(\rho - \bar\rho) + v\cdot\nabla(\rho-\bar\rho) + \xi\nabla\cdot(u-\bar u) + (v-\bar v)\cdot\nabla\bar\rho + (\xi-\bar\xi)\nabla\cdot\bar u = 0, \\ \partial_t(u-\bar u) + v\cdot\nabla(u-\bar u) + \dfrac{p'(\xi)}{\xi}\lambda^2\nabla(\rho-\bar\rho) + (v-\bar v)\cdot\nabla\bar u + \lambda^2\Big(\dfrac{p'(\xi)}{\xi} - \dfrac{p'(\bar\xi)}{\bar\xi}\Big)\nabla\bar\rho \\ = \dfrac{\nu}{\xi}(\Delta(u-\bar u) + \nabla(\nabla\cdot(u-\bar u))) + \dfrac{\mu_1}{\xi}\nabla\cdot(\tau-\bar\tau) \\ +\nu\Big(\dfrac{1}{\xi} - \dfrac{1}{\bar\xi}\Big)(\Delta\bar u + \nabla\nabla\cdot\bar u) + \mu_1\Big(\dfrac{1}{\xi}-\dfrac{1}{\bar\xi}\Big)\nabla\cdot\bar\tau, \\ \partial_t(\tau-\bar\tau) + v\cdot\nabla(\tau-\bar\tau) + a(\tau-\bar\tau) + \Big[\dfrac{1}{\xi}Q(\sigma,\nabla u) - \dfrac{1}{\bar\xi}Q(\bar\sigma,\nabla\bar u)\Big] + (v-\bar v)\cdot\nabla\bar\tau \\ = \dfrac{\mu_2}{\xi}\Gamma(\nabla(u-\bar u)) + \mu_2\Big(\dfrac{1}{\xi}-\dfrac{1}{\bar\xi}\Big)\Gamma(\bar u). \end{cases} \tag{3.49}$$

Proceeding as before in the derivation of (3.4) and in the estimation of (3.16)–(3.19), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2\frac{p'(\xi)}{\xi}|\rho-\bar\rho|^2 + \xi(\mu_2|u-\bar u|^2 + \mu_1|\tau-\bar\tau|^2)\Big]\,dx + a\mu_1\|\tau-\bar\tau\|^2 \\ &+\ \nu\mu_2(\|\nabla(u-\bar u)\|^2 + \|\nabla\cdot(u-\bar u)\|^2) \\ \leq{}& C(\|\rho-\bar\rho\|^2 + \|u-\bar u\|^2 + \|\tau-\bar\tau\|^2 + \|\xi-\bar\xi\|^2 + \|v-\bar v\|^2) \\ &+ (Q(\sigma,\nabla(u-\bar u)),\mu_1(\tau-\bar\tau)) + (Q(\sigma-\bar\sigma,\nabla\bar u),\mu_1(\tau-\bar\tau)) \\ &+ \Big(Q(\sigma,\nabla\bar u)\Big(\frac{1}{\xi}-\frac{1}{\bar\xi}\Big),\mu_1\xi(\tau-\bar\tau)\Big) \\ \leq{}& C(\|\rho-\bar\rho\|^2 + \|u-\bar u\|^2 + \|\tau-\bar\tau\|^2 + \|\xi-\bar\xi\|^2 + \|v-\bar v\|^2 + \|\sigma-\bar\sigma\|^2) + \frac{\nu\mu_2}{2}\|\nabla(u-\bar u)\|^2. \end{aligned}$$

Noting that $(U-\overline{U})(0) = 0$, by the Gronwall's inequality, we obtain

$$\max_{0\leq t\leq T_0} E(U(t)-\overline{U}(t)) \leq (CT_0\exp CT_0)\max_{0\leq t\leq T_0} E(V(t)-\overline{V}(t)). \tag{3.50}$$

We complete the proof of Lemma 3.3.

Before proving Theorem 2.1, we also need the following lemma.

Lemma 3.4 *Consider the incompressible system of Oldroyd type* (2.8) *with the initial condition* (2.9)*. Then there exists at most one smooth solution.*

Proof Assume that $(u,\ \tau)$ and $(\bar u,\ \bar\tau)$ are two smooth solutions of (1.1) with the same initial data (2.9). Then we have

$$\begin{cases} \nabla\cdot(u-\bar u)=0,\\ \partial_t(u-\bar u)+u\cdot\nabla(u-\bar u)+\nabla(q-\bar q)+(u-\bar u)\cdot\nabla\bar u=\nu\Delta(u-\bar u)+\mu_1\nabla\cdot(\tau-\bar\tau),\\ \partial_t(\tau-\bar\tau)+u\cdot\nabla(\tau-\bar\tau)+a(\tau-\bar\tau)+Q(\tau,\nabla(u-\bar u))+(u-\bar u)\cdot\nabla\bar\tau+Q(\tau-\bar\tau,\nabla\bar u)\\ =\mu_2\Gamma(u-\bar u)\end{cases}$$

for $0\le t\le t_0$. Thus proceeding as before we can get

$$\begin{aligned}&\frac12\frac{d}{dt}(\mu_2\|u-\bar u\|^2+\mu_1\|\tau-\bar\tau\|^2)+a\mu_1\|\tau-\bar\tau\|^2+\nu\mu_2\|\nabla u-\nabla\bar u\|^2\\ \le{}& C(\|u-\bar u\|^2+\|\tau-\bar\tau\|^2)+\frac{\nu\mu_2}{2}\|\nabla u-\nabla\bar u\|^2.\end{aligned}$$

Noting that $(u-\bar u,\ \tau-\bar\tau)(0)=0$, with the aid of Gronwall's inequality we obtain

$$\|u-\bar u\|^2+\|\tau-\bar\tau\|^2=0,\quad 0\le t\le t_0,$$

which means that $u=\bar u,\ \tau=\bar\tau$.

Proof of Theorem 2.1 For any fixed λ, the standard classical iteration procedure produces a sequence (with slight change in notation) $\{(\rho_j,\ u_j,\ \tau_j)\}_{j=0}^\infty$ which belongs to $C([0,T_0],H^s)\cap C^1([0,T_0],H^{s-1})$ and is guaranteed to converge to a limit $(\rho^\lambda,\ u^\lambda,\ \tau^\lambda)$ by the above Lemma 3.3. The convergent sequence satisfies the following equations

$$\begin{cases}\partial_t\rho_{j+1}+u_j\cdot\nabla\rho_{j+1}+\rho_j\nabla\cdot u_{j+1}=0,\\ \partial_t u_{j+1}+u_j\cdot\nabla u_{j+1}+\dfrac{p'(\rho_j)}{\rho_j}\lambda^2\nabla\rho_{j+1}=\dfrac{\nu}{\rho_j}(\Delta u_{j+1}+\nabla(\nabla\cdot u_{j+1}))+\dfrac{\mu_1}{\rho_j}\nabla\cdot\tau_{j+1},\\ \partial_t\tau_{j+1}+u_j\cdot\nabla\tau_{j+1}+a\tau_{j+1}+\dfrac{1}{\rho_j}Q(\sigma_j,\nabla u_{j+1})=\dfrac{\mu_2}{\rho_j}\Gamma(\nabla u_{j+1})\end{cases}$$

as well as the estimates

$$\begin{cases}E_s(U_j(t))+\displaystyle\int_0^t(\|\tau_j\|_s^2+\|\nabla u_j\|_s^2)dt\le 4(\|u_0\|_s^2+\|\tau_0\|_s^2),\\ E_{s-1}(\partial_tU_j(t))+\displaystyle\int_0^t(\|\nabla\partial_t u_j\|_{s-1}^2+\|\partial_t\tau_j\|_{s-1}^2)dt\le\kappa,\end{cases}$$

where $U_j=(\rho_j,\ u_j,\ \tau_j)^T$. It follows obviously that $(\rho^\lambda,\ u^\lambda,\ \tau^\lambda)\in L^\infty(0,T_0,H^s)\cap \mathrm{Lip}([0,T_0],H^{s-1})$ and satisfies the estimates (2.6). From the standard Sobolev interpolation inequalities

$$\|(\rho_j,u_j,\tau_j)-(\rho^\lambda,u^\lambda,\tau^\lambda)\|_{s-\delta}\le C\|(\rho_j,u_j,\tau_j)-(\rho^\lambda,u^\lambda,\tau^\lambda)\|^\theta(\|(\rho_j,u_j,\tau_j)\|_s+\|(\rho^\lambda,u^\lambda,\tau^\lambda)\|_s)^{1-\theta}$$

for an appropriate $\theta\in(0,1)$, where δ is a small positive constant. With the aid of (3.50), we also have $(\rho^\lambda,\ u^\lambda,\ \tau^\lambda)\in C([0,T_0],H^{s-\delta})$. By Sobolev imbedding theorem, $(\rho^\lambda,\ u^\lambda,\ \tau^\lambda)$ is a classical solution of the compressible Oldroyd model (1.3). This completes the proof of the uniform stability part of Theorem 2.1.

From the uniform stability estimates (2.6), we have, as $\lambda\to\infty$, that $\rho^\lambda\to 1$ in $L^\infty(0,T_0,H^s)$ $\cap\mathrm{Lip}([0,T_0],H^{s-1})$. Moreover, a standard compactness argument based on the Lions-Aubin lemma (see [19], for example) implies that any subsequence of u^λ and τ^λ has a subsequence (still denoted by u^λ and τ^λ) with a limit u and τ, respectively, with $(u,\tau) \in L^\infty(0,T_0,H^s)\cap C([0,T_0],H^{s-\delta})$, $(u_t,\tau_t)\in L^\infty(0,T_0,H^{s-1})$ satisfying (2.7). Now let $\phi(t,x)$ and $\varphi(t,x)$ be two smooth test functions with compact supports in $t\in[0,T_0]$ and $\nabla\cdot\phi=0$. Then we have

$$\begin{aligned}&\int_0^{T_0}\int_{\mathbf{T}^n}\phi\Big(\partial_t u^\lambda+u^\lambda\cdot\nabla u^\lambda-\frac{\nu}{\rho^\lambda}\Delta u^\lambda-\frac{\mu_1}{\rho^\lambda}\nabla\cdot\tau^\lambda-\frac{\nu}{\rho^\lambda}\nabla\nabla\cdot u^\lambda\Big)\,dxdt\\ =&\int_0^{T_0}\int_{\mathbf{T}^n}-\phi\lambda^2\nabla\int_1^\rho\frac{p'(\xi)}{\xi}\,d\xi\,dxdt=0\end{aligned}$$

and

$$\int_0^{T_0}\int_{\mathbf{T}^n}\varphi\Big(\partial_t\tau^\lambda+u^\lambda\cdot\nabla\tau^\lambda+a\tau^\lambda+\frac{1}{\rho^\lambda}Q(\tau^\lambda,\nabla u^\lambda)-\frac{\mu_2}{\rho^\lambda}\Gamma(\nabla u^\lambda)\Big)\,dxdt=0.$$

On the other hand, from the first equation of system (1.3) we have

$$\nabla\cdot u^\lambda=\partial_t\rho^\lambda+u^\lambda\cdot\nabla\rho^\lambda+(\rho^\lambda-1)\nabla\cdot u^\lambda. \tag{3.51}$$

Let $\lambda\to\infty$, we obtain that $(u,\ \tau)$ satisfies (2.8) and (2.9). From Lemma 3.4, by the uniqueness of the classical solution of system (2.8) with the initial data (2.9), it follows that the convergence is in fact the sequences u^λ and τ^λ themselves.

Remark 3.1 It follows from (3.51) that

$$\begin{aligned}\|\lambda\nabla\cdot u^\lambda\|_{s-1}&\le\|\lambda\rho_t^\lambda\|_{s-1}+\|\lambda u^\lambda\cdot\nabla\rho^\lambda\|_{s-1}+\|\lambda(\rho^\lambda-1)\nabla\cdot u^\lambda\|_{s-1}\\&\le C(\|\lambda\rho_t^\lambda\|_{s-1}+\|\lambda(\rho^\lambda-1)\|\le C\kappa\quad\text{for } 0\le t\le T_0.\end{aligned}$$

Noting Remark 2.1, we in fact have that

$$\lambda(\|\nabla\cdot u\|_{s-1}+\|\rho_t^\lambda\|_{s-1}+\|\rho^\lambda-1\|_s)+\lambda^2\Big(\|\nabla\rho^\lambda\|_{s-2}+\int_0^t\|\nabla\rho^\lambda\|_{s-1}^2\,dt\Big)\le C\kappa \tag{3.52}$$

for $0\le t\le T_0$.

4 Proof of Theorems 2.2 and 2.3

In this section we devote ourselves to getting a priori dispersive energy estimates for any fixed $\overline{T}>0$ and small initial displacements, and to proving Theorems 2.2 and 2.3. What we need to do is just to improve the previous estimates (3.16)–(3.41) to get the desired dispersive energy estimates.

Now we go back to equation (3.3) and replace the function $(\xi,\ v,\ \sigma)$ by $(\rho,\ u,\ \tau)$ to get a similar equality as (3.4)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^n}\mu_2\Big[\lambda^2\frac{p'(\rho)}{\rho}|D^\alpha(\rho-1)|^2+\rho(\mu_2|D^\alpha u|^2+\mu_1|D^\alpha\tau|^2)\Big]dx\\&+a\mu_1\|D^\alpha\tau\|^2+\nu\mu_2(\|\nabla D^\alpha u\|^2+\|\nabla\cdot D^\alpha u\|^2)=\sum_{1\le j\le 11}\mathrm{J}_j,\end{aligned} \tag{4.1}$$

where $\mathrm{J}_j,\ 1\le j\le 11$, have the similar forms as $\mathrm{I}_j,\ 1\le j\le 11$, in (3.4) with $(\xi,\ v,\ \sigma)$ being replaced by $(\rho,\ u,\ \tau)$.

Since we do not need the decay for estimating the time derivatives $E(\partial_t U(t))$, and inequalities (3.16)–(3.19) have been of the desired form, we only need to improve (3.20) and (3.28)–(3.34) to get the corresponding estimates for J_j.

First, by (2.10), (3.20) can be improved as follows:

$$|\mathrm{J}_{11}| = \mu_1 |(Q(\tau, \nabla u), \tau)| \le \frac{\nu\mu_2}{8}\|\nabla u\|^2 + C\varepsilon\|\tau\|^2. \tag{4.2}$$

Next, noting (2.10), we can improve (3.28) and (3.29) as follows:

$$\begin{aligned}
|\mathrm{J}_5| \le{}& C\lambda^{-1}\|\lambda\nabla^\alpha(\rho-1)\|(\|\nabla u\|_{s-1}\|\lambda^2\nabla\rho\|_\infty + \|\nabla u\|_\infty\|\lambda^2\nabla\rho\|_{s-1} \\
&+ \|\lambda^2\nabla\rho\|_{s-1}\|\|\nabla u\|_\infty + \|\lambda^2\nabla\rho\|_\infty\|\|\nabla u\|_{s-1}) \\
\le{}& C\lambda^{-1}(\|\lambda^2(\rho-1)\|_{s-1}^2 + \|\lambda(\rho-1)\|_s^2), \qquad (4.3)\\
|\mathrm{J}_6| \le{}& C\|u\|_s\|\nabla u\|_\infty\|\nabla u\|_{s-1} + C\|\tau\|_s(\|\nabla u\|_\infty\|\nabla\tau\|_{s-1} + \|\nabla\tau\|_\infty\|\nabla u\|_{s-1}) \\
\le{}& C\varepsilon(\|\nabla u\|_{s-1}^2 + \|\tau\|_s^2). \qquad (4.4)
\end{aligned}$$

By using Lemma 3.1 and Corollary 3.1, we have

$$\begin{aligned}
|\mathrm{J}_7| \le{}& C\lambda^{-1}\|u\|_s\|\lambda^2\nabla\rho\|_\infty\|\lambda\nabla\rho\|_{s-1} \le C\lambda^{-1}\|\lambda^2\nabla\rho\|_{s-2}(\|u\|_s^2 + \|\lambda(\rho-1)\|_s^2) \\
\le{}& C\lambda^{-1}(\|u\|_s^2 + \|\lambda(\rho-1)\|_s^2),
\end{aligned} \tag{4.5}$$

where in the last inequality we used (3.52).

Similarly, the rest inequalities (3.31)–(3.33) can be improved as follows:

$$|\mathrm{J}_8| \le C\lambda^{-1}\|u\|_s(\|\lambda\nabla\rho\|_\infty\|\Delta u\|_{s-1} + \|\lambda\nabla\rho\|_{s-1}\|\Delta u\|_\infty) \le C\lambda^{-1}(\|u\|_s^2 + \|\nabla u\|_s^2), \tag{4.6}$$

$$|\mathrm{J}_9| \le C\lambda^{-1}\|u\|_s(\|\lambda\nabla\rho\|_\infty\|\nabla\tau\|_{s-1} + \|\lambda\nabla\rho\|_{s-1}\|\nabla\tau\|_\infty) \le C\lambda^{-1}(\|u\|_s^2 + \|\tau\|_s^2), \tag{4.7}$$

$$|\mathrm{J}_{10}| \le C\lambda^{-1}\|\tau\|_s(\|\lambda\nabla\rho\|_\infty\|\nabla u\|_{s-1} + \|\lambda\nabla\rho\|_{s-1}\|\nabla u\|_\infty) \le C\lambda^{-1}(\|u\|_s^2 + \|\tau\|_s^2). \tag{4.8}$$

At last, noting $\rho - 1$ is small, we can improve (3.34) as follows:

$$|J_{11}| \le C\|\tau\|_s\Big(\Big\|\nabla\frac{\tau}{\rho}\Big\|_\infty\|\nabla u\|_{s-1} + \Big\|\nabla\frac{\tau}{\rho}\Big\|_{s-1}\|\nabla u\|_\infty\Big) \le C\varepsilon(\|\tau\|_s^2 + \|\nabla u\|_{s-1}^2). \tag{4.9}$$

Now adding up all these estimates (3.16)–(3.19) (with $(\xi,\ v,\ \sigma)$ being replaced by $(\rho,\ u,\ \tau)$) and (4.1)–(4.9) together and noting (2.2) and (3.52), we obtain

$$\frac{d}{dt}[\widetilde{E}(U(t)) + \widetilde{E}(\nabla^s U(t))] + (\|\nabla u\|_s^2 + \|\tau\|_s^2) \le C\lambda^{-1}E_s(U(t)), \tag{4.10}$$

which together with (3.43) gives

$$E_s(U(t)) + \int_0^t (\|\nabla u\|_s^2 + \|\tau\|_s^2)\ dt \le 4(\|u_0\|_s^2 + \|\tau_0\|_s^2) \tag{4.11}$$

provided λ is sufficiently large and $0 \le t \le T^\lambda$, with $T^\lambda = \frac{\lambda}{C}$.

Next we show the second inequality of (2.11). For this thought estimate, we do not need the decay. Thus all estimates are almost the same as (3.21)–(3.27) and (3.35)–(3.41) with $(\xi,\ v,\ \sigma)$ replaced by $(\rho,\ u,\ \tau)$. Thus (3.47) holds with $U(t) = (\rho,\ u,\ \tau)^T$. We complete the proof of Theorem 2.2.

By (2.11), we can proceed as in Section 3, by taking smooth test functions, to prove that the smooth limiting function (ρ, u, τ) and ∇q satisfy the incompressible Oldroyd-B system (1.1) in the time interval $[0, \overline{T}]$ with the initial data (2.9) satisfying the constraints (2.4) and (2.10). Since $\overline{T}$ is arbitrary, we have in fact obtained a unique global smooth solution of (1.1) and proved Theorem 2.3.

Acknowledgements The author would like to thank Professor Zhou Yi and Professor Liu Chun for their guidance and encouragements, and thank Professor Li Ta-tsien and Professor Qin Tiehu for their helpful suggestions and comments.

References

[1] Alinhac, S., Blowup for nonlinear hyperbolic equations, Birkhäuser, Boston, 1995.

[2] Chemin, J. Y. and Masmoudi, N., About lifespan of regular solutions of equations related to viscoelastic fluids, *SIAM J. Math. Anal.*, **33**(1), 2001, 84–112.

[3] Climent Ezquerra, B. and Guillén González, F., Global in time solutions for the Poiseuille flow of Oldroyd type in 3D domains, *Ann. Univ. Ferrara, Sez. VII* (*N. S.*), **47**, 2001, 23–40.

[4] Guillopé, C. and Saut, J. C., Existence results for the flow of viscoelastic fluids with a differential constitutive law, *Nonlinear Anal.*, **15**, 1990, 849–869.

[5] Hron, J., Málek, J., Nečas, J. and Rajagopal, K. R., Numerical simulations and global existence of solutions of two-dimensional flows of fluids with pressure- and shear- dependent viscosities, *Math. Comput. Simulation*, **61**(3-6), 2003, 297–315; MODELLING 2001, Pilsen.

[6] Joseph, D. D., Instability of the rest state of fluids of arbitrary grade greater than one, *Arch. Ration. Mech. Anal.*, **75**(3), 1980/81, 251–256.

[7] Kawashima, S. and Shibata, Y., Global existence and exponetial stability of small solutions to nonlinear viscoelasticy, *Comm. Math. Phys.*, **148**, 1992, 189–208.

[8] Klainerman, S., Uniform decay estimates and the Lorentz invariance of the classical wave equation, *Comm. Pure Appl. Math.*, **38**, 1985, 321–332.

[9] Klainerman, S., The null condition and global existence to nonlinear wave equations, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N. M., 1984), 293–326; Lectures in Appl. Math., Vol. 23, A. M. S., Providence, RI, 1986.

[10] Klainerman, S. and Majda, A., Singular limits of quasilinear hyperbolic system with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.*, **34**, 1981, 481–524.

[11] Lin, F. H., Liu, C. and Zhang, P., On hydrohynamics of viscoelastic fluids, *Comm. Pure Appl. Math.*, 2004, to appear.

[12] Lions, P. L. and Masmoudi, N., Global solutions for some Oldroyd models of non-Newtonian flows, *Chin. Ann. Math.*, **21B**(2), 2000, 131–146.

[13] Liu, C and Walkington, N. J., An Eulerian description of fluids containing visco-hyperelastic particles, *Arch. Rat. Mech Anal.*, **159**, 2001, 229–252.

[14] Málek, J., Nečas, J. and Rajagopal, K. R., Global analysis of solutions of the flows of fluids with pressure-dependent viscosities, *Arch. Ration. Mech. Anal.*, **165**(3), 2002, 243–269.

[15] Schowalter, W. R., Mechanics of Non-Newtonian Fluids, Pergamon Press, Oxford, 1978.

[16] Sideris, T. C., Nonresonance and global existence of prestressed nonlinear elastic waves, *Ann. of Math.*, **151**, 2000, 849–874.

[17] Sideris, T. C. and Thomases, B., Global existence for 3D incompressible isotropic elastodynamics via the incompressible limit, *Comm. Pure Appl. Math.*, **57**, 2004, 1–39.

[18] Slemrod, M., Constitutive relations for Rivlin-Erichsen fluids bases on generalized rational approximation, *Arch. Ration. Mech. Anal.*, **146**(1), 1999, 73–93.

[19] Teman, R., Navier-Stokes Equations, North Holland, Amsterdam, 1977.