Chin. Ann. Math.
27B(5), 2006, 581–594
DOI: 10.1007/s11401-004-0317-8

Chinese Annals of Mathematics, Series B
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Beckner Inequality on Finite- and Infinite-Dimensional Manifolds**

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Abstract By using the dimension-free Harnack inequality, the coupling method, and Bakry-Emery's argument, some explicit lower bounds are presented for the constant of the Beckner type inequality on compact manifolds. As applications, the Beckner inequality and the transportation cost inequality are established for a class of continuous spin systems. In particular, some results in [1, 2] are generalized.

Keywords Beckner inequality, Continuous spin systems, Transportation cost inequality
2000 MR Subject Classification 47D07, 60H10

1 Introduction

In 1989, W. Beckner [3] proved the following functional inequality for the standard Gaussian measure γ on $\mathbb{R}^d$:

$$\gamma(f^2) - \gamma(|f|^p)^{\frac{2}{p}} \le (2-p)\gamma(|\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d),\ p \in [1,2). \tag{1.1}$$

In this paper, we aim to study this type inequality on finite- and infinite-dimensional Riemannian manifolds.

Let M be a connected compact Riemannian manifold with diameter D and dimension d. Consider the operator $L := \Delta + \nabla V$ for some $V \in C^\infty(M)$ and let $\lambda(dx)$ be the normalized Riemannian volume measure. It is well known that $(L, C^\infty(M))$ is essentially self-adjoint operator on $L^2(\mu)$, where $\mu(dx) := Z(V)^{-1}e^{V(x)}\lambda(dx)$, $Z(V) := \int_M e^{V(x)}\lambda(dx) < \infty$. Let $\|\cdot\|$ be the Hilbert-Schmitt norm. If

$$(\mathrm{Ric} - \mathrm{Hess}_V)(\nabla f, \nabla f) + \|\mathrm{Hess}_f\|^2 \ge R|\nabla f|^2 + \frac{1}{n}(Lf)^2, \quad f \in C^2(M) \tag{1.2}$$

holds for some $R > 0$ and $n \in (2, \infty]$, then [4, Theorem 3.1] says that

$$\frac{1}{2-p}\big(\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}}\big) \le \frac{n-1}{nR}\mu(|\nabla f|^2) \tag{1.3}$$

holds for any $p \in [1, \frac{2n}{n-2}]$ and any $f \in C^\infty(M)$.

Manuscript received July 18, 2004. Revised February 21, 2006.
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**Project supported by the National Natural Science Foundation of China (No.10121101) and the Research Fund for the Doctoral Program of Higher Education (No.20040027009).

In this paper, we allow the curvature to be negative. For any $p \in [1,2)$, let $C_p(V) > 0$ be the largest positive constant such that

$$\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}} \le \frac{2-p}{C_p(V)}\mu(|\nabla f|^2), \quad f \in C^\infty(M). \tag{1.4}$$

In particular, $C_1(V)$ coincides with the spectral gap of L, i.e.,

$$C_1(V) = \inf\{\, \mu(|\nabla f|^2) : \mu(f^2) - \mu(f)^2 = 1,\ f \in C^\infty(M)\}.$$

Moreover, since

$$\lim_{p\to 2} \frac{\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}}}{2-p} = \frac{1}{2}\mathrm{Ent}_\mu(f^2) := \frac{1}{2}\mu\Big(f^2 \log \frac{f^2}{\mu(f^2)}\Big),$$

we may regard $C_2(V)$ as the log-Sobolev constant, i.e.,

$$C_2(V) = \inf\{\, 2\mu(|\nabla f|^2) : \mathrm{Ent}_\mu(f^2) = 1,\ f \in C^\infty(M)\}.$$

It is well known that $C_1(V) \ge C_2(V)$ (see [5, 6]). Moreover, since according to [7], for any $f \in L^2(\mu)$, $\frac{p\left[\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}}\right]}{2-p}$ is increasing in $p \in [1,2)$, therefore, one has $C_p(V) \ge \frac{p}{2}C_2(V)$. We remark that our estimates of $C_p(V)$ which will be stated below are stronger than those implied by $C_p(V) \ge \frac{p}{2}C_2(V)$ and known estimates of $C_2(V)$ included in [1, 2].

To estimate $C_p(V)$, we first follow the line of Wang [1] where $C_2(V)$ was estimated by using the Harnack inequality and the coupling method. Let $K(V) \in \mathbb{R}$ be such that

$$(\mathrm{Ric} - \mathrm{Hess}_V)(X,X) \ge -K(V)|X|^2, \quad X \in TM. \tag{1.5}$$

We have (see Theorem 2.2 below)

$$\begin{aligned} C_p(V) &\ge \sup_{\alpha \ge p} \frac{\sqrt{(1+mK(V)^+)^2 + 4\alpha m C_1(V)/p} - (1+mK(V)^+)}{2\alpha m/p} \\ &\ge \frac{\sqrt{(1+D^2K(V)^+)^2 + 4D^2C_1(V)} - (1+D^2K(V)^+)}{2D^2}, \quad p \in (1,2], \end{aligned} \tag{1.6}$$

where $m := \frac{(p-1)\alpha D^2}{p(\alpha-1)}$. If in particular $K(V) \le 0$, then (see Corollary 3.1 below)

$$C_p(V) \ge \frac{p^2 c_0}{4D^2}, \tag{1.7}$$

where $c_0 > 0$ solves $c^2 = \frac{32(1-e^{-c})}{p^2}$. These two estimates generalize Theorem 3.1 and Corollary 4.2 in [1] respectively.

To establish the Beckner type inequality for continuous spin systems, we follow the line of [2] to estimate $C_p(V)$ by using Bakry-Emery's argument (see Section 4 for details). As applications, we present explicit lower bound of $C_p(V)$ for the stochastic Ising models on $M^{\mathbb{Z}^m}$. Similarly to [8, Theorem 1.1], this implies the transportation cost inequality for the Gibbs state.

2 Estimates of $C_p(V)$ by Using the Harnack Inequality

In this section we modify the argument in [1] where $C_2(V)$ was estimated by using the dimension-free Harnack inequality established in [9]. To this end, we consider the non-linear equation

$$\frac{f-f^{p-1}}{2-p}-\varepsilon f^{p-1}=-\frac{1}{C_p^\varepsilon(V)}Lf,\quad f\geq 0,\ \mu(f^p)=1,\tag{2.1}$$

where $\varepsilon>0$, $p\in[1,2)$ and

$$C_p^\varepsilon(V):=\inf\Big\{\frac{(2-p)\mu(|\nabla f|^2)}{[\mu(f^2)-1-(2-p)\varepsilon]^+}:f\geq 0,\ \mu(f^p)=1\Big\}.$$

According to [10, 11] (see also [4, 12]), there exists a nontrivial solution to (2.1) which attains the infimum in the definition of $C_p^\varepsilon(V)$. Obviously, $C_p^\varepsilon(V)\to C_p(V)$ as $\varepsilon\downarrow 0$. Since when $\varepsilon=0$ the solution to (2.1) might be constant which does not provide any information of $C_p(V)$, we first consider the case that $\varepsilon>0$ and then let $\varepsilon\downarrow 0$.

Theorem 2.1 *Suppose that $V\in C^2(M)$ and ∂M is either convex or empty. Let $p\in(1,2)$. If $f>0$ solves (2.1) then*

$$(\sup f)^{p-\alpha tC_p^\varepsilon(V)}\leq\exp\Big(\frac{\alpha K(V)D^2}{2(\alpha-1)(1-e^{-2K(V)t})}-\varepsilon\alpha t(\sup f)^{p-2}\Big),\quad\alpha\geq p.\tag{2.2}$$

Proof Let x_0 and y_0 be respectively the maximum point and the minimum point of f, by (2.1) we obtain

$$\begin{aligned}P_tf(x_0)-P_sf(x_0)&=\int_s^t E^{x_0}Lf(x_u)du=-C_p^\varepsilon(V)\int_s^t E^{x_0}\Big(\frac{f(x_u)-f^{p-1}(x_u)}{2-p}-\varepsilon f^{p-1}(x_u)\Big)du\\&\geq\Big(\frac{(f^{p-2}(x_0)-1)C_p^\varepsilon(V)}{2-p}+\varepsilon f^{p-2}(x_0)\Big)\int_s^t P_uf(x_0)du,\quad t\geq s\geq 0.\end{aligned}$$

This implies

$$P_tf(x_0)\geq f(x_0)\exp\Big(\frac{(f^{p-2}(x_0)-1)C_p^\varepsilon(V)}{2-p}t+\varepsilon f^{p-2}(x_0)t\Big).\tag{2.3}$$

On the other hand, we have (see [9])

$$|P_tf(x)|^\alpha\leq P_t|f|^\alpha(y)\exp\Big(\frac{\alpha K(V)\rho(x,y)^2}{2(\alpha-1)(1-e^{-2K(V)t})}\Big),\quad\alpha>1.\tag{2.4}$$

Combining (2.3) with (2.4), we get

$$\begin{aligned}&f^\alpha(x_0)\exp\Big(\frac{(f^{p-2}(x_0)-1)C_p^\varepsilon(V)}{2-p}\alpha t+\varepsilon\alpha f^{p-2}(x_0)t\Big)\\&\leq P_tf^\alpha(y)\exp\Big(\frac{\alpha K(V)D^2}{2(\alpha-1)(1-e^{-2K(V)t})}\Big).\end{aligned}\tag{2.5}$$

Since $f(x_0)\geq 1$, there exists $r\in[p-2,0]$ such that

$$\frac{1-f^{p-2}(x_0)}{2-p}=f^r(x_0)\log f(x_0)\leq\log f(x_0).$$

Then for any $\alpha \geq p$,

$$\begin{aligned} &f^{\alpha}(x_0)\exp(-\alpha t C_p^{\varepsilon}(V)\log f(x_0)) \\ &\leq f^{\alpha-p}(x_0)P_t f^p(y)\exp\Big(\frac{\alpha K(V)D^2}{2(\alpha-1)(1-e^{-2K(V)t})} - \varepsilon\alpha f^{p-2}(x_0)t\Big). \end{aligned}$$

Since $\mu(f^p)=1$, we finish the proof by taking integral over y w.r.t. μ.

Next, it is easy to check that

$$\frac{K(V)}{1-e^{-2K(V)t}} \leq \frac{1}{2t} + K(V)^+, \quad t>0.$$

By taking $t = \frac{p}{2\alpha C_p^{\varepsilon}(V)}$, we obtain from (2.2) that

$$\log f(x_0) \leq \frac{\alpha D^2}{p(\alpha-1)}\Big(K(V)^+ + \frac{\alpha C_p^{\varepsilon}(V)}{p}\Big) - \frac{\varepsilon f^{p-2}(x_0)}{C_p^{\varepsilon}(V)}, \quad \alpha \geq p. \tag{2.6}$$

Theorem 2.2 *Under the assumption of Theorem* 2.1. *For any* $p \in (1,2)$ *we have*

$$C_p(V) \geq \sup_{\alpha \geq p} \frac{\sqrt{(1+mK(V)^+)^2 + 4\alpha m C_1(V)/p} - (1+mK(V)^+)}{2\alpha m/p}, \tag{2.7}$$

where $m := \frac{(p-1)\alpha D^2}{p(\alpha-1)}$. *Especially,* (2.7) *with* $\alpha = p$ *implies*

$$C_p(V) \geq \frac{\sqrt{(1+D^2K(V)^+)^2 + 4D^2C_1(V)} - (1+D^2K(V)^+)}{2D^2}. \tag{2.8}$$

Proof Let $f>0$ solves (2.1). By the spectral representation we have

$$\mu((Lf)^2) \geq C_1(V)\mu(|\nabla f|^2). \tag{2.9}$$

Next, by (2.1) we have

$$\begin{aligned} \mu((Lf)^2) &= \frac{C_p^{\varepsilon}(V)}{2-p}\mu(-(f-f^{p-1}-\varepsilon(2-p)f^{p-1})Lf) \\ &= \frac{C_p^{\varepsilon}(V)}{2-p}[\mu(-fLf) + (1+\varepsilon(2-p))\mu(f^{p-1}Lf)] \\ &= \frac{C_p^{\varepsilon}(V)}{2-p}[\mu(|\nabla f|^2) - (p-1)(1+\varepsilon(2-p))\mu(f^{p-2}|\nabla f|^2)] \\ &\leq \frac{C_p^{\varepsilon}(V)}{2-p}[-(p-1)(1+\varepsilon(2-p))(\min f^{p-2}) + 1]\mu(|\nabla f|^2). \end{aligned}$$

Then

$$\begin{aligned} C_1(V) &\leq \frac{C_p^{\varepsilon}(V)}{2-p}(1-(p-1)(1+\varepsilon(2-p))(\min f^{p-2})) \\ &= C_p^{\varepsilon}(V)\Big(1+(p-1)(1+\varepsilon(2-p))\frac{1-f(x_0)^{p-2}}{2-p} - (p-1)\varepsilon\Big) \\ &\leq C_p^{\varepsilon}(V)(1+(p-1)(1+\varepsilon(2-p))f(x_0)^r\log f(x_0) - (p-1)\varepsilon) \quad (\exists\, r\in[p-2,0]) \\ &\leq C_p^{\varepsilon}(V)(1+(p-1)(1+\varepsilon(2-p))\log f(x_0) - (p-1)\varepsilon). \end{aligned}$$

Combining this with (2.6), we arrive at

$$C_1(V) \le C_p^\varepsilon(V)\Big[1+(p-1)(1+(2-p)\varepsilon)\Big(\frac{\alpha D^2}{p(\alpha-1)}\Big(K(V)^+ + \frac{\alpha C_p^\varepsilon(V)}{p}\Big) - \frac{\varepsilon}{C_p^\varepsilon(V)} f^{p-2}(x_0)\Big) - (p-1)\varepsilon\Big].$$

By letting $\varepsilon \downarrow 0$, we obtain

$$C_1(V) \le C_p(V)\Big[1+\frac{(p-1)\alpha D^2}{p(\alpha-1)}\Big(K(V)^+ + \frac{\alpha C_p(V)}{p}\Big)\Big]. \tag{2.10}$$

Therefore, the desired assertion follows by solving (2.10).

3 Estimates of $C_p(V)$ by Using Coupling

The coupling method has been used successfully to estimate $C_1(V)$ and $C_2(V)$, see e.g. [1, 13] (refer to [14] for more details about coupling). By the approximation procedure as in Section 2, we may assume that a nonconstant solution to (2.1) for $\varepsilon = 0$ exists. The coupling method then works as follows.

Theorem 3.1 *Let $f > 0$ be a nonconstant solution to* (2.1) *with $\varepsilon = 0$. Define $\beta_1 := \sup f$ and $\beta_2 := \sup|1-(p-1)f^{p-2}|$. Let (x_t, y_t) be a coupling for the L-diffusion process with coupling time $T := \inf\{t \ge 0 : x_t = y_t\}$. Then we have*

$$C_p(V) \ge \frac{(2-p)(\beta_1-1)}{(\beta_1-\beta_1^{p-1})\sup\limits_{x,y\in M} E^{x,y}T}, \quad p \in (1,2). \tag{3.1}$$

Furthermore, if there exists $\bar\rho \in C(M\times M)$ with $\bar\rho \ge c\rho$ for some $c \ge 0$ such that

$$E^{x,y}\bar\rho(x_t,y_t) \le \bar\rho(x,y)e^{-\sigma t} \tag{3.2}$$

for some $\sigma > 0$ and all $t \ge 0$, then $C_p(V) \ge \frac{(2-p)\sigma}{\beta_2}$, $p \in (1,2)$.

Proof Let x_0 and y_0 be respectively the maximum point and the minimum point of f. Set $\delta(f) := \sup f - \inf f$. We have

$$\begin{aligned} 1 &= \frac{1}{f(x_0)-f(y_0)}\Big\{E^{x_0,y_0}[f(x_t)-f(y_t)] + \int_0^t E^{x_0,y_0}[Lf(x_s)-Lf(y_s)]ds\Big\} \\ &\le P^{x_0,y_0}(T>t) + \frac{\delta(f-f^{p-1})C_p(V)}{(2-p)(f(x_0)-f(y_0))}\int_0^t P^{x_0,y_0}(T>s)ds, \end{aligned} \tag{3.3}$$

By a simple calculation, it is easy to see that the function $u(x) := x - x^{p-1}$ reaches the minimum at $c := (p-1)^{\frac{1}{2-p}}$ in $x \in (0,\infty)$. Since $c < 1$, we have

$$\begin{aligned} \delta(f-f^{p-1}) &= [f(x_0)-f(x_0)^{p-1}] - [f(y_0)\vee c - (f(y_0)\vee c)^{p-1}] \\ &= \int_{f(y_0)\vee c}^{f(x_0)} [1-(p-1)t^{p-2}]dt \\ &\le (f(x_0)-f(y_0)\vee c)\frac{1}{\beta_1-1}\int_1^{\beta_1}[1-(p-1)t^{p-2}]dt \\ &\le (f(x_0)-f(y_0))\frac{\beta_1-\beta_1^{p-1}}{\beta_1-1}. \end{aligned} \tag{3.4}$$

Here we have used the fact that $f(y_0) \le 1$ and $\frac{1}{\beta_1 - r}\int_r^{\beta_1}[1-(p-1)t^{p-2}]dt$ is increasing in r. Combining (3.3) and (3.4) we obtain

$$C_p(V) \ge \frac{(2-p)(\beta_1 - 1)P^{x_0,y_0}(T \le t)}{(\beta_1 - \beta_1^{p-1})E^{x_0,y_0}T}.$$

This implies (3.1) by letting $t \uparrow \infty$.

To prove (3.2), let $x_\varepsilon \ne y_\varepsilon$ be such that

$$\frac{f(x_\varepsilon) - f(y_\varepsilon)}{\bar\rho(x_\varepsilon, y_\varepsilon)} \ge \sup_{x,y}\frac{f(x)-f(y)}{\bar\rho(x,y)} - \varepsilon \triangleq C - \varepsilon.$$

By the mean-value theorem we have

$$\frac{|(f(x)-f(x)^{p-1}) - (f(y)-f(y)^{p-1})|}{\bar\rho(x,y)} \le \beta_2 \frac{|f(x)-f(y)|}{\bar\rho(x,y)} \le C\beta_2.$$

Combining this with (3.2) and (2.1) with $\varepsilon = 0$ we obtain

$$\begin{aligned}
&(C-\varepsilon)\bar\rho(x_\varepsilon, y_\varepsilon) \le f(x_\varepsilon) - f(y_\varepsilon)\\
&\le E^{x_\varepsilon,y_\varepsilon}|f(x_t) - f(y_t)| + \frac{C_p(V)}{2-p}\int_0^t E^{x_\varepsilon,y_\varepsilon}|(f-f^{p-1})(x_s) - (f-f^{p-1})(y_s)|ds\\
&\le CE^{x_\varepsilon,y_\varepsilon}\bar\rho(x_t,y_t) + \frac{C\beta_2 C_p(V)}{2-p}\int_0^t E^{x_\varepsilon,y_\varepsilon}\bar\rho(x_s,y_s)ds\\
&\le C\bar\rho(x_\varepsilon,y_\varepsilon)\Big(e^{-\sigma t} + \frac{\beta_2(1-e^{-\sigma t})C_p(V)}{(2-p)\sigma}\Big).
\end{aligned}$$

The proof is then completed by letting $t \uparrow \infty$ and $\varepsilon \downarrow 0$.

Corollary 3.1 *If $K(V) \le 0$, we have $C_p(V) \ge \frac{p^2 c_0}{4D^2}$, where $c_0 > 0$ solves $c^2 = \frac{32(1-e^{-c})}{p^2}$.*

Proof For the coupling by reflection, we have (see [1, 9, 15])

$$E^{x,y}T \le \frac{D^2}{8} \quad \text{if } K(V) \le 0. \tag{3.5}$$

Taking $\alpha = 2$ and $\varepsilon = 0$ in (2.6), we have

$$\beta_1 \le \exp\Big(\frac{4D^2 C_p(V)}{p^2}\Big), \tag{3.6}$$

Theorem 3.1 yields that

$$C_p(V) \ge \frac{(2-p)(\beta_1 - 1)}{(\beta_1 - \beta_1^{p-1})\sup\limits_{x,y\in M} E^{x,y}T} \ge \frac{\beta_1 - 1}{(\beta_1 \log\beta_1)\sup\limits_{x,y\in M}E^{x,y}T}. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) we have

$$C_p(V) \ge \frac{2p^2}{D^4 C_p(V)}\Big(1 - e^{-\frac{4D^2C_p(V)}{p^2}}\Big).$$

Let $c = \frac{4D^2 C_p(V)}{p^2}$. Then $c > 0$ and

$$c^2 \ge \frac{32(1-e^{-c})}{p^2}.$$

Since $\frac{c^2}{1-e^{-c}}$ is increasing in $c > 0$, we conclude that $c \ge c_0$.

4 Estimates of $C_p(V)$ by Using Bakry-Emery's Argument

In order to establish the log-Sobolev inequality for continuous spin systems, Deuschel and Stroock [2] presented explicit estimates of $C_2(V)$ by using Bakry-Emery's argument. In this section we intend to extend their results for $C_p(V)$.

Theorem 4.1 *Let $\delta(V) := \sup V - \inf V$. We have*

$$C_p(V) \geq \frac{1}{d+2}\Big(3C_1(0)e^{-\delta(V)} - dK\Big((d+2)\frac{V}{d}\Big) - 2(1-e^{-\delta(V)})K(0)^+\Big).$$

Next, Theorem 4.1 makes sense only if $3C_1(0)e^{-\delta(V)} - dK\big((d+2)\frac{V}{d}\big) - 2(1-e^{-\delta(V)})K(0)^+ > 0$, but it will be better than the previous ones for small $K(0)$ and big D. Most importantly, if $K(0) \leq 0$ and $K(V) \leq 0$, then this estimate is nontrivial even when $d \to \infty$, so that it works also for the infinite dimensional case. This feature is crucial for the study of continuous spin systems.

As mentioned above, we shall adopt Bakry-Emery's argument to prove Theorem 4.1, so that we need to estimate the so called Γ_2 operator. To this end, we first present the following lemma.

Lemma 4.1 *For any strictly positive $f \in C^\infty(M)$,*

$$\lambda(f^{2(p-1)}\|\mathrm{Hess}_{f^{2-p}}\|^2) \geq \frac{(2-p)^2}{d+2}\lambda\Big(3(\Delta f)^2 - 2\mathrm{Ric}(\nabla f, \nabla f) + (p-1)(3p-5)\frac{|\nabla f|^4}{f^2}\Big). \quad (4.1)$$

Proof Since

$$\mathrm{Hess}_{f^{2-p}} = (2-p)f^{1-p}\mathrm{Hess}_f + (2-p)(1-p)f^{-p}\nabla f \otimes \nabla f,$$

one has

$$f^{2(p-1)}\|\mathrm{Hess}_{f^{2-p}}\|^2 = (2-p)^2\Big(\|\mathrm{Hess}_f\|^2 + 2(1-p)\frac{\mathrm{Hess}_f(\nabla f, \nabla f)}{f} + (1-p)^2\frac{|\nabla f|^4}{f^2}\Big).$$

Noting that

$$2\mathrm{Hess}_f(\nabla f, \nabla f) = \langle \nabla f, \nabla |\nabla f|^2\rangle,$$
$$\lambda\Big(\frac{\langle \nabla f, \nabla |\nabla f|^2\rangle}{f}\Big) = -\lambda(|\nabla f|^2 \Delta \log f) = -\lambda\Big(\frac{|\nabla f|^2\Delta f}{f}\Big) + \lambda\Big(\frac{|\nabla f|^4}{f^2}\Big),$$

we obtain

$$\begin{aligned}&\lambda(f^{2(p-1)}\|\mathrm{Hess}_{f^{2-p}}\|^2)\\ &= (2-p)^2\lambda\Big(\|\mathrm{Hess}_f\|^2 + (p-1)\frac{|\nabla f|^2\Delta f}{f} + (p-1)(p-2)\frac{|\nabla f|^4}{f^2}\Big). \quad (4.2)\end{aligned}$$

Moreover,

$$\begin{aligned}\lambda(f^{2(p-1)}(\Delta f^{2-p})^2) &= \lambda(f^{2(p-1)}((2-p)f^{1-p}\Delta f + (2-p)(1-p)f^{-p}|\nabla f|^2)^2)\\ &= (2-p)^2\lambda\Big((\Delta f)^2 - 2(p-1)\frac{|\nabla f|^2\Delta f}{f} + (p-1)^2\frac{|\nabla f|^4}{f^2}\Big).\end{aligned}$$

Combining this with $(\Delta f^{2-p})^2 \le d\|\mathrm{Hess}_{f^{2-p}}\|^2$, we arrive at

$$\begin{aligned}\lambda\Big(\frac{|\nabla f|^2\Delta f}{f}\Big) &= \frac{1}{2(p-1)}(\lambda((\Delta f)^2) + (p-1)^2\lambda(f^{-2}|\nabla f|^4)) - \frac{1}{(2-p)^2}\lambda(f^{2(p-1)}(\Delta f^{2-p})^2)\\ &\ge \frac{1}{2(p-1)}\Big(\lambda((\Delta f)^2) + (p-1)^2\lambda(f^{-2}|\nabla f|^4)\\ &\quad - \frac{d}{(2-p)^2}\lambda(f^{2(p-1)}\|\mathrm{Hess}_{f^{2-p}}\|^2)\Big).\end{aligned}$$

Since $\lambda(\Delta|\nabla f|^2) = 0$, the proof is finished by combining this with (4.2) and the Bochner-Weitzenböck formula:

$$\frac{1}{2}\big(\Delta|\nabla f|^2 - 2\langle\nabla f, \nabla\Delta f\rangle\big) = \|\mathrm{Hess}_f\|^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

Proof of Theorem 4.1 Assume that f is strictly positive and $\mu(f^p) = 1$. Let $f_t := P_t f^p$, $h_p(t) := \frac{\mu\big(f_t^{\frac{2}{p}} - f_t\big)}{2-p}$. Then

$$\begin{aligned}\frac{d}{dt}h_p(t) &= \frac{1}{2-p}\mu\Big(\Big(\frac{2}{p}f_t^{\frac{2}{p-1}} - 1\Big)Lf_t\Big) = -\frac{1}{2-p}\mu\Big(\Big\langle\nabla f_t, \nabla\Big(\frac{2}{p}f_t^{\frac{2}{p-1}} - 1\Big)\Big\rangle\Big)\\ &= -2\mu(\langle\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}}\rangle) = -2\mu(|\nabla f_t^{\frac{1}{p}}|^2).\end{aligned} \tag{4.3}$$

Thus,

$$h_p(0) \le -\frac{h_p'(0)}{2C_p(V)}. \tag{4.4}$$

Since $h_p(t) \to 0$ and $h_p'(t) \to 0$ as $t \to \infty$, (4.4) is implied by

$$h_p''(t) \ge -2C_p(V)h_p'(t), \quad t \ge 0. \tag{4.5}$$

Next, by the second equality in (4.3) and the integration by parts formula, we obtain

$$\begin{aligned}&-\frac{d^2}{dt^2}h_p(t) = \frac{2}{p(2-p)}\frac{d}{dt}\mu(\langle\nabla f_t, \nabla f_t^{\frac{2}{p-1}}\rangle)\\ &= \frac{2}{p(2-p)}\mu\Big(\Big\langle\frac{2-p}{p}\nabla f_t^{\frac{2}{p-2}}Lf_t, \nabla f_t\Big\rangle + \langle\nabla f_t^{\frac{2}{p-1}}, \nabla Lf_t\rangle\Big)\\ &= \frac{2}{p(2-p)}\mu\Big(\Big\langle\nabla\Big(Lf_t^{\frac{2}{p-1}} - \frac{2(2-p)(1-p)}{p^2}f_t^{\frac{2}{p-3}}|\nabla f_t|^2\Big), \nabla f_t\Big\rangle + \langle\nabla f_t^{\frac{2}{p-1}}, \nabla Lf_t\rangle\Big)\\ &= \frac{2}{p(2-p)}\mu\Big(-\frac{2(1-p)}{2-p}\langle\nabla(f_t^{1-\frac{2}{p}}|\nabla f_t^{\frac{2}{p-1}}|^2), \nabla f_t\rangle + \frac{2p}{2-p}f_t^{2-\frac{2}{p}}\langle\nabla Lf_t^{\frac{2}{p}-1}, \nabla f_t^{\frac{2}{p}-1}\rangle\Big).\end{aligned}$$

Since

$$\begin{aligned}&\mu(\langle\nabla(f_t^{1-\frac{2}{p}}|\nabla f_t^{\frac{2}{p}-1}|^2), \nabla f_t\rangle)\\ &= \mu(\langle(f_t^{1-\frac{2}{p}}\nabla|\nabla f_t^{\frac{2}{p}-1}|^2), \nabla f_t\rangle) + \mu(\langle|\nabla f_t^{\frac{2}{p}-1}|^2\nabla(f_t^{1-\frac{2}{p}}), \nabla f_t\rangle)\\ &= \frac{p}{2(p-1)}\mu(\langle\nabla|\nabla f_t^{\frac{2}{p}-1}|^2, \nabla f_t^{2-\frac{2}{p}}\rangle) + \frac{p-2}{p}\mu(f_t^{-\frac{2}{p}}|\nabla f_t^{\frac{2}{p}-1}|^2|\nabla f_t|^2),\end{aligned}$$

we arrive at

$$\frac{d^2}{dt^2}h_p(t) = \frac{4}{(2-p)^2}\mu(f_t^{2-\frac{2}{p}}\Gamma_2(f_t^{\frac{2}{p}-1}, f_t^{\frac{2}{p}-1})) + \frac{4(p-1)}{p^2(2-p)}\mu(f_t^{-\frac{2}{p}}|\nabla f_t^{\frac{2}{p}-1}|^2|\nabla f_t|^2), \tag{4.6}$$

where by the Bochner-Weitzenböck formula,

$$\Gamma_2(f,f) := \frac{1}{2}[L|\nabla f|^2 - 2(\langle \nabla f, \nabla Lf\rangle)] = \|\mathrm{Hess}_f\|^2 + (\mathrm{Ric} - \mathrm{Hess}_V)(\nabla f, \nabla f). \tag{4.7}$$

Without loss of generality, we assume that $\inf V = 0$, otherwise just replace V by $V - \inf V$. Combining (4.7) with Lemma 4.1, we obtain

$$\begin{aligned} h_p''(t) &= \frac{4}{(2-p)^2}[\mu(f_t^{2-\frac{2}{p}}(\mathrm{Ric}-\mathrm{Hess}_V)(\nabla f_t^{\frac{2}{p}-1}, \nabla f_t^{\frac{2}{p}-1})) + \mu(f_t^{2-\frac{2}{p}}\|\mathrm{Hess}_{f_t^{\frac{2}{p}-1}}\|^2)] \\ &\quad + \frac{4(p-1)}{p^2(2-p)}\mu(f_t^{-\frac{2}{p}}|\nabla f_t^{\frac{2}{p}-1}|^2|\nabla f_t|^2) \\ &\geq 4\mu((\mathrm{Ric}-\mathrm{Hess}_V)(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})) + \frac{4}{(d+2)Z(V)}\lambda\Big[3(\Delta f_t^{\frac{1}{p}})^2 - 2\mathrm{Ric}(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}}) \\ &\quad + (p-1)(3p-5)\frac{|\nabla f_t^{\frac{1}{p}}|^4}{f_t^{\frac{2}{p}}}\Big] + \frac{4(p-1)(2-p)}{Z(V)}\lambda\Big(\frac{|\nabla f_t^{\frac{1}{p}}|^4}{f_t^{\frac{2}{p}}}\Big) \\ &\geq 4\mu((\mathrm{Ric}-\mathrm{Hess}_V)(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})) + \frac{4}{(d+2)Z(V)}\lambda[3(\Delta f_t^{\frac{1}{p}})^2 - 2\mathrm{Ric}(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})]. \end{aligned}$$

It is well known by the spectral representation that

$$\lambda((\Delta f_t^{\frac{1}{p}})^2) \geq C_1(0)\lambda(|\nabla f_t^{\frac{1}{p}}|^2).$$

Since

$$\begin{aligned} -2\lambda(\mathrm{Ric}(\nabla f, \nabla f)) &\geq -2\lambda(e^V \mathrm{Ric}(\nabla f, \nabla f)^+) + 2\lambda(\mathrm{Ric}(\nabla f, \nabla f)^-) \\ &\geq Z(V)(-2\mu(\mathrm{Ric}(\nabla f, \nabla f)) - 2(1-e^{-\delta(V)})\mu(\mathrm{Ric}(\nabla f, \nabla f)^-)), \end{aligned}$$

we have

$$\begin{aligned} h_p''(t) &\geq 4\mu((\mathrm{Ric}-\mathrm{Hess}_V)(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})) + \frac{4}{d+2}[3C_1(0)e^{-\delta(V)}\mu(|\nabla f_t^{\frac{1}{p}}|^2) \\ &\quad - 2\mu(\mathrm{Ric}(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})) - 2(1-e^{-\delta(V)})\mu(\mathrm{Ric}(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})^-)] \\ &\geq 4\mu\Big(\Big(1 - \frac{2}{d+2}\Big)\mathrm{Ric} - \mathrm{Hess}_V\Big)(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}}) \\ &\quad + \frac{4}{d+2}[3C_1(0)e^{-\delta(V)}\mu(|\nabla f_t^{\frac{1}{p}}|^2) - 2(1-e^{-\delta(V)})\mu(\mathrm{Ric}(\nabla f_t^{\frac{1}{p}}, \nabla f_t^{\frac{1}{p}})^-)] \\ &\geq \frac{4}{d+2}\Big[3C_1(0)e^{-\delta(V)} - 2(1-e^{-\delta(V)})K(0)^+ - dK\Big((d+2)\frac{V}{d}\Big)\Big]\mu(|\nabla f_t^{\frac{1}{p}}|^2) \\ &= \frac{-2}{d+2}\Big[3C_1(0)e^{-\delta(V)} - 2(1-e^{-\delta(V)})K(0)^+ - dK\Big((d+2)\frac{V}{d}\Big)\Big]h_p'(t). \end{aligned}$$

Since $C_p(V)$ is the largest constant such that (4.5) holds, this implies the desired result.

5 Applications to Continuous Spin Systems

Let $\mathbb{Z}^m$ be the m-dimensional integer lattice with the metric $|\,i\,| := \max\limits_{1\leq k\leq m} |i_k|$. For $\Lambda \subset \mathbb{Z}^m$ we denote by $|\Lambda|$ the cardinality of Λ, and if $|\Lambda| < \infty$ we write $\Lambda \subset\subset \mathbb{Z}^m$. Let M be a compact connected Riemannian manifold and let $E := M^{\mathbb{Z}^m}$ be equipped with the product topology. Given $x, y \in E$ and $\Lambda \subset \mathbb{Z}^m$, define $x_\Lambda \times y_{\Lambda^c} \in E$ by

$$(x_\Lambda \times y_{\Lambda^c})_i = \begin{cases} x_i, & \text{if } i \in \Lambda, \\ y_i, & \text{if } i \in \Lambda^c. \end{cases}$$

Moreover, we let x_Λ be the projection of x onto M^Λ.

Let $\mathcal{F}C^\infty := \{f \in C^\infty(E) :$ there exists $\Lambda \subset\subset \mathbb{Z}^m$ and $\tilde{f} \in C^\infty(M^\Lambda)$ such that $f(x) = \tilde{f}(x_\Lambda)\}$. For simplicity, we identify a function $f \in \mathcal{F}C^\infty$ and the corresponding function $\tilde{f}$ on M^Λ.

Let $\mathfrak{U} = \{J_\Lambda : \varnothing \neq \Lambda \subset\subset \mathbb{Z}^m\}$ be a shift-invariant, finite range potential on E. That is, $J_\Lambda \in C^\infty(M^\Lambda)$ and $J_\Lambda(x) = J_{\Lambda+k}(\theta_k x)$ for any $\Lambda \subset\subset \mathbb{Z}^m$, $x \in E$, $k \in \mathbb{Z}^m$ and $\theta_k x := x_{\cdot - k}$, and there exists $R > 0$ such that $J_\Lambda = 0$ if $\sup\limits_{i,j\in\Lambda} |i - j| \geq R$. Given $k \in \mathbb{Z}^m$, let

$$H_k^\omega(x) := \sum_{\Lambda \ni k} J_\Lambda(x_k \times \omega_{\{k\}^c}), \quad \mu_k^\omega(dx) := \frac{\exp\left(-H_k^\omega(x)\right)}{Z_k^\omega} \prod_{i\in\mathbb{Z}^m} \lambda(dx_i), \quad x,\ \omega \in E,$$

where Z_k^ω is the normalization. A probability measure μ on E is called a Gibbs state with potential $\mathfrak{U}$ (denoted by $\mu \in \mathcal{G}(\mathfrak{U})$), if

$$\int_E f d\mu = \int_E \Big(\int_E f(x_k \times \omega_{\{k\}^c}) \mu_k^\omega(dx) \Big) \mu(d\omega), \quad f \in C(E),\ k \in \mathbb{Z}^m.$$

Next, we introduce the diffusion operator on E which is symmetric w.r.t. μ. To this end, first choose a family of vector field $X^1, \cdots, X^r$ on M such that $X^1(x), \cdots, X^r(x)$ spans $T_x M$ for any $x \in M$. For $k \in \mathbb{Z}^m$, $s := (s_1, \cdots, s_r) \in \mathbb{N}^r$, set

$$X_k^s = (X_k^1)^{s_1} \cdots (X_k^r)^{s_r} \quad \text{and} \quad |s| := \sum_{i=1}^r s_i,$$

where X_k^i is the vector field X^i acting on the k-th component. Next introduce the space $\mathfrak{G}(E)$ of $f \in C^\infty(E)$ with the property that

$$\|f\|_n := \|f\|_\infty + \sum_{k\in\mathbb{Z}^m} \sum_{1\leq|s|\leq n} \|X_k^s f\|_\infty < \infty, \quad n \geq 1.$$

Obviously, $\mathfrak{G}(E) \supset \mathcal{F}C^\infty$.

We then define the operator $L^\mathfrak{U}$ on $\mathfrak{G}(E)$ by

$$L^\mathfrak{U} f := \sum_{k\in\mathbb{Z}^m} (\Delta_k - \nabla_k H_k) f, \quad f \in \mathfrak{G}(E),$$

where $H_k := \sum\limits_{\Lambda \ni k} J_\Lambda$, Δ_k and ∇_k are the Laplacian and gradient operators on the i-th manifold M^k respectively.

It is classical that a probability measure μ on E is a Gibbs state with potential $\mathfrak{U}$ if and only if (cf. [2])

$$-\int_E f L^\mathfrak{U} g d\mu = \sum_{k\in\mathbb{Z}^m} \int_E \langle \nabla_k f, \nabla_k g \rangle d\mu, \quad f, g \in \mathfrak{G}(E).$$

In other words, for every $\mu \in \mathcal{G}(\mathfrak{U})$, $L^{\mathfrak{U}}$ is symmetric in $L^2(\mu)$.

According to [2, Theorem 2.2], $L^{\mathfrak{U}}$ generates a unique Markov semigroup $P_t^{\mathfrak{U}}$ on $C(E)$ preserving $\mathfrak{G}(E)$. Now, let $C_p(\mathfrak{U})$ be the largest positive constant such that

$$\mu(f^2) - \mu(|f|^p)^{\frac{2}{p}} \le \frac{2-p}{C_p(\mathfrak{U})}\mu(|\nabla f|^2), \quad f \in \mathfrak{G}(E),\ p \in [1,2). \tag{5.1}$$

To follow the argument in the last section, we first study the following Γ_2 operator:

$$\Gamma_2^{\mathfrak{U}}(f,f) := \frac{1}{2}\Big(\sum_{k\in\mathbb{Z}^m} L^{\mathfrak{U}}(|\nabla_k f|^2) - 2\sum_{k\in\mathbb{Z}^m}\langle \nabla_k(L^{\mathfrak{U}}f), \nabla_k f\rangle\Big), \quad f \in \mathfrak{G}(E). \tag{5.2}$$

Let Ric_k be the Ricci curvature tensor on M^k. For any $X \in TE$, let X_k be its projection on TM^k. For any $f \in C^\infty(E)$ and $X, Y \in TE$, define

$$\mathrm{Hess}_f^{k,l}(X,Y)(x) := \begin{cases} \mathrm{Hess}_{f(\,\cdot\,\times x_{\{k\}^c})}(X_k, Y_l)(x_k), & \text{if } k = l, \\ X_k \widetilde{Y}_l f(x), & \text{if } k \ne l, \end{cases}$$

where $\widetilde{Y}_l$ is a smooth vector field on M^l such that $\widetilde{Y}_l(x_l) = Y_l(x)$. Finally, set

$$\mathrm{Hess}_{\mathfrak{U}}^{k,l} = \sum_{\Lambda \ni \{k,l\}} \mathrm{Hess}_{J_\Lambda}^{k,l} \quad \text{for } k,l \in \mathbb{Z}^m.$$

Then $\Gamma_2^{\mathfrak{U}}(f,f)$ satisfies

$$\Gamma_2^{\mathfrak{U}}(f,f) = \sum_{k\in\mathbb{Z}^m} \Gamma_k^{\mathfrak{U}}(f,f) + R(f,f), \tag{5.3}$$

where

$$\Gamma_k^{\mathfrak{U}}(f,f) := \|\mathrm{Hess}_f^{k,k}\|^2 + (\mathrm{Ric}_k + \mathrm{Hess}_{k,k}(\mathfrak{U}))(\nabla_k f, \nabla_k f), \tag{5.4}$$

$$R(f,f) := \sum_{k\in\mathbb{Z}^m, l\in\mathbb{Z}^m\setminus\{k\}} \mathrm{Hess}_{\mathfrak{U}}^{k,l}(\nabla_k f, \nabla_l f). \tag{5.5}$$

Let

$$\beta(\mathfrak{U}) := \sup\Big\{\beta \in \mathbb{R} : R(f,f) \ge \beta \sum_{k\in\mathbb{Z}^m} |\nabla_k f|^2 \ \text{ for all } \ f \in \mathfrak{G}(E)\Big\}.$$

Define

$$\delta(\mathfrak{U}) := \sup\{\delta(H_0^\omega) : \omega \in E\} \quad \text{and} \quad K(\mathfrak{U}) := \sup\{K(H_0^\omega) : \omega \in E\}.$$

Furthermore denote

$$\alpha(\mathfrak{U}) := \frac{1}{d+2}\Big[3C(0)e^{-\delta(\mathfrak{U})} - dK\Big((d+2)\frac{\mathfrak{U}}{d}\Big) - 2(1 - e^{-\delta(\mathfrak{U})})K(0)^+\Big].$$

Then we have the following extension of Theorem 4.1.

Theorem 5.1 *If $\alpha(\mathfrak{U}) + \beta(\mathfrak{U}) > 0$ then there exists a unique $\mu \in \mathcal{G}(\mathfrak{U})$ and $C_p(\mathfrak{U}) \ge \alpha(\mathfrak{U}) + \beta(\mathfrak{U}) > 0$.*

Proof Let $\mu \in \mathcal{G}(\mathfrak{U})$ and $h_p(t) := \frac{\mu\left(f_t^{\frac{2}{p}} - f_t\right)}{2-p}$. According to the proof of Theorem 4.1, it suffice to prove that

$$h_p''(t) \geq C_p(\mathfrak{U})(-2h_p'(t)),$$

where $h_p(t)$ is defined as in the proof of Theorem 4.1 for positive $f \in \mathfrak{G}(E)$ and $P_t^{\mathfrak{U}}$. Since $P_t^{\mathfrak{U}}\mathfrak{G}(E) \subset \mathfrak{G}(E)$, the derivatives w.r.t. t is exchangeable with the integration of μ. Thus, by (5.2)–(5.5) and repeating the proof of Theorem 4.1, we conclude that for any extreme Gibbs state μ,

$$C_p(\mathfrak{U}) \geq \alpha(\mathfrak{U}) + \beta(\mathfrak{U}) > 0.$$

In particular, the log-Sobolev inequality holds and hence μ is unique according to [16].

In applications, it is convenient to replace $\beta(\mathfrak{U})$ by the more explicit quantity $\gamma(\mathfrak{U}) = \sum_{k\neq 0} \gamma(k)$, where

$$\gamma(k) := \sup\{\|\mathrm{Hess}_{\mathfrak{U}}^{0,k}(X_0, Y_k)\|_\infty : X, Y \in TE,\ |X|, |Y| \leq 1\}, \quad k \neq 0.$$

Due to the shift-invariance and Young's inequality, we have

$$\beta(\mathfrak{U}) \geq -\gamma(\mathfrak{U}).$$

Therefore, the following is a direct consequence of Theorem 5.1.

Corollary 5.1 *If* $\alpha(\mathfrak{U}) > \gamma(\mathfrak{U})$ *then* $C_p(\mathfrak{U}) \geq \alpha(\mathfrak{U}) - \gamma(\mathfrak{U})$.

Finally, we consider the transportation cost inequality for the Gibbs measure μ. Let ρ be the Riemannian distance on M and

$$\bar\rho(x, y) := \sqrt{\sum_{i\in\mathbb{Z}^m} \rho(x_i, y_i)^2}, \quad x, y \in E. \tag{5.6}$$

Let

$$W_p^{\bar\rho}(F\mu, \mu) := \inf_{\pi\in\mathcal{C}(F\mu,\mu)} \left\{ \int_{E\times E} \bar\rho(x, y)^p \pi(dx, dy) \right\}^{\frac{1}{p}}, \quad p \geq 1, \tag{5.7}$$

where $\mathcal{C}(F\mu, \nu)$ is the set of all couplings of $F\mu$ and μ.

As an application of the Theorem 5.1, we have the following result on transportation cost.

Corollary 5.2 *If* $\alpha(\mathfrak{U}) > \gamma(\mathfrak{U})$, *then the unique Gibbs state* μ *satisfies*

$$W_p^{\bar\rho}(F\mu, \mu) \leq p\sqrt{\frac{\mu(F^{\frac{2}{p}}) - 1}{(\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}))(2-p)}}\ , \quad F \geq 0,\ \mu(F) = 1,\ p \in [1, 2]. \tag{5.8}$$

Proof It suffices to prove for cylindrical function F. Let

$$\Lambda_n := \{i \in \mathbb{Z}^m : |\,i\,| \leq n\}$$

and μ_n be the projection of μ onto $M^{\Lambda_n}, n \geq 1$. Then there exists $n_0 \geq 1$ such that $F(x)$ depends only on $x_{\Lambda_{n_0}}$. Since Corollary 5.1 implies that

$$\mu_n(F^2) - \mu_n(|F|^p)^{\frac{2}{p}} \leq \frac{2-p}{\alpha(\mathfrak{U}) - \gamma(\mathfrak{U})} \mu_n(|\nabla_{\Lambda_n} F|^2), \quad F \in C^\infty(M^{\Lambda_n}),\ n \geq 1,$$

it follows from [8, Theorem 1.1] that

$$W_p^{\rho_n}(F\mu_n, \mu_n) \leq p\sqrt{\frac{\mu(F^{\frac{2}{p}}) - 1}{(2-p)(\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}))}}\ , \quad n \geq n_0, \tag{5.9}$$

where

$$\rho_n(x_{\Lambda_n}, y_{\Lambda_n}) := \Big(\sum_{i \in \Lambda_n} \rho(x_i, y_i)^2\Big)^{\frac{1}{2}}.$$

Let $\pi_n \in \mathcal{C}(F\mu_n, \mu_n)$ such that

$$\Big(\int \rho_n^p d\pi_n\Big)^{\frac{1}{p}} \leq p\sqrt{\frac{\mu(F^{\frac{2}{p}}) - 1}{(2-p)(\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}))}}\ , \quad n \geq n_0.$$

Define

$$\tilde{\pi}_n(dx, dy) := \pi_n(dx_{\Lambda_n}, dy_{\Lambda_n})\mu(dx_{\Lambda_n^c}|x_{\Lambda_n})\mu(dy_{\Lambda_n^c}|y_{\Lambda_n}).$$

Then $\tilde{\pi}_n \in \mathcal{C}(F\mu, \mu)$ and

$$\Big(\int \rho_n^p d\tilde{\pi}_n\Big)^{\frac{1}{p}} \leq p\sqrt{\frac{\mu(F^{\frac{2}{p}}) - 1}{(2-p)(\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}))}}\ , \quad n \geq n_0. \tag{5.10}$$

On the other hand, since $\{F\mu, \mu\}$ is tight as E is a Polish space under the product topology, so is $\mathcal{C}(F\mu, \mu)$. Thus, there exists $\pi \in \mathcal{P}(E \times E)$ such that $\tilde{\pi}_n \to \pi$ weakly for some subsequence $n_k \to \infty$. It is trivial that $\pi \in \mathcal{C}(F\mu, \mu)$ and (5.10) implies

$$\Big(\int \rho_n^p d\pi\Big)^{\frac{1}{p}} \leq p\sqrt{\frac{\mu(F^{\frac{2}{p}}) - 1}{(2-p)(\alpha(\mathfrak{U}) - \gamma(\mathfrak{U}))}}\ , \quad n \geq n_0.$$

Therefore, the proof is finished by letting $n \uparrow \infty$.

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