Chin. Ann. Math.
27B(6), 2006, 657–674
DOI: 10.1007/s11401-005-0472-6

Chinese Annals of Mathematics, Series B

Codimension 3 Non-resonant Bifurcations of Rough Heteroclinic Loops with One Orbit Flip***

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Abstract Heteroclinic bifurcations in four dimensional vector fields are investigated by setting up a local coordinates near a rough heteroclinic loop. This heteroclinic loop has a principal heteroclinic orbit and a non-principal heteroclinic orbit that takes orbit flip. The existence, nonexistence, coexistence and uniqueness of the 1-heteroclinic loop, 1-homoclinic orbit and 1-periodic orbit are studied. The existence of the two-fold or three-fold 1-periodic orbit is also obtained.

Keywords Bifurcation, Heteroclinic loop, Non-resonance, Orbit flip, Periodic orbit
2000 MR Subject Classification 34C37, 37C29, 34C23

1 Introduction

We have had many results on the bifurcations of principal homoclinic or heteroclinic loops in higher dimensional vector fields. For example, papers [1–4, 6, 7, 13, 18] deal with homoclinic bifurcations, and papers [8–10, 17, 19] deal with heteroclinic bifurcations. But few studies are concerned in the non-principal homoclinic (resp. heteroclinic) loops or invariant manifolds along the homoclinic (resp. heteroclinic) loops (see [12]). For example, [14] investigated codimension-two bifurcations of homoclinic orbits with an orbit flip. [11] studied codimension-two bifurcations of homoclinic orbits with an inclination flip. [5] studied codimension-three bifurcations in case that the resonance and either an orbit flip or an inclination flip hold simultaneously, and put forward some conjectures. [12] treated these conjectures on codimension-three resonant homoclinic flip bifurcations by numerical techniques. Because of the complexity, these non-principal homoclinic orbits and their associated bifurcations were mainly studied for 3-dimensional systems in the above mentioned references. Recently, we have considered codimension 3 homoclinic bifurcations in case that an orbit flip and an inclination flip hold simultaneously in [15]. We have also considered codimension 3 non-resonant bifurcations of homoclinic orbits with two inclination flips in [16]. In this paper, we study the codimension 3 bifurcations of rough heteroclinic loops that they are composed of a principal heteroclinic orbit and a non-principal heteroclinic orbit which takes orbit flip in 4-dimensional systems. It is worthy to be mentioned that the restriction on the dimension is not essential, the method

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***Project supported by the National Natural Science Foundation of China (No.10471087) and the Zhejiang Provincial Natural Science Foundation of China (No.Y605044).

used in this paper can be extended to any higher dimensional systems without any difficulty and the same conclusions can be deduced under the same hypotheses. The bifurcation results obtained here are also compared briefly with the relevant known results in the last section.

Consider the following C^r system and its unperturbed system

$$\dot{z} = f(z) + g(z,\mu), \tag{1.1}$$
$$\dot{z} = f(z), \tag{1.2}$$

where $r \geq 7,\ z \in \mathbb{R}^4,\ \mu \in \mathbb{R}^3,\ f(p_i) = 0,\ g(p_i,\mu) = g(z,0) = 0,\ f, g \in C^r$.

We need the following assumptions.

(H1) (Non-principal Hypothesis) System (1.2) has a heteroclinic loop

$$\Gamma = \Gamma_1 \cup \Gamma_2,$$

where $\Gamma_i = \{z = r_i(t) : t \in \mathbb{R}\},\ r_i(+\infty) = r_{i+1}(-\infty) = p_{i+1},\ r_3(t) = r_1(t),\ p_3 = p_1$, and the eigenvalues of $D_z f(p_i)$ are

$$-\rho_2^i, \quad -\rho_1^i, \quad \lambda_1^i, \quad \lambda_2^i,$$

which satisfy

$$-\rho_2^i < -\rho_1^i < 0 < \lambda_1^i < \lambda_2^i, \quad i = 1,2 \quad \text{and} \quad \frac{\rho_1^1 \rho_2^2}{\lambda_1^1 \lambda_1^2} \neq 1.$$

Let W_i^s and W_i^u be respectively the stable and unstable manifolds of p_i, $e_i^{\pm} = \lim\limits_{t\to\pm\infty} \frac{\dot{r}_i(-t)}{|\dot{r}_i(-t)|}$. Then

$$e_1^+ \in T_{p_1} W_1^u, \quad e_2^+ \in T_{p_2} W_2^u, \quad e_1^- \in T_{p_2} W_2^s \quad \text{and} \quad e_2^- \in T_{p_1} W_1^s$$

are unit eigenvectors corresponding to the eigenvalues $\lambda_1^1,\ \lambda_1^2,\ -\rho_2^2$ and $-\rho_1^1$ respectively.

Here, that $e_1^- \in T_{p_2} W_2^s$ is a unit eigenvector corresponding to the eigenvalue $-\rho_2^2$ means that Γ_1 enters the critical point p_2 in positive time along the strong stable direction of $T_{p_2} W_2^s$, that is to say, Γ_1 is a heteroclinic orbit with orbit flip, and so it is non-principal. While the assumption

$$\frac{\rho_1^1 \rho_2^2}{\lambda_1^1 \lambda_1^2} \neq 1$$

implies that Γ is a rough heteroclinic loop.

(H2) (Non-degenerate Hypothesis) $\dim(T_{r_i(t)} W_i^u \cap T_{r_i(t)} W_{i+1}^s) = 1$.

(H3) (Principal Hypothesis)

$$\begin{aligned}
\operatorname{span}(T_{r_1(t)} W_1^u,\ T_{r_1(t)} W_2^s,\ e_2^+) &= \mathbb{R}^4, \quad \text{as } t \gg 1,\\
\operatorname{span}(T_{r_2(t)} W_2^u,\ T_{r_2(t)} W_1^s,\ e_1^+) &= \mathbb{R}^4, \quad \text{as } t \gg 1,\\
\operatorname{span}(T_{r_1(t)} W_1^u,\ T_{r_1(t)} W_2^s,\ e_2^-) &= \mathbb{R}^4, \quad \text{as } t \ll -1,\\
\operatorname{span}(T_{r_2(t)} W_2^u,\ T_{r_2(t)} W_1^s,\ T_{p_2} W_2^{s-}) &= \mathbb{R}^4, \quad \text{as } t \ll -1,
\end{aligned}$$

where $T_{p_2}W_2^{s-}$ is a unit eigenvector associated with the eigenvalue $-\rho_1^2$.

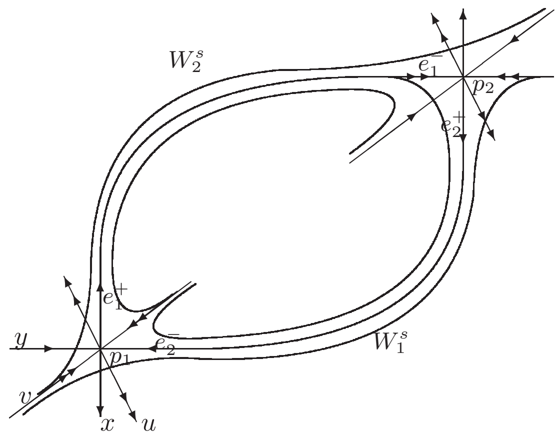

Figure 1

(For simplicity, we have only drawn stable manifolds W_1^s and W_2^s in this figure)

With the above assumptions, the heteroclinic loop Γ is of codimension-three. The assumption (H3) is also called the strong inclination property (cf. [2]), its genericity will be explained at the end of this section. (H1) and the last two hypotheses of (H3) are illustrated in Figure 1.

Now we devote to the establishment of the normal form of system (1.1) in a series steps.

Step 1 Introducing a linear transformation if necessary, system (1.1) takes the form

$$\begin{cases} \dot{x} = \lambda_1^i(\mu)x + O(2), \\ \dot{y} = -\rho_1^i(\mu)y + O(2), \\ \dot{u} = \lambda_2^i(\mu)u + O(2), \\ \dot{v} = -\rho_2^i(\mu)v + O(2), \end{cases} \tag{1.3}$$

where $\lambda_j^i(0) = \lambda_j^i,\ \rho_j^i(0) = \rho_j^i,\ i, j = 1, 2.$

Step 2 For system (1.3), the stable and unstable manifold theorem implies that there exist C^r manifolds $W_{p_i}^s$ and $W_{p_i}^u$ intersecting at the origin O, where $W_{p_i}^s = \{z = (x, y, u, v) \mid x = x_i(y, v), u = u_i(y, v), x_i(0, 0) = u_i(0, 0) = 0, \frac{\partial(x,u)}{\partial(y,v)} = 0, (y, v) \in U_i^s\}$ and $W_{p_i}^u = \{z = (x, y, u, v) \mid y = y_i(x, u), v = v_i(x, u), y_i(0, 0) = v_i(0, 0) = 0, \frac{\partial(y,v)}{\partial(x,u)} = 0, (x, u) \in U_i^u\}$ are local stable and unstable manifolds, respectively, $U_i^s \subset R^s = \{z \mid x = u = 0\},\ U_i^u \subset R^u = \{z \mid y = v = 0\},\ U_i^s \times U_i^u \subset U_i \subset U_i^1 \subset \mathbb{R}^4$, U_i and U_i^1 are sufficiently small neighborhoods of the origin O.

Next we want to straighten the local manifolds $W_{p_i}^s$ and $W_{p_i}^u$ so that $W_{p_i}^s = \{z \mid x = u = 0, z \in U_i\}$, $W_{p_i}^u = \{z \mid y = v = 0, z \in U_i\}$. Denote by $C_i^{s1}, C_i^{s2}, C_i^{u1}$ and C_i^{u2} the cones with

$$C_i^{s1} \subset C_i^{s2}, \quad C_i^{u1} \subset C_i^{u2}, \quad C_i^{s2} \cap C_i^{u2} = \{0\}, \quad W_{p_i}^s \subset C_i^{s1}, \quad W_{p_i}^u \subset C_i^{u1}.$$

Let $\widetilde{C}_i^{jk} = C_i^{jk} \cap U_i^{k-1}$ for $j = s, u, k = 1, 2$, and $U_i^0 = U_i$. Then we choose two C^∞ bump functions ϕ_i^s and ϕ_i^u such that

$$\phi_i^s(z)=\begin{cases}1 & \text{for } z\in\widetilde{C}_i^{s1},\\ 0 & \text{for } z\notin\widetilde{C}_i^{s2},\end{cases}\qquad \phi_i^u(z)=\begin{cases}1 & \text{for } z\in\widetilde{C}_i^{u1},\\ 0 & \text{for } z\notin\widetilde{C}_i^{u2},\end{cases}$$

and $\phi_i^s(z)\in(0,1)$ for $z\in\widetilde{C}_i^{s2}-\widetilde{C}_i^{s1}$, $\phi_i^u(z)\in(0,1)$ for $z\in\widetilde{C}_i^{u2}-\widetilde{C}_i^{u1}$. At last, taking the straightening coordinate transformation

$$\begin{aligned}&x\to x-\phi_i^s(z)x_i(y,v),\quad y\to y,\quad u\to u-\phi_i^s(z)u_i(y,v),\quad v\to v\qquad \text{as } z\in C_i^{s2},\\ &x\to x,\quad y\to y-\phi_i^u(z)y_i(x,u),\quad u\to u,\quad v\to v-\phi_i^u(z)v_i(x,u)\qquad \text{as } z\in C_i^{u2},\end{aligned}$$

system (1.3) is changed into the following form in U_i

$$\begin{cases}\dot{x}=x(\lambda_1^i(\mu)+o(1))+O(u)(O(y)+O(u)+O(v)),\\ \dot{y}=y(-\rho_1^i(\mu)+o(1))+O(v)(O(x)+O(u)+O(v)),\\ \dot{u}=u(\lambda_2^i(\mu)+o(1))+O(x)(O(x)+O(y)+O(v)),\\ \dot{v}=v(-\rho_2^i(\mu)+o(1))+O(y)(O(x)+O(y)+O(u)),\end{cases}\tag{1.4}$$

by the invariance of $W_{p_i}^s$ and $W_{p_i}^u$. System (1.4) is C^{r-1}.

Step 3 Similarly to the above, we may further straighten the C^{r-1} local strong stable manifold and the local strong unstable manifold so that $W_{p_i}^{ss}=W_i^{ss}\cap U_i=\{z\mid x=y=u=0,z\in U_i\}$, $W_{p_i}^{uu}=W_i^{uu}\cap U_i=\{z\mid x=y=v=0,z\in U_i\}$. Owing to the invariance of $W_{p_i}^{ss},W_{p_i}^{uu}$, system (1.4) locally becomes

$$\begin{cases}\dot{x}=x(\lambda_1^i(\mu)+o(1))+O(u)(O(y)+O(v)),\\ \dot{y}=y(-\rho_1^i(\mu)+o(1))+O(v)(O(x)+O(u)),\\ \dot{u}=u(\lambda_2^i(\mu)+o(1))+O(x)(O(x)+O(y)+O(v)),\\ \dot{v}=v(-\rho_2^i(\mu)+o(1))+O(y)(O(x)+O(y)+O(u)).\end{cases}\tag{1.5}$$

System (1.5) is C^{r-2}.

Remark 1.1 Now we explain concisely the genericity of hypotheses (H3). Notice that

$$\begin{aligned}&W_{p_2}^s=\mathrm{span}\{(0,1,0,0)^*,(0,0,0,1)^*\},\quad W_{p_2}^u=\mathrm{span}\{(1,0,0,0)^*,(0,0,1,0)^*\},\\ &W_{p_2}^{ss}=\mathrm{span}\{(0,0,0,1)^*\},\quad W_{p_2}^{uu}=\mathrm{span}\{(0,0,1,0)^*\},\quad e_2^+=(1,0,0,0)^*.\end{aligned}$$

On the other hand, both the hypothesis (H2) and the assumption on the orbit flip of Γ_1 mean that $T_{r_1(T_1^1)}W_1^u=\mathrm{span}\{(0,0,0,1)^*,(b_1,b_2,b_3,b_4)^*\}$ with $b_1^2+b_3^2\neq 0$. So generically, $b_3\neq 0$. Then (1.3) implies that $\frac{b_i}{b_3}\to 0$ exponentially as $T_1^1\to+\infty$ for $i=1,2,4$. This is just the meanings of the first assumption of (H3). The others of (H3) can be interpreted in the same way.

2 Local Coordinates and Bifurcation Equations

Set $A_i(t)=D_zf(r_i(t))$. We consider the linear system and its adjoint system

$$\dot{z}=A_i(t)z,\tag{2.1}$$

$$\dot{\psi}=-A_i^*(t)\psi.\tag{2.2}$$

Equation (1.5) is equivalent to that, locally speaking, the stable (resp. unstable) manifold is the y-v (resp. x-u) plane, and the strong stable (resp. strong unstable) manifold is the v- (resp. u-) axis. Thus (H1) implies that, for δ being small enough so that $\{z : |z| < 2\delta\} \subset U_i$, there exist $T_i^0, T_i^1 \gg 1$ such that $r_1(-T_1^0) = (\delta, 0, \delta_1^u, 0)^*$, $r_2(-T_2^0) = (\delta, 0, \delta_2^u, 0)^*$, $r_1(T_1^1) = (0, 0, 0, \delta)^*$, $r_2(T_2^1) = (0, \delta, 0, \delta_2^s)^*$. By the tangency of Γ_1 with the x-axis, we have $|\delta_1^u| = O(\delta^2)$. Similarly, there are $|\delta_2^u|, |\delta_2^s| = O(\delta^2)$.

Lemma 2.1 *Denote $r_i(t) = (r_1^i,\ r_2^i,\ r_3^i,\ r_4^i)^*(t)$. There exist $\overline{\omega}_{11}^1$ and $\overline{\omega}_{13}^2$ such that system (2.1) has a fundamental solution matrix $Z_i(t) = (z_1^i(t), z_0^i(t), z_2^i(t), z_3^i(t))$ satisfying*

$$z_1^i(t) \in (T_{r_i(t)} W_i^u)^c \cap (T_{r_i(t)} W_{i+1}^s)^c,$$

$$z_0^1(t) = \frac{-\dot{r}_1(t)}{|\dot{r}_1(T_1^1)|} \in T_{r_1(t)} W_1^u \cap T_{r_1(t)} W_2^s, \quad z_0^2(t) = \frac{-\dot{r}_2(t)}{|\dot{r}_2^2(T_2^1)|} \in T_{r_2(t)} W_2^u \cap T_{r_2(t)} W_1^s,$$

$$z_2^i(t) \in T_{r_i(t)} W_i^u \cap (T_{r_i(t)} W_{i+1}^s)^c,$$

$$z_3^i(t) \in (T_{r_i(t)} W_i^u)^c \cap T_{r_i(t)} W_{i+1}^s,$$

$$Z_1(-T_1^0) = \begin{pmatrix} \omega_{10}^1 & \omega_{00}^1 & 0 & \omega_{30}^1 \\ \omega_{11}^1 & 0 & 0 & \omega_{31}^1 \\ \omega_{12}^1 & \omega_{02}^1 & 1 & \omega_{32}^1 \\ 0 & 0 & 0 & \omega_{33}^1 \end{pmatrix}, \quad Z_1(T_1^1) = \begin{pmatrix} 1 & 0 & \omega_{20}^1 & 0 \\ \overline{\omega}_{11}^1 & 0 & \omega_{21}^1 & 1 \\ 0 & 0 & \omega_{22}^1 & 0 \\ 0 & 1 & \omega_{23}^1 & 0 \end{pmatrix},$$

$$Z_2(-T_2^0) = \begin{pmatrix} \omega_{10}^2 & \omega_{00}^2 & 0 & \omega_{30}^2 \\ \omega_{11}^2 & 0 & 0 & \omega_{31}^2 \\ \omega_{12}^2 & \omega_{02}^2 & 1 & \omega_{32}^2 \\ 0 & 0 & 0 & \omega_{33}^2 \end{pmatrix}, \quad Z_2(T_2^1) = \begin{pmatrix} 1 & 0 & \omega_{20}^2 & 0 \\ 0 & 1 & \omega_{21}^2 & 0 \\ 0 & 0 & \omega_{22}^2 & 0 \\ \overline{\omega}_{13}^2 & \omega_{03}^2 & \omega_{23}^2 & 1 \end{pmatrix},$$

where $|\omega_{jj}^i| \neq 0,\ j = 0,1,2,3,\ i = 1,2,$ and $|\overline{\omega}_{11}^1| \ll 1,\ |\overline{\omega}_{13}^2| \ll 1,\ |\omega_{03}^2| \ll 1,\ |(\omega_{00}^i)^{-1}\omega_{02}^i| \ll 1,\ i = 1,2,\ |(\omega_{11}^i)^{-1}\omega_{1j}^i| \ll 1,\ i = 1,2,\ j = 0,2,\ |(\omega_{22}^i)^{-1}\omega_{2j}^i| \ll 1,\ i = 1,2,\ j = 0,1,3,$ $|(\omega_{33}^i)^{-1}\omega_{3j}^i| \ll 1,\ i = 1,2,\ j = 0,1,2,$ as $T_i^0, T_i^1 \gg 1$.

Proof Here we only consider $i = 1$. Clearly, it follows from the expressions of the local invariant manifolds in U_1 that we can take $z_2^1(t)$, $z_3^1(t)$ satisfying $z_2^1(-T_1^0) = (0, 0, 1, 0)^*$, $z_3^1(T_1^1) = (0, 1, 0, 0)^*$. By the definition of $z_0^1(t)$ and the hypothesis on the orbit flip of Γ_1, we see $z_0^1(T_1^1)$ and $z_0^1(-T_1^0)$ must take the values as shown in $Z_1(T_1^1)$ and $Z_1(-T_1^0)$ with $\omega_{00}^1 \neq 0$. The hypothesis (H3) on the strong inclination property implies that $\omega_{22}^1 \neq 0$ and $\omega_{33}^1 \neq 0$.

Now we consider $z_1^1(T_1^1)$ and $z_1^1(-T_1^0)$. Based on the first hypothesis of (H3), we have $T_{r_1(T_1^1)} W_2^s = \operatorname{span}\{(0, 1, 0, 0)^*,\ (0, 0, 0, 1)^*\}$, $T_{r_1(T_1^1)} W_1^u = \operatorname{span}\{(0, 0, 0, 1)^*,\ (0, 0, 1, 0)^*\}$. Then, it is easy to see that we can take $\tilde{z}_1^1(t) \in (T_{r_1(t)} W_1^u)^c \cap (T_{r_1(t)} W_2^s)^c$ such that $\tilde{z}_1^1(T_1^1) = (1, 0, 0, 0)$ and $\tilde{z}_1^1(-T_1^0) = (\widetilde{\omega}_{10}^1, \widetilde{\omega}_{11}^1, \widetilde{\omega}_{12}^1, \widetilde{\omega}_{13}^1)$. If $\widetilde{\omega}_{13}^1 = 0$, then we set $z_1^1 = \tilde{z}_1^1(t)$. Otherwise, owing to $\omega_{33}^1 \neq 0$, we take $z_1^1(t) = \tilde{z}_1^1(t) - \widetilde{\omega}_{13}^1(\omega_{33}^1)^{-1} z_3^1(t) \in (T_{r_1(t)} W_1^u)^c \cap (T_{r_1(t)} W_2^s)^c$ with $\overline{\omega}_{11}^1 = -\widetilde{\omega}_{13}^1(\omega_{33}^1)^{-1}$, and $z_1^1(-T_1^0) = (\widetilde{\omega}_{10}^1 - \widetilde{\omega}_{13}^1(\omega_{33}^1)^{-1}\omega_{30}^1,\ \widetilde{\omega}_{11}^1 - \widetilde{\omega}_{13}^1(\omega_{33}^1)^{-1}\omega_{31}^1,\ \widetilde{\omega}_{12}^1 - \widetilde{\omega}_{13}^1(\omega_{33}^1)^{-1}\omega_{32}^1,\ 0)$. According to Liouville's formula, $\det Z(T_1^1) \neq 0$ implies $\det Z(-T_1^0) \neq 0$, and so $\omega_{11}^1 \neq 0$.

Now we show $|(\omega_{33}^1)^{-1}\omega_{3j}^1| \ll 1$ for $j = 0, 1, 2$. Let T_1^1 (resp. T_1^0) increase to $T_1^1 + T$ (resp. $T_1^0 + T$). Then

$$z_3^1(T_1^1 + T) = e^{-\rho_1^1 T} z_3^1(T_1^1), \quad z_3^1(-T_1^0 - T) = (\omega_{30}^1 e^{-\lambda_1^1 T}, \omega_{31}^1 e^{\rho_1^1 T}, \omega_{32}^1 e^{-\lambda_2^1 T}, \omega_{33}^1 e^{\rho_2^1 T}).$$

Reset $z_3^1(T_1^1+T)=(0,1,0,0)$. Then it is easy to see that ω_{33}^1 becomes $\omega_{33}^1 e^{(\rho_1^1+\rho_2^1)T}$ and the new components of $z_3^1(-T_1^0-T)$ satisfy $|(\omega_{33}^1)^{-1}\omega_{3j}^1| \to 0$ as $T\to+\infty$ for $j=0,1,2$. The others can be proved in the same way. Thus the proof is complete.

Denote $\Psi_i(t)=(Z_i^{-1}(t))^*=(\psi_1^i(t),\psi_0^i(t),\psi_2^i(t),\psi_3^i(t))$. Then, $\Psi_i(t)$ is a fundamental solution matrix of the adjoint system (2.2). Using the transformation

$$z=r_i(t)+(z_1^i(t),z_2^i(t),z_3^i(t))(n_1^i,n_2^i,n_3^i)^* \stackrel{\text{def}}{=} S_i(t), \quad t\in[-T_i^0,T_i^1]$$

in the neighborhood of Γ_i, we see that system (1.1) becomes

$$\begin{aligned}&\dot r_i(t)+\dot Z_i(t)(n_1^i,0,n_2^i,n_3^i)^*+Z_i(t)(\dot n_1^i,0,\dot n_2^i,\dot n_3^i)^*\\ &=f(r_i(t))+A_i(t)Z_i(t)(n_1^i,0,n_2^i,n_3^i)^*+g(r_i(t),\mu)+\text{h.o.t.}\end{aligned}$$

By $\dot r_i(t)=f(r_i(t))$ and $\dot Z_i(t)=A_i(t)Z_i(t)$, the above equation can be simplified to the following

$$Z_i(t)(\dot n_1^i,0,\dot n_2^i,\dot n_3^i)^*=g(r_i(t),\mu)+\text{h.o.t.}$$

Multiplying two sides of the equation by $\Psi_i^*(t)$ and utilizing $\Psi_i^*(t)Z_i(t)=I$, we get

$$\dot n_j^i(t)=\psi_j^i(t)g(r_i(t),\mu)+\text{h.o.t.}, \quad j=1,2,3. \tag{2.3}$$

Equation (2.3) produces a map $P_1^i: S_1^i\to S_0^i$, where $S_1^i=\{z=S_i(-T_i^0): |z|<\frac{3}{2}\delta\}$, $S_0^i=\{z=S_i(T_i^1): |z|<\frac{3}{2}\delta\}$. Integrating two sides of equation (2.3) from $-T_i^0$ to T_i^1, we get

$$n_j^i(T_i^1)=n_j^i(-T_i^0)+M_j^i\mu+\text{h.o.t.}, \quad j=1,2,3, \tag{2.4}$$

where $M_j^i=\int_{-T_i^0}^{T_i^1}\psi_j^i(t)g_\mu(r_i(t),0)dt$, $j=1,2,3$.

Lemma 2.2 $M_1^1=\displaystyle\int_{-\infty}^{\infty}\psi_1^1(t)g_\mu(r_1(t),0)\,dt, \quad M_1^2=\int_{-\infty}^{\infty}\psi_1^2(t)g_\mu(r_2(t),0)\,dt,$
$M_3^1=\displaystyle\int_{-\infty}^{\infty}\psi_3^1(t)g_\mu(r_1(t),0)\,dt.$

Proof We first have $r_1(t)=(0,0,0,r_1^4(t))$, as $t\ge T_1^1$, where $|r_1^4(t)|=O(\delta)$. Then equation (1.5) implies that $g_\mu(r_1(t),0)=(0,0,0,g_1^4(t))$, $|g_1^4(t)|=O(\delta)$ as $t\ge T_1^1$, and that

$$A_1(t)=\begin{pmatrix}\lambda_1^1+O(\delta) & 0 & O(\delta) & 0\\ O(\delta) & -\rho_1^1+O(\delta) & O(\delta) & 0\\ O(\delta) & 0 & \lambda_2^1+O(\delta) & 0\\ O(\delta) & O(\delta) & O(\delta) & -\rho_2^1+O(\delta)\end{pmatrix} \quad \text{as } t\ge T_1^1.$$

Denote $\psi_1^1(t)=(a(t),b(t),c(t),d(t))^*$. Based on $\Psi_1^*(T_1^1)Z_1(T_1^1)=I$, we see that $b(T_1^1)=d(T_1^1)=0$, $a(T_1^1)=1$, $c(T_1^1)=-(\omega_{22}^1)^{-1}\omega_{20}^1$. We solve equation (2.2) with the initial value $(a(T_1^1),b(T_1^1),c(T_1^1),d(T_1^1))$, and get $b(t)=d(t)=0$ as $t\ge T_1^1$. Hence, we obtain $\psi_1^1(t)g_\mu(r_1(t),0)\equiv 0$ as $t\ge T_1^1$.

Similarly, we have $r_1(t)=(r_1^1(t),0,r_1^3(t),0)$, $g_\mu(r_1(t),0)=(g_1^1(t),0,g_1^3(t),0)$, $|r_1^1(t)|=O(\delta)$, $|r_1^3(t)|=O(\delta^2)$, $|g_1^1(t)|=O(\delta)$, $|g_1^3(t)|=O(\delta^2)$ as $t\le -T_1^0$ and

$$A_1(t)=\begin{pmatrix}\lambda_1^1+O(\delta) & O(\delta) & O(\delta) & O(\delta)\\ 0 & -\rho_1^1+O(\delta) & 0 & O(\delta)\\ O(\delta) & O(\delta) & \lambda_2^1+O(\delta) & O(\delta)\\ 0 & O(\delta) & 0 & -\rho_2^1+O(\delta)\end{pmatrix} \quad \text{as } t\le -T_1^0.$$

So we can also show that the first and third components of $\psi_1^1(t)$ are equal to zero for $t \le -T_1^0$. Therefore, we still have $\psi_1^1(t) g_\mu(r_1(t), 0) \equiv 0$ as $t \le -T_1^0$. The first equality holds. The others can be proved in the same method. The proof is complete.

Next consider the maps $P_0^i : S_0^{i+1} \to S_1^i,\ q_0^{i+1} \stackrel{\text{def}}{=} (x_0^{i+1}, y_0^{i+1}, u_0^{i+1}, v_0^{i+1}) \mapsto q_1^i \stackrel{\text{def}}{=} (x_1^i, y_1^i, u_1^i, v_1^i)$ induced by the flow of system (1.3) in the neighborhood U_i, where $S_0^3 = S_0^1,\ q_0^3 = q_0^1$. To ensure the differentiability of the maps P_0^i at the origin, let $s_i = e^{-\lambda_1^i(\mu)\tau_i}$, where τ_i be the time flying from q_0^{i+1} to q_1^i. Omitting all higher terms we get (see [19])

$$x_0^{i+1} = x_1^i s_i, \quad y_1^i = s_i^{\frac{\rho_1^i(\mu)}{\lambda_1^i(\mu)}} y_0^{i+1}, \quad u_0^{i+1} = u_1^i s_i^{\frac{\lambda_2^i(\mu)}{\lambda_1^i(\mu)}}, \quad v_1^i = s_i^{\frac{\rho_2^i(\mu)}{\lambda_1^i(\mu)}} v_0^{i+1}. \tag{2.5}$$

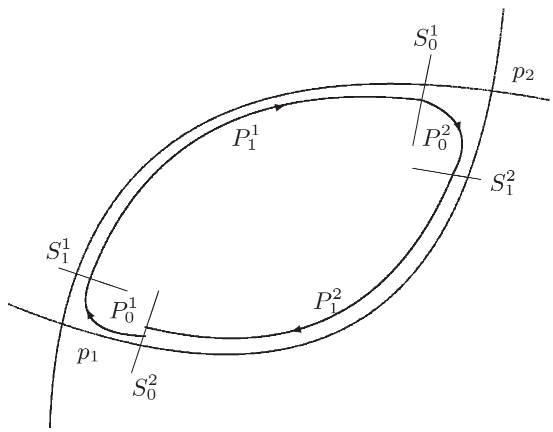

Figure 2

To establish the Poincaré map, we need to build up first the relationship between q_0^i, q_1^i and their new coordinates $q_0^i(n_1^i, n_2^i, n_3^i),\ q_1^i(n_1^i, n_2^i, n_3^i)$. Using the following formulas

$$(x_0^i, y_0^i, u_0^i, v_0^i) = r_i(T_i^1) + z_1(T_i^1) n_1^i + z_2(T_i^1) n_2^i + z_3(T_i^1) n_3^i,$$
$$(x_1^i, y_1^i, u_1^i, v_1^i) = r_i(-T_i^0) + z_1^i(-T_i^0) n_1^i + z_2^i(-T_i^0) n_2^i + z_3^i(-T_i^0) n_3^i,$$

and the expressions of $Z_i(T_i^1), Z_i(-T_i^0)$, we obtain

$$\begin{aligned}
n_1^1 &= x_0^1 - \omega_{20}^1(\omega_{22}^1)^{-1} u_0^1 \approx \delta s_2 - \omega_{20}^1(\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0,\\
n_2^1 &= (\omega_{22}^1)^{-1} u_0^1 \approx (\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0,\\
n_3^1 &= y_0^1 - \omega_{21}^1(\omega_{22}^1)^{-1} u_0^1 - \overline{\omega}_{11}^1 x_0^1 + \overline{\omega}_{11}^1 \omega_{20}^1 (\omega_{22}^1)^{-1} u_0^1\\
&\approx y_0^1 - \omega_{21}^1(\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0 - \overline{\omega}_{11}^1 \delta s_2 + \overline{\omega}_{11}^1 \omega_{20}^1 (\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0,\\
v_0^1 &= \delta + \omega_{23}^1 n_2^1 \approx \delta, \qquad\qquad (2.6)\\
n_1^2 &= x_0^0 - \omega_{20}^2(\omega_{22}^2)^{-1} u_0^0 \approx \delta s_1 - \omega_{20}^2(\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1,\\
n_2^2 &= (\omega_{22}^2)^{-1} u_0^0 \approx (\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1,\\
n_3^2 &= v_0^0 - \delta_2^s - \omega_{23}^2(\omega_{22}^2)^{-1} u_0^0 - \overline{\omega}_{13}^2 x_0^0 + \overline{\omega}_{13}^2 \omega_{20}^2 (\omega_{22}^2)^{-1} u_0^0
\end{aligned}$$

$$\approx v_0^0 - \delta_2^s - \omega_{23}^2(\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1 - \overline{\omega}_{13}^2 \delta s_1 + \overline{\omega}_{13}^2 \omega_{20}^2 (\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1,$$

$$y_0^0 = \delta + \omega_{21}^2 n_2^2 \approx \delta,$$

and

$$\begin{aligned}
n_1^1 &= (\omega_{11}^1)^{-1} y_1^1 - (\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} v_1^1 \approx (\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} \delta - (\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0,\\
n_2^1 &= u_1^1 - \delta_1^u - \omega_{32}^1(\omega_{33}^1)^{-1} v_1^1 - \omega_{12}^1(\omega_{11}^1)^{-1} y_1^1 + \omega_{12}^1(\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} v_1^1\\
&\approx u_1^1 - \delta_1^u - \omega_{32}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 - \omega_{12}^1(\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} \delta + \omega_{12}^1(\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0,\\
n_3^1 &= (\omega_{33}^1)^{-1} v_1^1 \approx (\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0,\\
x_1^1 &= \delta + \omega_{10}^1 n_1^1 + \omega_{30}^1 n_3^1 \approx \delta,\\
n_1^2 &= (\omega_{11}^2)^{-1} y_1^0 - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} v_1^0 \approx (\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta,\\
n_2^2 &= u_1^0 - \delta_2^u - \omega_{32}^2(\omega_{33}^2)^{-1} v_1^0 - \omega_{12}^2(\omega_{11}^2)^{-1} y_1^0 + \omega_{12}^2(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} v_1^0\\
&\approx u_1^0 - \delta_2^u - \omega_{32}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta - \omega_{12}^2(\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 + \omega_{12}^2(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta,\\
n_3^2 &= (\omega_{33}^2)^{-1} v_1^0 \approx (\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta,\\
x_1^0 &= \delta + \omega_{10}^2 n_1^2 + \omega_{30}^2 n_3^2 \approx \delta.
\end{aligned} \tag{2.7}$$

Now, by (2.4), (2.5) and (2.7) we get the expression of the map $P_i \stackrel{\text{def}}{=} P_1^i \circ P_0^i$ as follows:

$$\begin{aligned}
n_1^1(T_1^1) &= (\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} \delta - (\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 + M_1^1\mu + \text{h.o.t.},\\
n_2^1(T_1^1) &= u_1^1 - \delta_1^u - \omega_{32}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 - \omega_{12}^1(\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} \delta + \omega_{12}^1(\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0\\
&\quad + M_2^1\mu + \text{h.o.t.},\\
n_3^1(T_1^1) &= (\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 + M_3^1\mu + \text{h.o.t.},\\
n_1^2(T_2^1) &= (\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta + M_1^2\mu + \text{h.o.t.},\\
n_2^2(T_2^1) &= u_1^0 - \delta_2^u - \omega_{32}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta - \omega_{12}^2(\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 + \omega_{12}^2(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta\\
&\quad + M_2^2\mu + \text{h.o.t.},\\
n_3^2(T_2^1) &= (\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} \delta + M_3^2\mu + \text{h.o.t.}
\end{aligned} \tag{2.8}$$

Combining equalities (2.6) and (2.8) we get the successor functions

$$\begin{aligned}
G_1^1 &\stackrel{\text{def}}{=} (\omega_{11}^1)^{-1}\delta s_1^{\frac{\rho_1^1}{\lambda_1^1}} - (\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 - \delta s_2 + \omega_{20}^1(\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0\\
&\quad + M_1^1\mu + \text{h.o.t.},
\end{aligned}$$

$$\begin{aligned}
G_1^2 &\overset{\text{def}}{=} (\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_1^2}} - \delta s_1 + \omega_{20}^2(\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1 \\
&\quad + M_1^2\mu + \text{h.o.t.}, \\
G_2^1 &\overset{\text{def}}{=} u_1^1 - \delta_1^u - \omega_{32}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 - \omega_{12}^1(\omega_{11}^1)^{-1}\delta s_1^{\frac{\rho_1^1}{\lambda_1^1}} + \omega_{12}^1(\omega_{11}^1)^{-1}\omega_{31}^1(\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 \\
&\quad - (\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0 + M_2^1\mu + \text{h.o.t.}, \\
G_2^2 &\overset{\text{def}}{=} u_1^0 - \delta_2^u - \omega_{32}^2(\omega_{33}^2)^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_1^2}} - \omega_{12}^2(\omega_{11}^2)^{-1} s_2^{\frac{\rho_1^2}{\lambda_1^2}} y_0^1 + \omega_{12}^2(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}}\delta \\
&\quad - (\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1 + M_2^2\mu + \text{h.o.t.}, \\
G_3^1 &\overset{\text{def}}{=} (\omega_{33}^1)^{-1} s_1^{\frac{\rho_2^1}{\lambda_1^1}} v_0^0 - y_0^1 + \omega_{21}^1(\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0 + \overline{\omega}_{11}^1\delta s_2 - \overline{\omega}_{11}^1\omega_{20}^1(\omega_{22}^1)^{-1} s_2^{\frac{\lambda_2^2}{\lambda_1^2}} u_1^0 \\
&\quad + M_3^1\mu + \text{h.o.t.}, \\
G_3^2 &\overset{\text{def}}{=} (\omega_{33}^2)^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_1^2}} - v_0^0 + \delta_2^s + \omega_{23}^2(\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1 + \overline{\omega}_{13}^2\delta s_1 - \overline{\omega}_{13}^2\omega_{20}^2(\omega_{22}^2)^{-1} s_1^{\frac{\lambda_2^1}{\lambda_1^1}} u_1^1 \\
&\quad + M_3^2\mu + \text{h.o.t.}
\end{aligned} \tag{2.9}$$

3 The Main Results and Their Proofs

Assume that all hypotheses in Section 1 are valid. To investigate the existence of the heteroclinic loop, homoclinic orbit and periodic orbit of system (1.1) near Γ, we need only to consider the solution of the bifurcation equation $G \overset{\text{def}}{=} (G_1^1, G_1^2, G_2^1, G_2^2, G_3^1, G_3^2) = 0$, which satisfies $s_1 = s_2 = 0,\ s_1 = 0,\ s_2 > 0$ or $s_1 > 0,\ s_2 = 0$ and $s_1 > 0,\ s_2 > 0$, respectively.

Due to $G_3^2 = 0$ we have

$$v_0^0 = \delta_2^s + M_3^2\mu + \overline{\omega}_{13}^2\delta s_1 + (\omega_{33}^2)^{-1}\delta s_2^{\frac{\rho_2^2}{\lambda_1^2}} + O\Big(s_1^{\frac{\lambda_2^1}{\lambda_1^1}}\Big).$$

Substituting v_0^0 into $G_3^1 = 0$, we get

$$y_0^1 = M_3^1\mu + (\omega_{33}^1)^{-1}(\delta_2^s + M_3^2\mu) s_1^{\frac{\rho_2^1}{\lambda_1^1}} + \overline{\omega}_{11}^1\delta s_2 + o\Big(s_1^{\frac{\rho_2^1}{\lambda_1^1}}\Big) + O\Big(s_2^{\frac{\lambda_2^2}{\lambda_1^2}}\Big).$$

Then, we have the following equations by substituting $v_0^0,\ y_0^1$ into G_2^i

$$\begin{aligned}
u_1^1 &= \delta_1^u - M_2^1\mu + \omega_{12}^1(\omega_{11}^1)^{-1}\delta s_1^{\frac{\rho_1^1}{\lambda_1^1}} + O\Big(s_1^{\frac{\rho_2^1}{\lambda_1^1}}\Big) + O\Big(s_2^{\frac{\lambda_2^2}{\lambda_1^2}}\Big), \\
u_1^0 &= \delta_2^u - M_2^2\mu + \omega_{12}^2(\omega_{11}^2)^{-1} M_3^1\mu s_2^{\frac{\rho_1^2}{\lambda_1^2}} + O\Big(s_1^{\frac{\lambda_2^1}{\lambda_1^1}}\Big) + o\Big(s_2^{\frac{\rho_1^2}{\lambda_1^2}}\Big) + \text{h.o.t.}
\end{aligned}$$

Therefore, by substituting y_0^1, u_1^0, u_1^1 and v_0^0 into G_1^i, we obtain

$$\begin{cases}
s_1 = \delta^{-1}M_1^2\mu + (\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu s_2^{\frac{\rho_1^2}{\lambda_1^2}} - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} + \overline{\omega}_{11}^1(\omega_{11}^2)^{-1}\delta s_2^{\frac{\rho_1^2+\lambda_1^2}{\lambda_1^2}} \\
\qquad + (\omega_{11}^2)^{-1}(\omega_{33}^1)^{-1}(\delta_2^s + M_3^2\mu) s_1^{\frac{\rho_2^1}{\lambda_1^1}} s_2^{\frac{\rho_1^2}{\lambda_1^2}} + o\Big(s_1^{\frac{\rho_2^1}{\lambda_1^1}}\Big) O\Big(s_2^{\frac{\rho_1^2}{\lambda_1^2}}\Big) + O\Big(s_2^{\frac{\rho_1^2+\lambda_2^2}{\lambda_1^2}}\Big), \\
s_2 = \delta^{-1}M_1^1\mu + (\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} + \text{h.o.t.}
\end{cases} \tag{3.1}$$

Since we only consider the non-resonant bifurcations, it can be divided into the following three cases:

$$\text{I} \;\; \frac{\rho_1^1}{\lambda_1^1} > 1, \quad \text{II} \;\; \frac{\rho_1^1}{\lambda_1^1} < 1 < \frac{\rho_2^1}{\lambda_1^1}, \quad \text{III} \;\; \frac{\rho_2^1}{\lambda_1^1} < 1.$$

We only discuss case I in this paper. In fact, the following propositions have already revealed that the orbit flip may influence the bifurcation behavior associated with heteroclinic loop with one orbit flip. For case I, equation (3.1) can be simplified to

$$\begin{cases} s_1 = \delta^{-1} M_1^2 \mu + (\omega_{11}^2)^{-1} \delta^{-1} M_3^1 \mu s_2^{\frac{\rho_1^2}{\lambda_1^2}} - (\omega_{11}^2)^{-1} \omega_{31}^2 (\omega_{33}^2)^{-1} s_2^{\frac{\rho_2^2}{\lambda_1^2}} + \overline{\omega}_{11}^1 (\omega_{11}^2)^{-1} \delta s_2^{\frac{\rho_1^2+\lambda_1^2}{\lambda_1^2}} \\ \qquad + O\Big(s_2^{\frac{\rho_1^2+\lambda_2^2}{\lambda_1^2}} \Big) + \text{h.o.t.} \overset{\text{def}}{=} f(s_2), \\ s_2 = \delta^{-1} M_1^1 \mu + (\omega_{11}^1)^{-1} s_1^{\frac{\rho_1^1}{\lambda_1^1}} + \text{h.o.t.} \overset{\text{def}}{=} g(s_1). \end{cases} \tag{3.2}$$

Theorem 3.1 *Suppose that M_1^1 and M_1^2 are independent. Then the following are true.*

(1) *There exists a curve $C \overset{\text{def}}{=} \{\mu : M_1^1 \mu + o(|\mu|) = M_1^2 \mu + o(|\mu|) = 0\}$, such that there is a unique heteroclinic loop $\Gamma^\mu = \Gamma_1^\mu \cup \Gamma_2^\mu$ of system (1.1) in the neighborhood of Γ as $\mu \in C$ and $0 < |\mu| \ll 1$. Moreover, if $y_0^1 = M_3^1 \mu + \text{h.o.t.} \neq 0$, then Γ_1^μ is not orbit flip;*

(2) *If $\rho_1^2 > \lambda_1^2$, then the heteroclinic loop, 1-homoclinic orbit and 1-periodic orbit of system (1.1) can not be coexistent near Γ, which means that there is not any 1-homoclinic orbit and 1-periodic orbit as $\mu \in C$, $0 < |\mu| \ll 1$;*

(3) *If $\omega_{11}^1 < 0$ and $\mu \in C$, $0 < |\mu| \ll 1$, then system (1.1) has not any 1-periodic orbit near Γ^μ;*

(4) *If $\rho_1^2 + \lambda_1^2 < \rho_2^2$, $\omega_{11}^1 > 0$ and $\mu \in C$, $0 < |\mu| \ll 1$, then*

(i) *system (1.1) has not any 1-periodic orbit near Γ^μ as $\frac{\rho_1^1 \rho_1^2}{\lambda_1^1 \lambda_1^2} > 1$;*

(ii) *system (1.1) has a unique (resp. not any) 1-periodic orbit near Γ^μ as $\frac{\rho_1^1 \rho_1^2}{\lambda_1^1 \lambda_1^2} < 1$ and $\omega_{11}^2 M_3^1 \mu >$ (resp. $<$) 0;*

(5) *If $\rho_1^2 + \lambda_1^2 > \rho_2^2$, $\omega_{11}^1 > 0$ and $\mu \in C$, $0 < |\mu| \ll 1$, then*

(i) *system (1.1) has not any 1-periodic orbit near Γ^μ as $\frac{\rho_1^1 \rho_1^2}{\lambda_1^1 \lambda_1^2} > 1$;*

(ii) *system (1.1) has a unique (resp. not any) 1-periodic orbit near Γ^μ as $\frac{\rho_1^1 \rho_2^2}{\lambda_1^1 \lambda_1^2} > 1 > \frac{\rho_1^1 \rho_1^2}{\lambda_1^1 \lambda_1^2}$ and $\omega_{11}^2 M_3^1 \mu >$ (resp. $<$) 0;*

(iii) *system (1.1) has a unique (resp. not any) 1-periodic orbit near Γ^μ as $\frac{\rho_1^1 \rho_2^2}{\lambda_1^1 \lambda_1^2} < 1$ and $\omega_{31}^2 \omega_{33}^2 M_3^1 \mu >$ (resp. $<$) 0.*

Proof (1) If $s_1 = s_2 = 0$, then equations in (3.2) become $M_1^2 \mu + \text{h.o.t.} = M_1^1 \mu + \text{h.o.t.} = 0$. Thus the existence of Γ^μ follows immediately from the Implicity Function Theorem. By the definition, Γ_1^μ is orbit flip if and only if the solution of $G = 0$ satisfies $y_0^1 = 0$ (that is, the y component of $\Gamma_1^\mu \cap S_0^1 \subset W_{p_2}^s$ should be zero).

(2) In case $\rho_1^2 > \lambda_1^2$, it can be deduced from the Implicity Function Theorem that equation (3.2) has a unique small solution (s_1, s_2) as $0 < |\mu| \ll 1$.

(3) In this case, the second equation in (3.2) has not any positive solutions obviously.

(4) After eliminating s_2 in (3.2) we get

$$s_1 = (\omega_{11}^2)^{-1} (\omega_{11}^1)^{-\frac{\rho_1^2}{\lambda_1^2}} \delta^{-1} M_3^1 \mu s_1^{\frac{\rho_1^1 \rho_1^2}{\lambda_1^1 \lambda_1^2}} + \text{h.o.t.}$$

Conclusion (4) holds clearly.

(5) Eliminating s_2 in (3.2) now leads to

$$s_1=(\omega_{11}^2)^{-1}(\omega_{11}^1)^{-\frac{\rho_1^2}{\lambda_1^2}}\delta^{-1}M_3^1\mu s_1^{\frac{\rho_1^1\rho_1^2}{\lambda_1^1\lambda_1^2}}-(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}(\omega_{11}^1)^{-\frac{\rho_2^2}{\lambda_1^2}}s_1^{\frac{\rho_1^1\rho_2^2}{\lambda_1^1\lambda_1^2}}+\text{h.o.t.} \tag{3.3}$$

The first two conclusions of (5) then become easy to check. Under the condition of (iii), (3.3) can be rewritten as

$$s_1^{\frac{\rho_1^1(\rho_2^2-\rho_1^2)}{\lambda_1^1\lambda_1^2}}=\delta^{-1}(\omega_{11}^1)^{\frac{\rho_2^2-\rho_1^2}{\lambda_1^2}}(\omega_{31}^2)^{-1}\omega_{33}^2M_3^1\mu+\text{h.o.t.} \tag{3.4}$$

Thus, the third conclusion also follows.

Remark 3.1 If $M_1^i\neq 0$, then there exists a surface $\Sigma_i\overset{\text{def}}{=}\{\mu: M_1^i\mu+o(|\mu|)=0\}$ such that there is a unique heteroclinic orbit Γ_i^μ of system (1.1) in the neighborhood of Γ_i as $\mu\in\Sigma_i$ (see [9]). From the proof of Theorem 3.1 in [9], we can see that there is not any 1-periodic orbit as $\mu\in C$ if the original heteroclinic orbit Γ_1 is not orbit flip.

Theorem 3.2 (1) *If $M_1^1\neq 0$, then there exists a surface*

$$\Sigma^2\overset{\text{def}}{=}\Big\{\mu: M_1^1\mu+(\omega_{11}^1)^{-1}\delta(\delta^{-1}M_1^2\mu)^{\frac{\rho_1^1}{\lambda_1^1}}+\text{h.o.t.}=0,\ M_1^2\mu>0\Big\},$$

such that system (1.1) *has a unique orbit Γ_μ^2 homoclinic to p_2 in the neighborhood of Γ as $\mu\in\Sigma^2$ and $0<|\mu|\ll 1$, and, it is not orbit flip as $y_0^1=M_3^1\mu+\text{h.o.t.}\neq 0$.*

(2) *If $M_1^2\neq 0$, then there exists a surface*

$$\begin{aligned}\Sigma^1\overset{\text{def}}{=}\Big\{\mu:\ &\omega_{11}^2M_1^2\mu+M_3^1\mu(\delta^{-1}M_1^1\mu)^{\frac{\rho_1^2}{\lambda_1^2}}-\delta\omega_{31}^2(\omega_{33}^2)^{-1}(\delta^{-1}M_1^1\mu)^{\frac{\rho_2^2}{\lambda_1^2}}\\&+\delta^2\overline{\omega}_{11}^1(\delta^{-1}M_1^1\mu)^{\frac{\rho_1^2+\lambda_1^2}{\lambda_1^2}}+\text{h.o.t.}=0,\ M_1^1\mu>0\Big\},\end{aligned}$$

such that system (1.1) *has a unique orbit Γ_μ^1 homoclinic to p_1 in the neighborhood of Γ as $\mu\in\Sigma^1$ and $0<|\mu|\ll 1$.*

Proof When $s_2=0$, equation (3.2) becomes

$$\begin{cases}s_1=\delta^{-1}M_1^2\mu+\text{h.o.t.},\\ 0=\delta^{-1}M_1^1\mu+(\omega_{11}^1)^{-1}s_1^{\frac{\rho_1^1}{\lambda_1^1}}+\text{h.o.t.}\end{cases}$$

When $s_1=0$, the equations become

$$\begin{cases}0=\delta^{-1}M_1^2\mu+(\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu s_2^{\frac{\rho_1^2}{\lambda_1^2}}-(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}s_2^{\frac{\rho_2^2}{\lambda_1^2}}\\ \qquad+\overline{\omega}_{11}^1(\omega_{11}^2)^{-1}\delta s_2^{\frac{\rho_1^2+\lambda_1^2}{\lambda_1^2}}+O\Big(s_2^{\frac{\rho_1^2+\lambda_2^2}{\lambda_1^2}}\Big),\\ s_2=\delta^{-1}M_1^1\mu+\text{h.o.t.}\end{cases}$$

Therefore, conclusions (1) and (2) hold.

Theorem 3.3 *Suppose $\rho_2^2<\rho_1^2+\lambda_1^2$, $\rho_1^2>\lambda_1^2$, $\omega_{31}^2\omega_{33}^2M_3^1\mu<0$ and $|\mu|\ll 1$. Then the following are true.*

(1) *If $\omega_{11}^1<0$ and $M_1^1\mu<0$, then system (1.1) has not any 1-periodic orbit near Γ;*

(2) *If $\omega_{11}^1>0$ and $M_1^1\mu>0$, then system (1.1) has a unique 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu>h_1(\mu)$, has a unique 1-homoclinic orbit near Γ as $\delta^{-1}M_1^2\mu=h_1(\mu)$, and has not any 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu<h_1(\mu)$, where $h_1(\mu)=(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}(\delta^{-1}M_1^1\mu)^{\frac{\rho_2^2}{\lambda_1^2}}-(\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu(\delta^{-1}M_1^1\mu)^{\frac{\rho_1^2}{\lambda_1^2}}+\text{h.o.t.}$;*

(3) *If $\omega_{11}^1>0$ and $M_1^1\mu<0$, then system (1.1) has a unique 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu>(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$, has a unique 1-homoclinic orbit near Γ as $\delta^{-1}M_1^2\mu=(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}+\text{h.o.t.}$, and has not any 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu<(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$;*

(4) *If $\omega_{11}^1<0$ and $M_1^1\mu>0$, then system (1.1) has not any 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu>(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$, has a unique 1-homoclinic orbit near Γ as $\delta^{-1}M_1^2\mu=(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}+\text{h.o.t.}$, has a unique 1-periodic orbit near Γ as $h_1(\mu)<\delta^{-1}M_1^2\mu<(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$, has a unique 1-homoclinic orbit near Γ as $\delta^{-1}M_1^2\mu=h_1(\mu)$, and has not any 1-periodic orbit near Γ as $\delta^{-1}M_1^2\mu<h_1(\mu)$.*

Proof The case (1) is obvious.

For other cases, we see that $f(s_2)$, $g^{-1}(s_2)$ are monotonous, and the curve $s_1=f(s_2)$ (resp. $s_1=g^{-1}(s_2)$) intersects the s_1 (resp. s_2) axis at $s_1=s_1^*=\delta^{-1}M_1^2\mu$ (resp. $s_2=s_2^*=\delta^{-1}M_1^1\mu$). Further, the curve $s_1=g^{-1}(s_2)$ intersects the s_1 axis at $s_1=\bar{s}_1\overset{\text{def}}{=}(-\delta^{-1}\omega_{11}^1M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$ as $\omega_{11}^1M_1^1\mu<0$.

In case (2), we get a positive solution $s_2=g(s_1)>0$ if $s_1\ge 0$. Now eliminating s_2 in (3.2), we have

$$\begin{aligned}F(s_1)\overset{\text{def}}{=}\ &s_1-\delta^{-1}M_1^2\mu-(\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu\Big(\delta^{-1}M_1^1\mu+(\omega_{11}^1)^{-1}s_1^{\frac{\rho_1^1}{\lambda_1^1}}\Big)^{\frac{\rho_1^2}{\lambda_1^2}}\\&+(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}\Big(\delta^{-1}M_1^1\mu+(\omega_{11}^1)^{-1}s_1^{\frac{\rho_1^1}{\lambda_1^1}}\Big)^{\frac{\rho_2^2}{\lambda_1^2}}+\text{h.o.t.}\end{aligned}\tag{3.5}$$

Because of $\frac{\rho_1^1}{\lambda_1^1},\frac{\rho_1^2}{\lambda_1^2},\frac{\rho_2^2}{\lambda_1^2}>1$, we get $F'(s_1)\approx 1>0$,

$$F(0)=-\delta^{-1}M_1^2\mu-(\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu(\delta^{-1}M_1^1\mu)^{\frac{\rho_1^2}{\lambda_1^2}}+(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}(\delta^{-1}M_1^1\mu)^{\frac{\rho_2^2}{\lambda_1^2}}+\text{h.o.t.}$$

If $F(0)<0$, i.e., $\delta^{-1}M_1^2\mu>h_1(\mu)$, then equation (3.5) has a unique small positive solution; if $F(0)>0$, then equation (3.5) has not any small positive solution; if $F(0)=0$, then equation (3.5) has a unique nonnegative solution $s_1=0$. Hence (2) holds (see the following Figure 3(a), where $f(s_2^*)=-F(0)>0$).

Under condition (3), $s_1=g^{-1}(s_2)>0$ if $s_2\ge 0$. Substituting it into the first equation of (3.2), we obtain

$$\begin{aligned}G(s_2)\overset{\text{def}}{=}\ &s_2-\delta^{-1}M_1^1\mu-(\omega_{11}^1)^{-1}\Big(\delta^{-1}M_1^2\mu+(\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu s_2^{\frac{\rho_1^2}{\lambda_1^2}}\\&-(\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}s_2^{\frac{\rho_2^2}{\lambda_1^2}}\Big)^{\frac{\rho_1^1}{\lambda_1^1}}+\text{h.o.t.}\end{aligned}\tag{3.6}$$

Similarly to the above, because of $\frac{\rho_1^1}{\lambda_1^1},\ \frac{\rho_1^2}{\lambda_1^2},\ \frac{\rho_2^2}{\lambda_1^2} > 1$, one has $G'(s_2) \approx 1 > 0$,

$$G(0) = -\delta^{-1}M_1^1\mu - (\omega_{11}^1)^{-1}(\delta^{-1}M_1^2\mu)^{\frac{\rho_1^1}{\lambda_1^1}} + \text{h.o.t.}$$

If $G(0) < 0$, i.e., $\delta^{-1}M_1^2\mu > (-\delta^{-1}\omega_{11}^1 M_1^1\mu)^{\frac{\lambda_1^1}{\rho_1^1}}$, then equation (3.6) has a unique small positive solution; if $G(0) > 0$, then equation (3.6) has not any small positive solution; if $G(0) = 0$, then equation (3.6) has a unique nonnegative solution $s_2 = 0$. Hence (3) holds (see the following Figure 3(b), where $g(s_1^*) = -G(0) > 0$).

The proof of case (4) is similar to that of case (2) and (3).

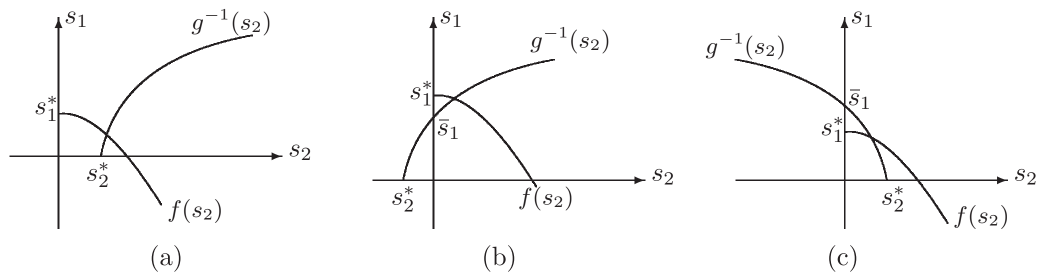

Figure 3

Remark 3.2 If $\rho_1^2 < \lambda_1^2 < \rho_2^2$, or $\rho_2^2 < \lambda_1^2$ and $\omega_{31}^2\omega_{33}^2 M_3^1\mu < 0$, then we can obtain some similar conclusions.

Next, we show that system (1.1) may have the three-fold 1-periodic orbit in the following theorem. Set

$$f(s_2) = \delta^{-1}M_1^2\mu + (\omega_{11}^2)^{-1}\delta^{-1}M_3^1\mu s_2^{\frac{\rho_1^2}{\lambda_1^2}} - (\omega_{11}^2)^{-1}\omega_{31}^2(\omega_{33}^2)^{-1}s_2^{\frac{\rho_2^2}{\lambda_1^2}} + \text{h.o.t.}$$

Then

$$f'(s_2) = \frac{\rho_1^2 M_3^1\mu}{\lambda_1^2\delta\omega_{11}^2}s_2^{\frac{\rho_1^2-\lambda_1^2}{\lambda_1^2}} - \frac{\rho_2^2\omega_{31}^2}{\lambda_1^2\omega_{11}^2\omega_{33}^2}s_2^{\frac{\rho_2^2-\lambda_1^2}{\lambda_1^2}} + \text{h.o.t.}$$

If $f'(s_2) = 0$, then $s_2 = \big(\frac{\rho_1^2\omega_{33}^2 M_3^1\mu}{\rho_2^2\delta\omega_{31}^2}\big)^{\frac{\lambda_1^2}{\rho_2^2-\rho_1^2}} + \text{h.o.t.} \overset{\text{def}}{=} \bar{s}$,

$$\begin{aligned} f(\bar{s}) &= \delta^{-1}M_1^2\mu + (\omega_{11}^2)^{-1}\bar{s}^{\frac{\rho_1^2}{\lambda_1^2}}\Big(\delta^{-1}M_3^1\mu - \omega_{31}^2(\omega_{33}^2)^{-1}\bar{s}^{\frac{\rho_2^2-\rho_1^2}{\lambda_1^2}}\Big) + \text{h.o.t.} \\ &= \delta^{-1}M_1^2\mu + \frac{(\rho_2^2-\rho_1^2)M_3^1\mu}{\delta\rho_2^2\omega_{11}^2}\bar{s}^{\frac{\rho_1^2}{\lambda_1^2}} + \text{h.o.t.} \\ &= \delta^{-1}M_1^2\mu + \frac{(\rho_2^2-\rho_1^2)\omega_{31}^2}{\rho_1^2\omega_{11}^2\omega_{33}^2}\bar{s}^{\frac{\rho_2^2}{\lambda_1^2}} + \text{h.o.t.}, \\ f''(\bar{s}) &= \bar{s}^{\frac{\rho_1^2-2\lambda_1^2}{\lambda_1^2}}\Big(\frac{\rho_1^2(\rho_1^2-\lambda_1^2)M_3^1\mu}{(\lambda_1^2)^2\delta\omega_{11}^2} - \frac{\rho_2^2(\rho_2^2-\lambda_1^2)\omega_{31}^2}{(\lambda_1^2)^2\omega_{11}^2\omega_{33}^2}\bar{s}^{\frac{\rho_2^2-\rho_1^2}{\lambda_1^2}}\Big) + \text{h.o.t.} \end{aligned}$$

$$= \frac{\rho_1^2(\rho_1^2-\rho_2^2)M_3^1\mu}{(\lambda_1^2)^2\delta\omega_{11}^2}\bar{s}^{\frac{\rho_1^2-2\lambda_1^2}{\lambda_1^2}} + \text{h.o.t.}$$
$$= -\frac{\rho_2^2(\rho_2^2-\rho_1^2)\omega_{31}^2}{(\lambda_1^2)^2\omega_{11}^2\omega_{33}^2}\bar{s}^{\frac{\rho_2^2-2\lambda_1^2}{\lambda_1^2}} + \text{h.o.t.} = O\Big(|M_3^1\mu|^{\frac{\rho_2^2-2\lambda_1^2}{\rho_2^2-\rho_1^2}}\Big).$$

Thus we can rewrite $f(s_2)$ as

$$f(s_2) = f(\bar{s}) + \frac{1}{2}f''(\bar{s})(s_2-\bar{s})^2 + \text{h.o.t.}$$

Now we substitute the second equation in (3.2) into $f(s_2)$. Then the first equation in (3.2) becomes

$$s_1 = f(\bar{s}) + \frac{1}{2}f''(\bar{s})\Big(\delta^{-1}M_1^1\mu + (\omega_{11}^1)^{-1}s_1^{\frac{\rho_1^1}{\lambda_1^1}} - \bar{s}\Big)^2 + \text{h.o.t.},$$

i.e.,

$$A(\mu) + B(\mu)s_1 + C(\mu)s_1^{\frac{\rho_1^1}{\lambda_1^1}} + s_1^{\frac{2\rho_1^1}{\lambda_1^1}} + \text{h.o.t.} = 0, \tag{3.7}$$

where $A(\mu) = 2(\omega_{11}^1)^2(f''(\bar{s}))^{-1}f(\bar{s}) + (\omega_{11}^1)^2(\delta^{-1}M_1^1\mu - \bar{s})^2,\ B(\mu) = -2(\omega_{11}^1)^2(f''(\bar{s}))^{-1},\ C(\mu) = 2\omega_{11}^1(\delta^{-1}M_1^1\mu - \bar{s})$.

In the following, we always assume $\omega_{11}^1 > 0,\ M_1^1\mu > 0$, which means $s_2 = g(s_1) > 0$ for $0 \le s_1 \ll 1$. Thus, to consider the homoclinic and periodic orbit bifurcation from Γ, it suffices to consider the nonnegative small solution $s_1 \ge 0$ of equation (3.7). Let

$$F(t) = A(\mu) + B(\mu)t + C(\mu)t^{\frac{\rho_1^1}{\lambda_1^1}} + t^{\frac{2\rho_1^1}{\lambda_1^1}} + \text{h.o.t.},$$
$$p = \frac{6F'(t_3)}{F'''(t_3)},\quad q = \frac{6F(t_3)}{F'''(t_3)},\quad t_3 = \Big[-\frac{(\rho_1^1-\lambda_1^1)C(\mu)}{2(2\rho_1^1-\lambda_1^1)}\Big]^{\frac{\lambda_1^1}{\rho_1^1}} + \text{h.o.t.},$$
$$F(t_3) = A(\mu) + B(\mu)t_3 - \frac{3\rho_1^1-\lambda_1^1}{\rho_1^1-\lambda_1^1}t_3^{\frac{2\rho_1^1}{\lambda_1^1}} + \text{h.o.t.},$$
$$F'(t_3) = B(\mu) + \frac{(\rho_1^1)^2C(\mu)}{\lambda_1^1(2\rho_1^1-\lambda_1^1)}t_3^{\frac{\rho_1^1-\lambda_1^1}{\lambda_1^1}} + \text{h.o.t.} = B(\mu) - \frac{2(\rho_1^1)^2}{\lambda_1^1(\rho_1^1-\lambda_1^1)}t_3^{\frac{2\rho_1^1-\lambda_1^1}{\lambda_1^1}} + \text{h.o.t.},$$
$$F'''(t_3) = \frac{\rho_1^1(\rho_1^1-\lambda_1^1)}{(\lambda_1^1)^3}t_3^{\frac{\rho_1^1-3\lambda_1^1}{\lambda_1^1}}[(\rho_1^1-2\lambda_1^1)C(\mu) + 4(2\rho_1^1-\lambda_1^1)t_3^{\frac{\rho_1^1}{\lambda_1^1}}] + \text{h.o.t.}$$
$$= -\frac{(\rho_1^1)^2(\rho_1^1-\lambda_1^1)C(\mu)}{(\lambda_1^1)^3}t_3^{\frac{\rho_1^1-3\lambda_1^1}{\lambda_1^1}} + \text{h.o.t.} = \frac{2(\rho_1^1)^2(2\rho_1^1-\lambda_1^1)}{(\lambda_1^1)^3}t_3^{\frac{2\rho_1^1-3\lambda_1^1}{\lambda_1^1}} + \text{h.o.t.}$$

Theorem 3.4 *Suppose that* $2\rho_1^1 < 3\lambda_1^1,\ \rho_2^2 < \lambda_1^2,\ \omega_{31}^2\omega_{33}^2M_3^1\mu > 0,\ \omega_{11}^1 > 0,\ M_1^1\mu > 0$ *and* $0 < |\mu| \ll 1$. *Then the following are true.*

(1) *In case* $C(\mu) > 0$, *we have*

(i) *if* $B(\mu) > 0$, *then the system* (1.1) *has a unique* (*resp. not any*) 1*-periodic orbit near* Γ *as* $A(\mu) < 0$ (*resp.* > 0).

(ii) *if* $B(\mu) < 0$ *and* $A(\mu) < 0$, *then the system* (1.1) *has a unique* 1*-periodic orbit near* Γ.

(iii) *if* $B(\mu) < 0$ *and* $A(\mu) > 0$, *then the system* (1.1) *has not any* 1*-periodic orbit near* Γ *as* $F(t_0) > 0$, *has a unique two-fold* 1*-periodic orbit near* Γ *as* $F(t_0) = 0$, *has exactly two*

1-periodic orbits near Γ *as* $F(t_0) < 0$*, where* t_0 *is a unique small positive solution of equation* $F'(t) = 0$.

(2) *In case* $C(\mu) < 0$*, we have*

(i) *if* $p > 0$*, then system* (1.1) *has a unique* (*resp. not any*) 1*-periodic orbit near* Γ *as* $pt_3 - q + t_3^3 > 0$ (*resp.* < 0), *and has an orbit homoclinic to the point* p_1 *as* $pt_3 - q + t_3^3 + \text{h.o.t.} = 0$.

(ii) *if* $p = 0$*, then system* (1.1) *has a unique three-fold* 1*-periodic orbit near* Γ *as* $q = 0$ (*that is,* μ *is situated in a codimension* 2 *bifurcation curve* Σ_1 *defined by* $\big[-\frac{\lambda_1^1 A(\mu)}{(2\rho_1^1-\lambda_1^1)}\big]^{\frac{\lambda_1^1}{2\rho_1^1}} + \text{h.o.t.} = \big[\frac{\lambda_1^1(\rho_1^1-\lambda_1^1)B(\mu)}{2(\rho_1^1)^2}\big]^{\frac{\lambda_1^1}{2\rho_1^1-\lambda_1^1}} + \text{h.o.t.} = \big[-\frac{(\rho_1^1-\lambda_1^1)C(\mu)}{2(2\rho_1^1-\lambda_1^1)}\big]^{\frac{\lambda_1^1}{\rho_1^1}} + \text{h.o.t.}$), *has a unique* 1*-periodic orbit near* Γ *as* $q < 0$ *or* $0 < q < t_3^3$*, has not any* 1*-periodic orbit near* Γ *as* $q \geq t_3^3 + \text{h.o.t.}$*, and has an orbit homoclinic to the point* p_1 *as* $q = t_3^3 + \text{h.o.t.}$

(iii) *if* $p < 0$ *and* $t_3 - \sqrt{-\frac{p}{3}} + \text{h.o.t.} \leq 0$*, then system* (1.1) *has exactly one* 1*-periodic orbit near* Γ *as* $-t_3^3 < pt_3 - q$*, has exactly one* 1*-periodic orbit and one orbit homoclinic to the point* p_1 *near* Γ *as* $-t_3^3 + \text{h.o.t.} = pt_3 - q$*, has exactly two* 1*-periodic orbits near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} < pt_3 - q < -t_3^3$*, has exactly one two-fold* 1*-periodic orbit near* Γ *as* $pt_3 - q = p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + \text{h.o.t.}$*, has not any* 1*-periodic orbit near* Γ *as* $pt_3 - q < p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3}$.

(iv) *if* $p < 0$ *and* $t_3 - \sqrt{-\frac{p}{3}} > 0$*, then system* (1.1) *has exactly one* 1*-periodic orbit near* Γ *as* $p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3} < pt_3 - q$*, has exactly one two-fold and one simple* 1*-periodic orbits near* Γ *as* $p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3} + \text{h.o.t.} = pt_3 - q$*, has exactly three* 1*-periodic orbits near* Γ *as* $-t_3^3 < pt_3 - q < p(t_3 - \sqrt{-\frac{p}{3}}) - \sqrt{-(\frac{p}{3})^3}$*, has two* 1*-periodic orbits and one orbit homoclinic to the point* p_1 *as* $-t_3^3 + \text{h.o.t.} = pt_3 - q$*, has two* 1*-periodic orbits near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} < pt_3 - q < -t_3^3$*, has one two-fold* 1*-periodic orbit near* Γ *as* $p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3} + \text{h.o.t.} = pt_3 - q$*, has not any* 1*-periodic orbit near* Γ *as* $pt_3 - q < p(t_3 + \sqrt{-\frac{p}{3}}) + \sqrt{-(\frac{p}{3})^3}$.

Proof We first consider case (1). When $A(\mu), B(\mu), C(\mu)$ are all positive (or negative), we have $F(t) \neq 0$ for small $t \in \mathbb{R}^+$. When $B(\mu), C(\mu)$ are all positive (or negative), but $A(\mu)B(\mu) < 0$, we have $F'(t) \neq 0$ for $t \in \mathbb{R}^+$, and $F(0)F(\hat{t}) = A(\mu)\big(B(\mu)\hat{t} + C(\mu)\hat{t}^{\frac{\rho_1^1}{\lambda_1^1}} + \text{h.o.t.}\big) < 0$, where $\hat{t} = (-A(\mu))^{\frac{\lambda_1^1}{2\rho_1^1}}$. Therefore (i) holds.

For case (ii), because of $F'(0)F'(\bar{t}) = B(\mu)\big(\frac{\rho_1^1}{\lambda_1^1}C(\mu)\bar{t}^{\frac{\rho_1^1-\lambda_1^1}{\lambda_1^1}} + \text{h.o.t.}\big) < 0$ and $F''(t) > 0$ for small $t \in \mathbb{R}^+$, equation $F'(t) = 0$ has a unique small positive solution $t = t_0 \in (0, \bar{t})$, where $\bar{t} = \big(-\frac{\lambda_1^1 B(\mu)}{2\rho_1^1}\big)^{\frac{\lambda_1^1}{2\rho_1^1-\lambda_1^1}}$. Hence, $F'(t) < 0$ for $t \in (0, t_0)$ and $F'(t) > 0$ for $t > t_0$. On the other hand, equation $A(\mu) + B(\mu)s_1 + s_1^{\frac{2\rho_1^1}{\lambda_1^1}} + \text{h.o.t.} = 0$, has a unique small positive solution $t = \tilde{t}$. In fact, the straight line $F_1(s_1) = A(\mu) + B(\mu)s_1 = 0$ must intersects the curve $F_2(s_1) = -s_1^{\frac{2\rho_1^1}{\lambda_1^1}} + \text{h.o.t.} = 0$ at a unique point $s_1 = s'$, and $F_1(s') = F_2(s') \to 0$ as $\mu \to 0$. Thereby, $F(0)F(\tilde{t}) = A(\mu)\big(C(\mu)\tilde{t}^{\frac{\rho_1^1}{\lambda_1^1}} + \text{h.o.t.}\big) < 0$. By the continuity of function $F(t)$, equation (3.7) has a unique small positive solution $t^* \in (t_0, \tilde{t})$. (ii) holds.

For case (1)(iii), we note that $t = t_0$ is a two-fold solution of equation (3.7) as $F(t_0) = 0$. Thereby, (iii) also holds.

Next, we consider case (2)(i)–(iv).

Solving equation $F''(t)=0$, we get its unique small positive solution

$$t=t_3=\Big[-\frac{(\rho_1^1-\lambda_1^1)C(\mu)}{2(2\rho_1^1-\lambda_1^1)}\Big]^{\frac{\lambda_1^1}{\rho_1^1}}+\text{h.o.t.}\quad \text{as } C(\mu)<0.$$

Hence equation (3.7) is equivalent to

$$\begin{aligned}F(t)&=F(t_3)+F'(t_3)(t-t_3)+\frac{1}{6}F'''(t_3)(t-t_3)^3+\text{h.o.t.}\\&=\frac{1}{6}F'''(t_3)[q+p(t-t_3)+(t-t_3)^3+\text{h.o.t.}]\\&=0.\end{aligned}\tag{3.8}$$

Clearly, the zero points of $F(t)$ are corresponding to the intersections of the line L: $H_0(t)=-p(t-t_3)-q$ with the curve C: $H(t)=(t-t_3)^3+\text{h.o.t.}$ Thus, it is easy to see that claim (i) is true. To show (ii), we need only to notice that if $F'(t_3)=p=0$, then we have

$$t_3=\Big[-\frac{(\rho_1^1-\lambda_1^1)C(\mu)}{2(2\rho_1^1-\lambda_1^1)}\Big]^{\frac{\lambda_1^1}{\rho_1^1}}+\text{h.o.t.}=\Big[\frac{\lambda_1^1(\rho_1^1-\lambda_1^1)B(\mu)}{2(\rho_1^1)^2}\Big]^{\frac{\lambda_1^1}{2\rho_1^1-\lambda_1^1}}+\text{h.o.t.}\overset{\text{def}}{=}t_4;$$

if $t_3=t_4$ then

$$F''(t_3)=F''(t_4)=F'(t_3)=F'(t_4)=0$$

and

$$F(t_3)=F(t_4)=A(\mu)+B(\mu)t_3-\frac{3\rho_1^1-\lambda_1^1}{\rho_1^1-\lambda_1^1}t_3^{\frac{2\rho_1^1}{\lambda_1^1}}+\text{h.o.t.}=A(\mu)+\frac{2\rho_1^1-\lambda_1^1}{\lambda_1^1}t_4^{\frac{2\rho_1^1}{\lambda_1^1}}+\text{h.o.t.},$$

thereby $F(t)=F'(t)=F''(t)=0$ as $t=t_3=t_4=\big[-\frac{\lambda_1^1A(\mu)}{(2\rho_1^1-\lambda_1^1)}\big]^{\frac{\lambda_1^1}{2\rho_1^1}}+\text{h.o.t.}\overset{\text{def}}{=}t_5$.

Now we show (iii) and (iv). Owing to

$$\begin{aligned}F'''(t_3)&=\frac{\rho_1^1(\rho_1^1-\lambda_1^1)}{(\lambda_1^1)^3}t_3^{\frac{\rho_1^1-3\lambda_1^1}{\lambda_1^1}}\Big[(\rho_1^1-2\lambda_1^1)C(\mu)+4(2\rho_1^1)-\lambda_1^1t_3^{\frac{\rho_1^1}{\lambda_1^1}}\Big]+\text{h.o.t.}\\&=-\frac{(\rho_1^1)^2(\rho_1^1-\lambda_1^1)C(\mu)}{(\lambda_1^1)^3}t_3^{\frac{\rho_1^1-3\lambda_1^1}{\lambda_1^1}}+\text{h.o.t.}=\frac{2(\rho_1^1)^2(2\rho_1^1-\lambda_1^1)}{(\lambda_1^1)^3}t_3^{\frac{2\rho_1^1-3\lambda_1^1}{\lambda_1^1}}+\text{h.o.t.}\end{aligned}$$

we see that the condition $2\rho_1^1<3\lambda_1^1$ ensures $|p|,|q|\ll 1$ as $|\mu|\ll 1$. If $p<0$, then (3.8) implies that $F'(t)=0$ has exactly two small solutions $t^\pm\approx t_3\pm\sqrt{-\frac{p}{3}}$ as $|\mu|\ll 1$. It means the curve C has two tangent lines $L^\pm$: $H_0^\pm(t)=-p(t-t^\pm)\pm\sqrt{-(\frac{p}{3})^3}$, which are parallel to the line L. The lines $L^\pm$ intersect the vertical axis at points $H^\pm(0,pt^\pm\pm\sqrt{-(\frac{p}{3})^3}\,)$, respectively. Moreover, we can show that the point $C_0(0,-t_3^3+\text{h.o.t.})$ is situated between points H^- and H^+ as $t^-=t_3-\sqrt{-\frac{p}{3}}>0$. In fact, if $t_3>\sqrt{-\frac{p}{3}}$, then $pt^++\sqrt{-(\frac{p}{3})^3}=pt_3-2\sqrt{-(\frac{p}{3})^3}<pt_3-2t_3^3<-2t_3^3$. Therefore, conclusions (iii) and (iv) hold (see the following Figure 4). The proof is complete.

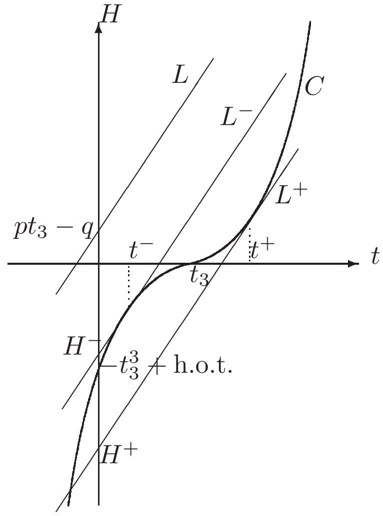

Figure 4

Remark 3.3 Clearly if $\omega_{11}^1 < 0$, $M_1^1\mu < 0$, and $0 < |\mu| \ll 1$, then system (1.1) has not any 1-periodic orbit near Γ. If $\omega_{11}^1 M_1^1 \mu < 0$, then $s_2 = g(s_1)$ can be rewritten as the following

$$s_2 = g(s_1) = \frac{\rho_1^1}{\lambda_1^1 \omega_{11}^1} s^{*\frac{\rho_1^1 - \lambda_1^1}{\lambda_1^1}} (s_1 - s^*) + \text{h.o.t.},$$

where $s^* = (-\delta^{-1}\omega_{11}^1 M_1^1 \mu)^{\frac{\lambda_1^1}{\rho_1^1}}$ + h.o.t., so system (1.1) can bifurcate two 1-periodic orbits at most near Γ.

4 Conclusions

We have known that a rough homoclinic loop can produce at most one 1-periodic orbit, and a non-resonant codimension 2 homoclinic loop with an orbit flip can yield at most two 1-periodic orbit (cf. [12, 5]). As for the rough non-twisted (i.e., $\omega_{11}^1\omega_{11}^2 > 0$) heteroclinic loop bifurcation without orbit flip and inclination flip, it follows from [9] that the persisted heteroclinic loop can not be coexistent with the 1-periodic orbit, and the original loop can bifurcate at most one (resp. two) 1-periodic orbit as $\frac{\rho_1^1}{\lambda_1^1} > 1$, $\frac{\rho_1^2}{\lambda_1^2} > 1$ $\big($resp. $\frac{\rho_1^2}{\lambda_1^2} < 1\big)$. While we have shown in this paper that, for the rough non-resonant heteroclinic loop with an orbit flip, on the one hand, the non-coexistence and the uniqueness are still valid in case $\frac{\rho_1^1}{\lambda_1^1} > 1$, $\frac{\rho_1^2}{\lambda_1^2} > 1$; on the other hand, the persisted heteroclinic loop (by Theorem 3.1, which is not orbit flip) can be coexistent with a 1-periodic orbit, three 1-periodic orbits can be produced simultaneously from the original loop, and much more complicated bifurcation phenomenon can occur in case $\frac{\rho_1^1}{\lambda_1^1} > 1$, $\frac{\rho_1^2}{\lambda_1^2} < 1$.

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