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Exact Controllability for Nonautonomous First Order Quasilinear Hyperbolic Systems**

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Abstract By means of the theory on the semi-global C^1 solution to the mixed initial-boundary value problem (IBVP) for first order quasilinear hyperbolic systems, we establish the exact controllability for general nonautonomous first order quasilinear hyperbolic systems with general nonlinear boundary conditions.

Keywords Nonautonomous quasilinear hyperbolic system, Mixed initial-boundary value problem, Semi-global C^1 solution, Exact controllability
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1 Introduction

Consider the following first order quasilinear hyperbolic system in the characteristic form

$$l_i(t,x,u)\Big(\frac{\partial u}{\partial t}+\lambda_i(t,x,u)\frac{\partial u}{\partial x}\Big)=\mu_i(t,x,u),\quad i=1,\cdots,n, \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$ is an unknown vector function of (t,x), $l_i(t,x,u)=(l_{i1}(t,x,u),\cdots,l_{in}(t,x,u))$ $(i=1,\cdots,n)$, and

$$\begin{aligned}\mu_i(t,x,u)=a_i(t,x,u)\Big(\frac{\partial b_i(t,x)}{\partial t}+\lambda_i(t,x,u)\frac{\partial b_i(t,x)}{\partial x}\Big)\\ +f_i(t,x,u)+c_i(t,x),\quad i=1,\cdots,n.\end{aligned} \tag{1.2}$$

We assume that $l_i(t,x,u),\lambda_i(t,x,u),a_i(t,x,u),b_i(t,x),f_i(t,x,u)$ and $c_i(t,x)$ $(i=1,\cdots,n)$ are all C^1 functions with respect to their arguments and on the domain under consideration,

$$\det|l_{ij}(t,x,u)|\neq 0 \tag{1.3}$$

and

$$f_i(t,x,0)\equiv 0,\quad i=1,\cdots,n. \tag{1.4}$$

The following discussion is still valid if a_i and b_i $(i=1,\cdots,n)$ are vector functions.

Suppose that $r_i(t,x,u)=(r_{i1}(t,x,u),\cdots,r_{in}(t,\,(x,u))^T$ $(i=1,\cdots,n)$ satisfy

$$l_i(t,x,u)r_j(t,x,u)\equiv\delta_{ij},\quad i,j=1,\cdots,n, \tag{1.5}$$

$$r_i^T(t,x,u)r_i(t,x,u)\equiv 1,\qquad i=1,\cdots,n. \tag{1.6}$$

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and $r_i(t,x,u)$ have the same regularity as $l_i(t,x,u)$.

Moreover, we suppose that on the domain under consideration

$$\lambda_p(t,x,u) < \lambda_q(t,x,u) \equiv 0 < \lambda_r(t,x,u), \tag{1.7}$$
$$p = 1, \cdots, l;\ q = l+1, \cdots, m;\ r = m+1, \cdots, n.$$

We give the initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le L \tag{1.8}$$

and the final condition

$$t = T: \quad u = \psi(x), \quad 0 \le x \le L. \tag{1.9}$$

The boundary conditions are of the form

$$x = 0: \quad v_r = G_r(t, v_1, \cdots, v_l, v_{l+1}, \cdots, v_m) + H_r(t), \qquad r = m+1, \cdots, n, \tag{1.10}$$
$$x = L: \quad v_p = G_p(t, v_{l+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + H_p(t), \quad p = 1, \cdots, l, \tag{1.11}$$

where

$$v_i = l_i(t,x,u)u, \quad i = 1, \cdots, n. \tag{1.12}$$

Without loss of generality, we assume that

$$G_p(t, 0 \cdots, 0) \equiv G_r(t, 0, \cdots, 0) \equiv 0, \quad p = 1, \cdots, l;\ r = m+1, \cdots, n. \tag{1.13}$$

Li Tatsien, Rao Bopeng and Yu Lixin etc. have established the exact controllability for general autonomous quasilinear hyperbolic systems in [3, 5–8]. This paper deals with the corresponding results for general nonautonomous first order quailinear hyperbolic systems, in which we adopt an idea initiated by Qin Tiehu for one-side controllability, when the number of the positive eigenvalues is different from that of the negative ones.

We first study the semi-global C^1 solution to the mixed initial-boundary value problem (IBVP) (1.1), (1.8) and (1.10)-(1.11) in §2. Then we consider the exact controllability for this problem in the cases that there is no zero eigenvalue and there are some zero eigenvalues respectively in §3 and §4.

2 Existence and Uniqueness of Semi-global C^1 Solution

In order to get the semi-global C^1 solution to the IBVP (1.1), (1.8) and (1.10)-(1.11) (Problem I), we need to introduce another type of boundary conditions as follows:

$$x = 0: \quad \tilde{v}_r = g_r(t, \tilde{v}_1, \cdots, \tilde{v}_l, \tilde{v}_{l+1}, \cdots, \tilde{v}_m) + h_r(t), \qquad r = m+1, \cdots, n, \tag{2.1}$$
$$x = L: \quad \tilde{v}_p = g_p(t, \tilde{v}_{l+1}, \cdots, \tilde{v}_m, \tilde{v}_{m+1}, \cdots, \tilde{v}_n) + h_p(t), \quad p = 1, \cdots, l, \tag{2.2}$$

where

$$\tilde{v}_i = l_i(t,x,\varphi(x))u, \quad i = 1, \cdots, n. \tag{2.3}$$

Without loss of generality, we assume that

$$g_p(t, 0 \cdots, 0) \equiv g_r(t, 0, \cdots, 0) \equiv 0, \quad p = 1, \cdots, l;\ r = m+1, \cdots, n. \tag{2.4}$$

Correspondingly, we denote the IBVP (1.1), (1.8) and (2.1)-(2.2) as Problem II. In order to prove the equivalence of Problem I and Problem II when $|u|$ is sufficiently small, it suffices to show that the boundary conditions (1.10)-(1.11) can be replaced by the boundary conditions (2.1)-(2.2) respectively, provided that $|u|$ is sufficiently small.

Similar to [4, 11], we have the following two lemmas.

Lemma 2.1 *Suppose that $l_i, g_p, h_p, g_r, h_r, G_p, H_p, G_r$ and H_r ($i = 1, \cdots, n;\ p = 1, \cdots, l;$ $r = m+1, \cdots, n$) are all C^1 functions with respect to their arguments. When $|u| \le \varepsilon_0$ ($\varepsilon_0 > 0$ is a suitably small number), if the boundary conditions* (1.10)-(1.11) *are replaced by the boundary conditions* (2.1)-(2.2) *respectively, then Problem* I *is equivalent to Problem* II.

Lemma 2.2 *Under the hypotheses of Lemma* 2.1*, the functions $h(t) = (h_1(t), \cdots, h_l(t),$ $h_{m+1}(t), \cdots, h_n(t))$ and $H(t) = (H_1(t), \cdots, H_l(t), H_{m+1}(t), \cdots, H_n(t))$ in two equivalent boundary conditions* (2.1)-(2.2) *and* (1.10)-(1.11) *satisfy the following properties: for any given l_i, g_p, g_r, G_p and G_r ($i = 1, \cdots, n;\ p = 1 \cdots, l;\ r = m+1, \cdots, n$), there exist two positive constants C_1 and C_2 depending only on ε_0, such that on the domain under consideration we have*

$$C_1 \|h\|_0 \le \|H\|_0 \le C_2 \|h\|_0, \tag{2.5}$$

$$\|h\|_1 \to 0 \Leftrightarrow \|H\|_1 \to 0, \tag{2.6}$$

where $\|\cdot\|_0$ and $\|\cdot\|_1$ stand for the C^0 norm and C^1 norm respectively.

By means of the method given in [9, 10], we have the existence and uniqueness of the local C^1 solution to Problem II.

Lemma 2.3 *Suppose that $l_i, \lambda_i, a_i, b_i, f_i, c_i, g_p, h_p, g_r, h_r$ ($i = 1, \cdots, n;\ p = 1, \cdots, l;\ r =$ $m+1, \cdots, n$) and φ are all C^1 functions. Suppose that* (1.3)-(1.4)*,* (1.7) *and* (2.4) *hold and the conditions of C^1 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$ respectively. Then, for any given $l_i, \lambda_i, a_i, f_i, g_p$ and g_r ($i = 1, \cdots, n;\ p = 1, \cdots, l;\ r = m+1, \cdots, n$), there exists a positive constant $\delta = \delta(\|\varphi\|_1, \|b\|_1, \|c\|_1, \|h\|_1) > 0$, such that Problem* II *admits a unique C^1 solution $u = u(t, x)$ on $R(\delta) = \{(t, x) \mid 0 \le t \le \delta,\ 0 \le x \le L\}$. Moreover, when $\|\varphi\|_1, \|b\|_1, \|c\|_1$ and $\|h\|_1$ are sufficiently small, the C^1 norm of $u = u(t, x)$ is also sufficiently small. In particular, we have*

$$|u(t, x)| \le \varepsilon_0, \quad \forall\, (t, x) \in R(\delta), \tag{2.7}$$

where ε_0 is given in Lemma 2.1.

Then, by Lemmas 2.1 and 2.2 we get

Lemma 2.4 *Suppose that $l_i, \lambda_i, a_i, b_i, f_i, c_i, G_p, H_p, G_r, H_r$ ($i = 1, \cdots, n;\ p = 1, \cdots, l;\ r =$ $m+1, \cdots, n$) and φ are all C^1 functions. Suppose that* (1.3)-(1.4)*,* (1.7) *and* (1.13) *hold and the conditions of C^1 compatibility are satisfied at the points $(t, x) = (0, 0)$ and $(0, L)$ respectively. Then, for any given $l_i, \lambda_i, a_i, f_i, G_p$ and G_r ($i = 1, \cdots, n;\ p = 1, \cdots, l;\ r = m+1, \cdots, n$), when*

$\|\varphi\|_1, \|b\|_1, \|c\|_1$ and $\|H\|_1$ are sufficiently small, there exists $\delta = \delta(\|\varphi\|_1, \|b\|_1, \|c\|_1, \|H\|_1) > 0$ such that Problem I *admits a unique C^1 solution $u = u(t,x)$ with small C^1 norm on $R(\delta)$. In particular,* (2.7) *holds.*

Theorem 2.1 *Under the hypotheses of Lemma* 2.4*, for any given $T_0 > 0$, Problem* I *admits a unique C^1 solution $u = u(t,x)$ (so-called the semi-global C^1 solution) with sufficiently small C^1 norm on the domain*

$$R(T_0) = \{(t,x) \mid 0 \le t \le T_0,\ 0 \le x \le L\}, \tag{2.8}$$

provided that $\|\varphi\|_{C^1[0,L]}, \|b\|_{C^1[R(T_0)]}, \|c\|_{C^1[R(T_0)]}$ and $\|H\|_{C^1[0,T_0]}$ are sufficiently small (depending on T_0).

Proof By Lemma 2.4, it suffices to prove that for any C^1 solution $u = u(t,x)$ to Problem I on the domain

$$R(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\} \tag{2.9}$$

with $0 < T < T_0$, we have the following uniform a priori estimate:

$$\|u(t,\cdot)\|_1 \triangleq \|u(t,\cdot)\|_0 + \|u_x(t,\cdot)\|_0 \le C(T_0), \quad \forall\, t \in [0,T], \tag{2.10}$$

where $C(T_0)$ is a sufficiently small positive constant independent of T but possibly depending on T_0.

Let $v = (v_1, \cdots, v_n),\ w = (w_1, \cdots, w_n)$, where v_i $(i = 1, \cdots, n)$ are given by (1.12) and

$$w_i = l_i(t,x,u)u_x, \quad i = 1, \cdots, n. \tag{2.11}$$

By (1.5)-(1.6), it is enough to estimate $\|v(t,\cdot)\|_0$ and $\|w(t,\cdot)\|_0$.

Similar to [1, 2, 11], we have

$$\frac{dv_i}{d_i t} = \beta_i(t,x,u) + \sum_{j=1}^n \beta_{ij}(t,x,u)v_j + \sum_{j,k=1}^n \beta_{ijk}(t,x,u)v_j w_k, \quad i = 1, \cdots, n, \tag{2.12}$$

$$\frac{dw_i}{d_i t} = \gamma_i(t,x,u) + \sum_{j=1}^n \gamma_{ij}(t,x,u)w_j + \sum_{j,k=1}^n \gamma_{ijk}(t,x,u)w_j w_k, \quad i = 1, \cdots, n, \tag{2.13}$$

where

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(t,x,u)\frac{\partial}{\partial x} \tag{2.14}$$

denotes the directional derivative along the i-th characteristic,

$$\beta_i(t,x,u) = a_i(t,x,u)\frac{db_i(t,x)}{d_i t} + f_i(t,x,u) + c_i(t,x), \tag{2.15}$$

$$
\begin{aligned}
\beta_{ij}(t,x,u) = &-\sum_{k=1}^{n} l_i(t,x,u)\nabla_u r_j(t,x,u) r_k(t,x,u)\Big[a_k(t,x,u)\frac{db_k(t,x)}{d_k t} \\
&+ f_k(t,x,u) + c_k(t,x)\Big] + \frac{dl_i(t,x,\cdot)}{d_i t} r_j(t,x,u),
\end{aligned}
\tag{2.16}
$$

$$
\beta_{ijk}(t,x,u) = (\lambda_k(t,x,u) - \lambda_i(t,x,u)) l_i(t,x,u)\nabla_u r_j(t,x,u) r_k(t,x,u); \tag{2.17}
$$

$$
\begin{aligned}
\gamma_i(t,x,u) = &\sum_{k=1}^{n} l_i(t,x,u)\frac{\partial r_k(t,x,\cdot)}{\partial x}\Big[a_k(t,x,u)\frac{db_k(t,x)}{d_k t} \\
&+ f_k(t,x,u) + c_k(t,x)\Big] + \frac{d}{d_i t}\Big(a_i(t,x,u)\frac{\partial b_i(t,x)}{\partial x}\Big) \\
&- \frac{\partial a_i(t,x,\cdot)}{\partial t}\frac{\partial b_i(t,x)}{\partial x} + \frac{\partial a_i(t,x,\cdot)}{\partial x}\frac{\partial b_i(t,x)}{\partial t} \\
&+ a_i(t,x,u)\frac{\partial \lambda_i(t,x,\cdot)}{\partial x}\frac{\partial b_i(t,x)}{\partial x} + \frac{\partial f_i(t,x,\cdot)}{\partial x} + \frac{\partial c_i(t,x)}{\partial x},
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
\gamma_{ij}(t,x,u) = &\sum_{k=1}^{n} l_i(t,x,u)(\nabla_u r_k(t,x,u) r_j(t,x,u) - \nabla_u r_j(t,x,u) r_k(t,x,u)) \\
&\cdot \Big[a_k(t,x,u)\frac{db_k(t,x)}{d_k t} + f_k(t,x,u) + c_k(t,x)\Big] \\
&+ \nabla_u f_i(t,x,u) r_j(t,x,u) + \nabla_u a_i(t,x,u)\frac{db_i(t,x)}{d_i t} r_j(t,x,u) \\
&+ a_i(t,x,u)\nabla_u \lambda_i(t,x,u)\frac{\partial b_i(t,x)}{\partial x} r_j(t,x,u) + \frac{\partial l_i(t,x,\cdot)}{\partial t} r_j(t,x,u) \\
&- \lambda_j(t,x,u) l_i(t,x,u)\frac{\partial r_j(t,x,\cdot)}{\partial x} - \frac{\partial \lambda_i(t,x,\cdot)}{\partial x}\delta_{ij},
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
\gamma_{ijk}(t,x,u) = &\frac{1}{2}[(\lambda_j(t,x,u) - \lambda_k(t,x,u)) l_i(t,x,u)\nabla_u r_k(t,x,u) r_j(t,x,u) \\
&- \nabla_u \lambda_k(t,x,u) r_j(t,x,u)\delta_{ik} + (j \mid k)],
\end{aligned}
\tag{2.20}
$$

in which $\frac{dl_i(t,x,\cdot)}{d_i t}$ and $\frac{\partial r_k(t,x,\cdot)}{\partial x}$ etc. are the corresponding derivatives regarding u as parameter, and the symbol $(j \mid k)$ in (2.20) stands for all terms obtained by changing j and k in the previous terms.

For the time being we assume that

$$
|v(t,x)| \le \frac{\eta_0}{n}, \quad |w(t,x)| \le \eta_1, \quad \forall\,(t,x) \in R(T), \tag{2.21}
$$

where η_0, η_1 are suitably small positive constants (the validity of this hypothesis will be shown later). Then, by (1.5)-(1.6) and (1.12) we have

$$
|u(t,x)| \le \eta_0, \quad \forall\,(t,x) \in R(T). \tag{2.22}
$$

Integrating the i-th equation in (2.12)-(2.13) along the i-th characteristic and using the boundary conditions if necessary (cf. [4, 11]), we have

$$|v(t,x)| \le C\max\{\|\varphi\|_{C^0[0,L]}, \|b\|_{C^0[R(T_0)]}, \|c\|_{C^0[R(T_0)]}, \|H\|_{C^0[0,T_0]}\}, \quad \forall\,(t,x)\in R(T), \tag{2.23}$$

$$|w(t,x)| \le C\max\Big\{\|\varphi'\|_{C^0[0,L]}, \|b\|_{C^1[R(T_0)]}, \|c\|_{C^1[R(T_0)]}, \|H'\|_{C^0[R(T_0)]}, \\ d(\eta_0), \|f\|_0, \Big\|\frac{\partial f(t,x,\cdot)}{\partial x}\Big\|_0\Big\}, \quad \forall\,(t,x)\in R(T), \tag{2.24}$$

where $C>1$ is a positive constant independent of T,

$$d(\eta_0)\to 0, \quad \text{as } \eta_0\to 0 \tag{2.25}$$

and

$$\|f\|_0 = \sup_{\substack{(t,x)\in R(T)\\ |u|\le\eta_0}} |f(t,x,u)|, \quad \Big\|\frac{\partial f(t,x,\cdot)}{\partial x}\Big\|_0 = \sup_{\substack{(t,x)\in R(T)\\ |u|\le\eta_0}} \Big|\frac{\partial f(t,x,\cdot)}{\partial x}\Big|. \tag{2.26}$$

Noting (1.4), we have

$$\|f\|_0\to 0, \quad \Big\|\frac{\partial f(t,x,\cdot)}{\partial x}\Big\|_0 \to 0, \quad \text{as } \eta_0\to 0. \tag{2.27}$$

Hence, both $\|v(t,\cdot)\|_0$ and $\|w(t,\cdot)\|_0$ $(0\le t\le T)$ are sufficiently small and η_0 can be chosen to be sufficiently small, provided that $\|\varphi\|_{C^1[0,L]}, \|b\|_{C^1[R(T_0)]}, \|c\|_{C^1[R(T_0)]}$ and $\|H\|_{C^1[0,T_0]}$ are sufficiently small. This implies not only (2.10) but also the validity of hypothesis (2.21). The proof of Theorem 2.1 is finished.

Remark 2.1 Under the hypotheses of Theorem 2.1, when $\frac{\partial G_p}{\partial t}, \frac{\partial G_r}{\partial t}$ $(p=1,\cdots,l;\ r=m+1,\cdots,n)$ satisfy the local Lipschitz condition with respect to their arguments except t and $\frac{\partial f_i}{\partial x}$ $(i=1,\cdots,n)$ satisfy the local Lipschitz condition with respect to u, the C^1 solution $u=u(t,x)$ to Problem I satisfies the following estimate

$$\|u\|_{C^1[R(T_0)]} \le C_0\max\{\|\varphi\|_{C^1[0,L]}, \|b\|_{C^1[R(T_0)]}, \|c\|_{C^1[R(T_0)]}, \|H\|_{C^1[R(T_0)]}\}, \tag{2.28}$$

where C_0 is a positive constant.

3 Exact Boundary Controllability for Nonautonomous First Order Quasilinear Hyperbolic Systems — Case Without Zero Eigenvalues

Assume that system (1.1) has no zero eigenvalues:

$$\lambda_r(t,x,u) < 0 < \lambda_s(t,x,u), \quad r=1,\cdots,m;\ s=m+1,\cdots,n. \tag{3.1}$$

Correspondingly, the boundary conditions are of the form:

$$x=0: \quad v_s = G_s(t,v_1,\cdots,v_m)+H_s(t), \qquad s=m+1,\cdots,n, \tag{3.2}$$

$$x=L: \quad v_r = G_r(t,v_{m+1},\cdots,v_n)+H_r(t), \quad r=1,\cdots,m \tag{3.3}$$

with

$$G_r(t,0\cdots,0)\equiv G_s(t,0,\cdots,0)\equiv 0, \quad r=1,\cdots,m;\ s=m+1,\cdots,n, \tag{3.4}$$

where v_i $(i = 1, \cdots, n)$ are given by (1.12).

In this case, we can realize the exact controllability only by boundary controls acting on $x = L$ and/or $x = 0$. By means of the theory on the semi-global C^1 solution in §2, we can establish the following two-sides exact boundary controllability by solving some well-posed IBVPs.

Theorem 3.1 (Two-Sides Control) *Suppose that* $l_i, \lambda_i, a_i, b_i, f_i, c_i$ *and* G_i $(i = 1, \cdots, n)$ *are all* C^1 *functions. Suppose furthermore that* (1.3)-(1.4), (3.1) *and* (3.4) *hold. Let*

$$R(T) = \{(t, x) \mid 0 \le t \le T,\ 0 \le x \le L\}. \tag{3.5}$$

If there exists $T > 0$ *such that*

$$\int_0^T \min_{i=1,\cdots,n} \inf_{0 \le x \le L} |\lambda_i(t, x, 0)|\, dt > L \tag{3.6}$$

and $\|(b_i, c_i)\|_{C^1[R(T)]}$ $(i = 1, \cdots, n)$ *is sufficiently small, then, for any given initial data* φ *and final data* ψ *with sufficiently small* C^1 *norm, there exist boundary controls* $H_i(t)$ $(i = 1, \cdots, n)$ *with small* C^1 *norm, such that the* IBVP (1.1), (1.8) *and* (3.2)-(3.3) *admits a unique* C^1 *solution* $u = u(t, x)$ *with small* C^1 *norm on* $R(T)$, *which verifies exactly the final condition* (1.9).

Proof By (3.6), there exists $\varepsilon_1 > 0$ so small that

$$\int_0^T \min_{i=1,\cdots,n} \inf_{\substack{0 \le x \le L, \\ |u| \le \varepsilon_1}} |\lambda_i(t, x, u)|\, dt > L. \tag{3.7}$$

Taking T_1, T_2 such that

$$\int_0^{T_1} \min_{i=1,\cdots,n} \inf_{\substack{0 \le x \le L, \\ |u| \le \varepsilon_1}} |\lambda_i(t, x, u)|\, dt = \int_{T-T_2}^T \min_{i=1,\cdots,n} \inf_{\substack{0 \le x \le L, \\ |u| \le \varepsilon_1}} |\lambda_i(t, x, u)|\, dt = \frac{L}{2}, \tag{3.8}$$

we have $T_1 < T - T_2$.

As in [5], we can construct a C^1 solution $u = u(t, x)$ to system (1.1) on $R(T)$, which satisfies the initial condition (1.8) and the final condition (1.9) simultaneously.

First we take some suitable artificial boundary conditions

$$x = 0 : \quad v_s = \eta_s(t), \quad s = m + 1, \cdots, n, \tag{3.9}$$

$$x = L : \quad v_r = \eta_r(t), \quad r = 1, \cdots, m, \tag{3.10}$$

where v_i $(i = 1, \cdots, n)$ are still given by (1.12), such that the forward IBVP (see Figure 1) for system (1.1) with the initial condition (1.8) and the boundary conditions (3.9)-(3.10) admits a unique semi-global C^1 solution $u = u_f(t, x)$ with small C^1 norm on

$$R_f = \{(t, x) \mid 0 \le t \le T_1,\ 0 \le x \le L\}. \tag{3.11}$$

Similarly, we can also take some other suitable artificial boundary conditions at $x = 0$ and $x = L$, such that the backward IBVP (see Figure 1) for system (1.1) with the final condition (1.9) and these boundary conditions admits a unique semi-global C^1 solution $u = u_b(t, x)$ with small C^1 norm on

$$R_b = \{(t, x) \mid T - T_2 \le t \le T,\ 0 \le x \le L\}. \tag{3.12}$$

Since $T_1 < T - T_2$, the domains R_f and R_b never intersect. Then there exists a C^1 function $\gamma(t)$ with small $C^1[0,T]$ norm such that

$$\gamma(t) = \begin{cases} u_f\Big(t, \dfrac{L}{2}\Big), & 0 \le t \le T_1, \\ u_b\Big(t, \dfrac{L}{2}\Big), & T - T_2 \le t \le T. \end{cases} \tag{3.13}$$

Now we change the status of the variables t and x. Then the leftward (resp. rightward) IBVP (see Figure 2) for system (1.1) with the initial condition

$$x = \frac{L}{2}: \quad u = \gamma(t), \quad 0 \le t \le T \tag{3.14}$$

and the following boundary conditions induced by (1.8) and (1.9) respectively:

$$t = 0: \quad v_r = l_r(0, x, \varphi(x))\varphi(x), \quad 0 \le x \le \frac{L}{2}, \quad r = 1, \cdots, m, \tag{3.15}$$

$$t = T: \quad v_s = l_s(T, x, \psi(x))\psi(x), \quad 0 \le x \le \frac{L}{2}, \quad s = m+1, \cdots, n, \tag{3.16}$$

$$(\text{resp.} \quad t = 0: \quad v_s = l_s(0, x, \varphi(x))\varphi(x), \quad \frac{L}{2} \le x \le L, \quad s = m+1, \cdots, n, \tag{3.17}$$

$$t = T: \quad v_r = l_r(T, x, \psi(x))\psi(x), \quad \frac{L}{2} \le x \le L, \quad r = 1, \cdots, m) \tag{3.18}$$

admits a unique semi-global C^1 solution $u = u_l(t,x)$ (resp. $u = u_r(t,x)$) with small C^1 norm on

$$R_l = \Big\{(t,x) \Big| 0 \le t \le T,\ 0 \le x \le \frac{L}{2}\Big\} \quad \Big(\text{resp. } R_r = \Big\{(t,x) \Big| 0 \le t \le T,\ \frac{L}{2} \le x \le L\Big\}\Big). \tag{3.19}$$

In particular, we have

$$|u_l(t,x)| \le \varepsilon_1, \quad \forall\, (t,x) \in R_l \quad (\text{resp. } |u_r(t,x)| \le \varepsilon_1, \quad \forall\, (t,x) \in R_r). \tag{3.20}$$

Let

$$u = u(t,x) = \begin{cases} u_l(t,x), & (t,x) \in R_l, \\ u_r(t,x), & (t,x) \in R_r. \end{cases} \tag{3.21}$$

We need only to check that $u = u(t,x)$ satisfies the initial condition (1.8) and the final condition (1.9) simultaneously.

By definition, both $u_l(t,x)$ and $u_f(t,x)$ satisfy the same one-side IBVP for system (1.1) with the initial condition

$$x = \frac{L}{2}: \quad u = \gamma(t), \quad 0 \le t \le T_1 \tag{3.22}$$

and the boundary condition (3.15). By (3.8), the maximum determinate domain of this one-side IBVP should contain the interval $[0, \frac{L}{2}]$ on x-axis (see Figure 3). Hence, we get

$$u_l(0,x) = u_f(0,x) = \varphi(x), \quad 0 \le x \le \frac{L}{2} \tag{3.23}$$

by the uniqueness of C^1 solution. Similarly, we have

$$u_r(0,x) = u_f(0,x) = \varphi(x), \quad \frac{L}{2} \le x \le L. \tag{3.24}$$

Thus, $u = u(t,x)$ satisfies the initial condition (1.8). In a similar way we get that $u = u(t,x)$ also verifies the final condition (1.9).

Taking the boundary controls as

$$H_s(t) = (v_s - G_s(t, v_1, \cdots, v_m))|_{x=0}, \qquad s = m+1, \cdots, n, \tag{3.25}$$

$$H_r(t) = (v_r - G_r(t, v_{m+1}, \cdots, v_n))|_{x=L}, \quad r = 1, \cdots, m, \tag{3.26}$$

where v_i $(i = 1, \cdots, n)$ are obtained from (1.12) and (3.21), we get the desired exact boundary controllability.

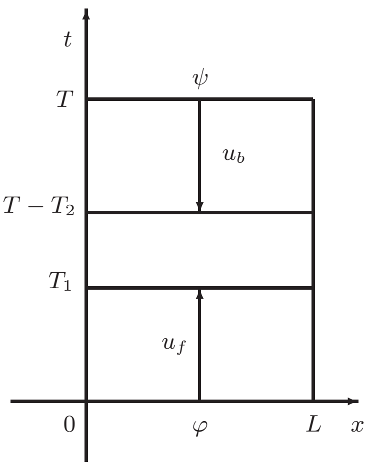

Figure 1

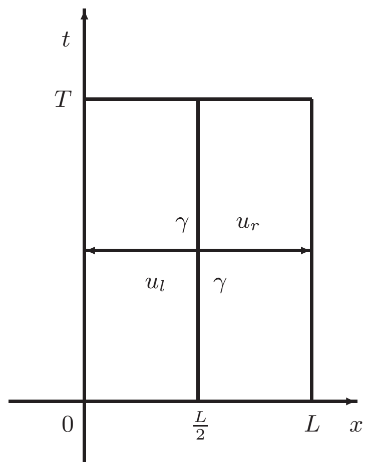

Figure 2

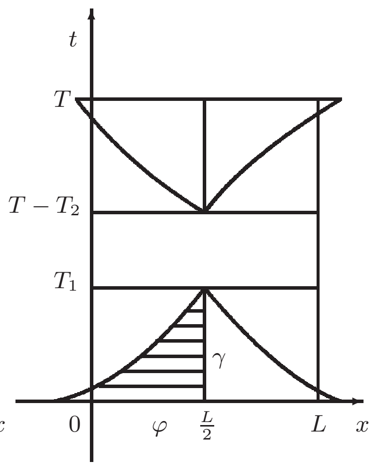

Figure 3

In the case of one-side control, without loss of generality, we may assume that the number of the positive eigenvalues is not bigger than that of the negative ones:

$$\overline{m} \triangleq n - m \le m, \quad \text{i.e., } n \le 2m. \tag{3.27}$$

Theorem 3.2 (One-Side Control) *Suppose that* $l_i, \lambda_i, a_i, b_i, f_i, c_i, G_i$ $(i = 1, \cdots, n)$ *and* H_s $(s = m+1, \cdots, n)$ *are all* C^1 *functions. Suppose furthermore that* (1.3)-(1.4), (3.1), (3.4) *and* (3.27) *hold. Suppose finally that the boundary condition* (3.2) *can be equivalently rewritten in a neighborhood of* $u = 0$ *as*

$$x = 0: \quad v_{\bar r} = \overline{G}_{\bar r}(t, v_{\overline{m}+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + \overline{H}_{\bar r}(t), \quad \bar r = 1, \cdots, \overline{m} \tag{3.28}$$

with

$$\overline{G}_{\bar r}(t, 0, \cdots, 0) \equiv 0, \quad \bar r = 1, \cdots, \overline{m}. \tag{3.29}$$

If there exist $T_1, T > 0$ *such that*

$$\int_0^{T_1} \min_{r=1,\cdots,m} \inf_{0\le x\le L} |\lambda_r(t,x,0)|\ dt > L, \tag{3.30}$$

$$\int_{T_1}^{T} \min_{s=m+1,\cdots,n} \inf_{0\le x\le L} |\lambda_s(t,x,0)|\ dt > L, \tag{3.31}$$

when $\|(b_i,c_i)\|_{C^1[R(T)]}$ $(i=1,\cdots,n)$ *and* $\|H_s\|_{C^1[0,T]}$ $(s=m+1,\cdots,n)$ *are sufficiently small, for any given initial data* φ *and final data* ψ *with small* C^1 *norm, satisfying the conditions of* C^1 *compatibility at the points* $(t,x)=(0,0)$ *and* $(T,0)$ *respectively, there exist boundary controls* $H_r(t)$ $(r=1,\cdots,m)$ *with small* C^1 *norm, such that the* IBVP (1.1), (1.8) *and* (3.2)-(3.3) *admits a unique semi-global* C^1 *solution* $u=u(t,x)$ *with small* C^1 *norm on* $R(T)$, *which verifies exactly the final condition* (1.9).

Proof By (3.30)-(3.31), there exists an $\varepsilon_1 > 0$ such that

$$\int_0^{T_1} \min_{r=1,\cdots,m} \inf_{\substack{0\le x\le L\\ |u|\le\varepsilon_1}} |\lambda_r(t,x,u)| dt > L, \tag{3.32}$$

$$\int_{T_1}^{T} \min_{s=m+1,\cdots,n} \inf_{\substack{0\le x\le L\\ |u|\le\varepsilon_1}} |\lambda_s(t,x,u)| dt > L. \tag{3.33}$$

Taking T_2 such that

$$\int_{T-T_2}^{T} \min_{s=m+1,\cdots,n} \inf_{\substack{0\le x\le L\\ |u|\le\varepsilon_1}} |\lambda_s(t,x,u)|\ dt = L, \tag{3.34}$$

we have $T_1 < T - T_2$.

First we choose some suitable artificial boundary conditions (3.10) at $x=L$, such that the forward IBVP (see Figure 4) for system (1.1) with the initial condition (1.8), the boundary conditions (3.2) and (3.10) admits a unique semi-global C^1 solution $u=u_f(t,x)$ with small C^1 norm on R_f.

Similarly, besides (3.2) (or (3.28)) we choose some additional artificial boundary conditions at $x=0$,

$$x=0: \quad v_r = \bar{\eta}_r(t), \quad r=\overline{m}+1,\cdots,m, \tag{3.35}$$

and some suitable artificial boundary condition at $x=L$,

$$x=L: \quad v_s = \bar{\eta}_s(t), \quad s=m+1,\cdots,n, \tag{3.36}$$

such that the backward IBVP (see Figure 4) for system (1.1) with the final condition (1.9) and these boundary conditions admits a unique semi-global C^1 solution $u=u_b(t,x)$ with small C^1 norm on R_b.

Since $T_1 < T-T_2$, the domains R_f and R_b never intersect. Then there exists a C^1 function $\gamma(t)$ with small $C^1[0,T]$ norm such that

$$\gamma(t) = \begin{cases} u_f(t,0), & 0\le t\le T_1, \\ u_b(t,0), & T-T_2\le t\le T, \end{cases} \tag{3.37}$$

and $\gamma(t)$ satisfies (3.2) on the whole interval $[0, T]$.

Now we change the status of the variables t and x. Then the rightward IBVP (see Figure 5) for system (1.1) with the initial condition

$$x = 0: \quad u = \gamma(t), \quad 0 \le t \le T \tag{3.38}$$

and the following boundary conditions induced by (1.8) and (1.9) respectively:

$$t = 0: \quad v_s = l_s(0, x, \varphi(x))\varphi(x), \quad 0 \le x \le L, \quad s = m+1, \cdots, n, \tag{3.39}$$

$$t = T: \quad v_r = l_r(T, x, \psi(x))\psi(x), \quad 0 \le x \le L, \quad r = 1, \cdots, m, \tag{3.40}$$

where v_i $(i = 1, \cdots, n)$ are given by (1.12), admits a unique semi-global C^1 solution $u = u(t, x)$ with small C^1 norm on $R(T) = \{(t, x) \mid 0 \le t \le T,\ 0 \le x \le L\}$. In particular, we have

$$|u(t, x)| \le \varepsilon_1, \quad \forall\, (t, x) \in R(T). \tag{3.41}$$

Consequently, $u(t, x)$ and $u_f(t, x)$ satisfy the same one-side IBVP for system (1.1) with the initial condition

$$x = 0: \quad u = \gamma(t), \quad 0 \le t \le T_1 \tag{3.42}$$

and the boundary condition (3.39). By (3.32), the maximum determinate domain of this one-side IBVP should contain the interval $[0, L]$ on x-axis (see Figure 6). Hence, we get

$$u(0, x) = u_f(0, x) = \varphi(x), \quad 0 \le x \le L \tag{3.43}$$

by the uniqueness of C^1 solution. Similarly, $u = u(t, x)$ also verifies the final condition (1.9).

Taking

$$H_r(t) = (v_r - G_r(t, v_{m+1}, \cdots, v_n))|_{x=L}, \quad r = 1, \cdots, m \tag{3.44}$$

as the boundary controls, where v_i $(i = 1, \cdots, n)$ are obtained by (1.12) and $u = u(t, x)$, we complete the proof of Theorem 3.2.

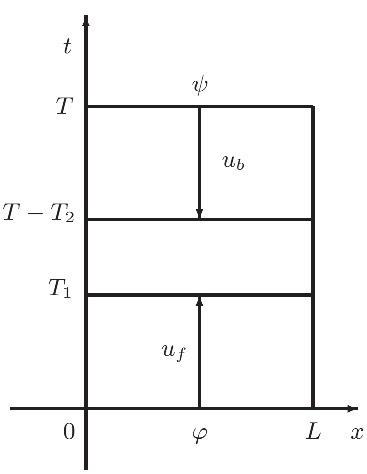

Figure 4

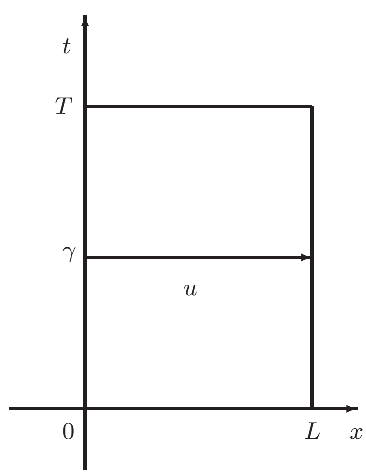

Figure 5

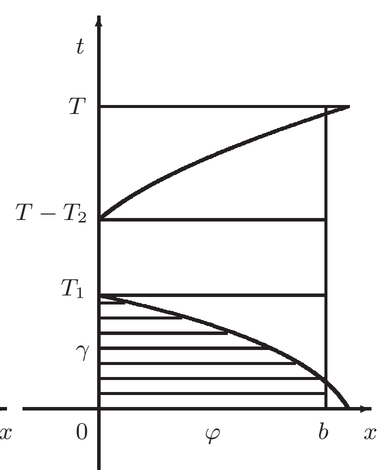

Figure 6

4 Exact Controllability for Nonautonomous First Order Quasilinear Hyperbolic Systems — Case with Zero Eigenvalues

Assume that system (1.1) has some zero eigenvalues, i.e., (1.7) holds. In order to realize the exact controllability, we should use not only suitable boundary controls acting on $x = 0$ and/or $x = L$ but also suitable internal controls on those equations which correspond to zero eigenvalues in (1.1). Consider the system

$$\begin{cases} l_p(t,x,u)\Big(\dfrac{\partial u}{\partial t} + \lambda_p(t,x,u)\dfrac{\partial u}{\partial x}\Big) = \mu_p(t,x,u), & p = 1,\cdots,l, \quad (4.1) \\ l_q(t,x,u)\dfrac{\partial u}{\partial t} = \mu_q(t,x,u) + \chi_q(t,x), & q = l+1,\cdots,m, \quad (4.2) \\ l_r(t,x,u)\Big(\dfrac{\partial u}{\partial t} + \lambda_r(u)\dfrac{\partial u}{\partial x}\Big) = \mu_r(t,x,u), & r = m+1,\cdots,n, \quad (4.3) \end{cases}$$

where

$$\chi_q(t,x) = \bar{a}_q(t,x)\frac{\partial \bar{b}_q(t,x)}{\partial t} + \bar{c}_q(t,x), \quad q = l+1,\cdots,m, \tag{4.4}$$

in which $\bar{a}_q, \bar{b}_q$ and $\bar{c}_q$ $(q = l+1,\cdots,m)$ are all C^1 functions of (t,x).

Similar to [8], we have the following

Theorem 4.1 (Two-Side and Internal Control) *Suppose that $l_i, \lambda_i, a_i, b_i, f_i, c_i, G_p$ and G_r $(i = 1,\cdots,n;\ p = 1,\cdots,l;\ r = m+1,\cdots,n)$ are all C^1 functions with respect to their arguments. Suppose furthermore that* (1.3)-(1.4), (1.7) *and* (1.13) *hold. If there exits $T > 0$ such that*

$$\int_0^T \min_{s=1,\cdots,l;\ m+1,\cdots,n} \inf_{0\le x\le L} |\lambda_s(t,x,0)|\, dt > L \tag{4.5}$$

and $\|(b_i,c_i)\|_{C^1[R(T)]}$ $(i = 1,\cdots,n)$ are sufficiently small, then, for any given initial data φ and final data ψ with sufficiently small C^1 norm, there exist boundary controls $H_p(t)$, $H_r(t)$ $(p = 1,\cdots,l;\ r = m+1,\cdots,n)$ with small C^1 norm and internal controls $\chi_q(t,x)$ $(q = l+1,\cdots,m)$ with "small" C^1 norm (see Remark 4.1 *for the precise meaning), such that the* IBVP (4.1)-(4.3), (1.8) *and* (1.10)-(1.11) *admits a unique semi-global C^1 solution $u = u(t,x)$ with small C^1 norm on $R(T) = \{(t,x) \mid 0 \le t \le T,\ 0 \le x \le L\}$, which verifies exactly the final condition* (1.9).

For one-side control, we still assume that positive eigenvalues are less than the negative ones:

$$\overline{m} \triangleq n - m \le l, \quad \text{i.e., } n \le l + m. \tag{4.6}$$

We have

Theorem 4.2 *Suppose that $l_i, \lambda_i, a_i, b_i, f_i, c_i, G_p, G_r$ and H_r $(i = 1,\cdots,n;\ p = 1,\cdots,l;\ r = m+1,\cdots,n)$ are all C^1 functions. Suppose furthermore that* (1.3)-(1.4), (1.7), (1.13) *and* (4.6) *hold. Suppose finally that the boundary condition* (1.10) *can be equivalently rewritten in a neighborhood of $u = 0$ as*

$$x = 0: \quad v_{\bar{p}} = \overline{G}_{\bar{p}}(t, v_{\overline{m}+1},\cdots,v_m,v_{m+1},\cdots,v_n) + \overline{H}_{\bar{p}}(t), \quad \bar{p} = 1,\cdots,\overline{m} \tag{4.7}$$

with

$$\overline{G}_{\bar{p}}(t,0,\cdots,0)\equiv 0, \quad \bar{p}=1,\cdots,\overline{m}. \tag{4.8}$$

If there exist $T_1, T>0$ *such that*

$$\int_0^{T_1} \min_{p=1,\cdots,m} \inf_{0\le x\le L} |\lambda_p(t,x,0)|dt > L, \tag{4.9}$$

$$\int_{T_1}^{T} \min_{r=m+1,\cdots,n} \inf_{0\le x\le L} |\lambda_r(t,x,0)|dt > L, \tag{4.10}$$

when $\|(b_i,c_i)\|_{C^1[R(T)]}$ $(i=1,\cdots,n)$ *and* $\|H_r\|_{C^1[0,T]}$ $(r=m+1,\cdots,n)$ *are sufficiently small, for any given initial data* φ *and final data* ψ *with sufficiently small* C^1 *norm, satisfying the conditions of* C^1 *compatibility* (*except* (4.2)) *at the points* $(t,x)=(0,0)$ *and* $(T,0)$ *respectively, there exist boundary controls* $H_p(t)$ $(p=1,\cdots,l)$ *with small* C^1 *norm and internal controls* $\chi_q(t,x)$ $(q=l+1,\cdots,m)$ *with "small"* C^1 *norm* (*see Remark* 4.1 *for the precise meaning*), *such that the* IBVP (4.1)-(4.3), (1.8) *and* (1.10)-(1.11) *admits a unique semi-global* C^1 *solution* $u=u(t,x)$ *with small* C^1 *norm on* $R(T)$, *which verifies exactly the final condition* (1.9).

Remark 4.1 The internal controls $\chi_q(t,x)$ $(q=l+1,\cdots,m)$ taken in Theorem 4.1 and Theorem 4.2 have the form of (4.4), where $\bar{a}_q, \bar{b}_q$, $\bar{c}_q$ $(q=l+1,\cdots,m)$ are all C^1 functions on $R(T)$ and the C^1 norms of $\bar{b}_q$ and $\bar{c}_q$ $(q=l+1,\cdots,m)$ are suitably small.

5 Remarks

Remark 5.1 Theorems 3.1 and 3.2 (or Theorems 4.1 and 4.2) show the exact controllability for general nonautonomous first order quasilinear hyperbolic systems. The assumptions (3.6) and (3.30)-(3.31) (or (4.5) and (4.9)-(4.10)) make essential restrictions on the behavior of the eigenvalues $\lambda_i(t,x,u)$ $(i=1,\cdots,n)$ with respect to t (see [12]).

Remark 5.2 Theorems 3.1-3.2 and Theorems 4.1-4.2 generalized all the results on the exact controllability for autonomous first order quasilinear hyperbolic systems in [3, 5–8].

Remark 5.3 Similar results hold, if the negative eigenvalues are less than the positive ones $(n\ge 2m)$. Similar to Theorem 3.2, we can establish the exact controllability by using suitable boundary controls $H_s(t)$ $(s=m+1,\cdots,n)$ at $x=0$, but the restrictions (3.30) and (3.31) on $T_1,\ T$ should be replaced by

$$\int_0^{T_1} \min_{s=m+1,\cdots,n} \inf_{0\le x\le L} |\lambda_s(t,x,0)|\ dt > L \tag{5.1}$$

and

$$\int_{T_1}^{T} \min_{r=1,\cdots,m} \inf_{0\le x\le L} |\lambda_r(t,x,0)|\ dt > L \tag{5.2}$$

respectively. Similar to Theorem 4.2, some suitable boundary controls $H_r(t)$ $(r=m+1,\cdots,n)$ can be acted on $x=0$, but the restrictions (4.9) and (4.10) on T_1, T should be replaced by

$$\int_0^{T_1} \min_{r=m+1,\cdots,n} \inf_{0\le x\le L} |\lambda_r(t,x,0)|\ dt > L \tag{5.3}$$

and

$$\int_{T_1}^{T} \min_{p=1,\cdots,m} \inf_{0\le x\le L} |\lambda_p(t,x,0)|\, dt > L \tag{5.4}$$

respectively. This means that the one-side boundary controls should be acted on the side where there are more boundary conditions and the number of the boundary controls is equal to the number of the boundary conditions on this side. In the special case where these two numbers are the same, the one-side boundary controls can be acted on either side.

Remark 5.4 The exact control time T given by (3.6) and (3.30)-(3.31) (or (4.5) and (4.9)-(4.10)) in Theorems 3.1 and 3.2 (or Theorems 4.1 and 4.2) is optimal.

Remark 5.5 The controls used to realize the exact controllability are not unique.

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