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Asymptotic Normality of LS Estimate in Simple Linear EV Regression Model

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Abstract Though EV model is theoretically more appropriate for applications in which measurement errors exist, people are still more inclined to use the ordinary regression models and the traditional LS method owing to the difficulties of statistical inference and computation. So it is meaningful to study the performance of LS estimate in EV model. In this article we obtain general conditions guaranteeing the asymptotic normality of the estimates of regression coefficients in the linear EV model. It is noticeable that the result is in some way different from the corresponding result in the ordinary regression model.

Keywords EV model, LS estimate, Asymptotic normality
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1 Introduction

Though many variables in applications suffer from measurement error, the EV model, which contains measurement error as an element in the model, has not yet gain popular use. The reason is apparently the complexity involved in its statistical inference and computation. For example with random independent variable we may have the identification problem (see [1, 2]), while when the independent variable is considered as unknown constant, the parameters in the model increase steadily with the sample size. Still more conditions are required in EV models. δ_1 and ε_1 are assumed to be independently normally distributed with common variance (see [3]). $(\delta_1, \varepsilon_1)$ is assumed to have spherical symmetric distribution (see [4]). Replicated observations are available (see [5]). Hence, though theoretically EV model is more appropriate in many circumstances, people are still inclined to turn to the ordinary regression models and use the traditional LS method in dealing with the problem of estimation. Therefore it is a meaningful question to study the behavior of LS estimate when we really have an EV model. The main purpose of this article is to study the asymptotic normality of LS estimate. For simplicity of presentation we restrict ourselves to the case of simple linear model, which can be described as:

$$\text{(A)}\quad \begin{cases} \eta_i = \alpha + \beta x_i + \varepsilon_i,\ \xi_i = x_i + \delta_i, & 1 \le i \le n, \\ (\varepsilon_i, \delta_i), & 1 \le i \le n,\ \text{i.i.d.}, \\ E\varepsilon_1 = E\delta_1 = 0,\ E\delta_1^2 = \sigma_1^2,\ E\varepsilon_1^2 = \sigma_2^2, & 0 < \sigma_1^2,\ \sigma_2^2 < \infty. \end{cases}$$

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Here (ξ_i, η_i), $1 \le i \le n$ are observable, while x_i, $1 \le i \le n$, α, β, σ_1^2, and σ_2^2 are unknown parameters.

From (A) we have

$$\eta_i = \alpha + \beta\xi_i + \nu_i, \quad \nu_i = \varepsilon_i - \beta\delta_i, \quad 1 \le i \le n. \tag{1.1}$$

Considering formally (1.1) as a usual regression model of η_i on ξ_i, we get the LS estimates of β and α as

$$\hat{\beta}_n = \frac{\sum\limits_{i=1}^{n}(\xi_i - \bar{\xi}_n)(\eta_i - \bar{\eta}_n)}{\sum\limits_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2}, \quad \hat{\alpha}_n = \bar{\eta}_n - \hat{\beta}_n\bar{\xi}_n. \tag{1.2}$$

Here $\bar{\eta}_n \equiv n^{-1}\sum\limits_{i=1}^{n}\eta_i$, $\bar{\xi}_n$ and $\bar{\delta}_n$ are defined similarly.

In an earlier paper [6] we studied the consistency of $\hat{\beta}_n$ and $\hat{\alpha}_n$, and showed that under model (A), the necessary and sufficient condition for $\hat{\beta}_n$ being strong and weak consistent estimate of β is

$$n^{-1}S_n \to \infty, \quad S_n = \sum_{i=1}^{n}(x_i - \bar{x}_n)^2. \tag{1.3}$$

Thus, as in the ordinary regression model, strong and weak consistency of $\hat{\beta}_n$ are equivalent.

In this paper, we study the asymptotic normality of LS estimates under model (A) with condition (1.3).

2 Asymptotic Normality of $\hat{\beta}_n$

From (1.2) we have

$$\begin{aligned}\hat{\beta}_n = \beta &+ \Big[\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2\Big]^{-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)\varepsilon_i - \beta\Big[\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2\Big]^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)\delta_i \\ &- \beta\Big[\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2\Big]^{-1}\sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2.\end{aligned} \tag{2.1}$$

In this section we will prove the following theorem.

Theorem 2.1 *Under model* (A), *suppose that δ_1 and ε_1 are independent, δ_1 has fourth order moment, and ε_1 has third order moment. If* (1.3) *holds and*

$$S_n^{-\frac{1}{2}}\max_{1\le i\le n}|x_i - \bar{x}_n| \to 0 \quad \text{as } n \to \infty, \tag{2.2}$$

then

$$\sqrt{S_n}(\hat{\beta}_n - \beta + nA_n^{-1}\beta\sigma_1^2) \xrightarrow{L} N(0, \sigma_2^2 + \beta^2\sigma_1^2), \tag{2.3}$$

where $A_n = S_n + n\sigma_1^2$.

We first put forward a preliminary fact.

Lemma 2.1 *Suppose that $\omega_1, \omega_2, \cdots$ are random variable sequence with zero mean and finite variance, and constant sequence $\{x_i\}$ satisfies $S_n \to \infty$. Then*

$$S_n^{-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)\omega_i \to 0 \quad a.s. \quad as\ n \to \infty.$$

This is a special case of a result in [2].

Turn back to the proof of the theorem. Write $T_n = \sum\limits_{i=1}^{n} (\xi_i - \bar{\xi}_n)^2$. From (2.1) we have

$$T_n(\hat{\beta}_n - \beta) + n\beta\sigma_1^2 = \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)\varepsilon_i - \beta \sum_{i=1}^{n} (x_i - \bar{x}_n)\delta_i - \beta \Big[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2 \Big]. \tag{2.4}$$

Let $A_n = S_n + n\sigma_1^2$. Then

$$W_n \equiv \frac{n}{\sqrt{T_n}} \Big(\frac{T_n}{A_n} - 1 \Big) = n \frac{\Big[\sum\limits_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2 \Big] + 2 \sum\limits_{i=1}^{n} (x_i - \bar{x}_n)\delta_i}{A_n \sqrt{T_n}}.$$

Lemma 2.1 and (1.3) imply $S_n^{-1} T_n \to 1$ a.s. While

$$\sqrt{\frac{T_n}{S_n}}\, W_n = \frac{n}{A_n} \frac{\sum\limits_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2}{\sqrt{S_n}} + 2 \frac{n}{A_n} \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x}_n)\delta_i}{\sqrt{S_n}}. \tag{2.5}$$

Because $E\delta_1^4 < \infty$, when $n \to \infty$, the distribution of $n^{-\frac{1}{2}} \Big[\sum\limits_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2 \Big]$ converges to a normal distribution. Also from (1.3) we have

$$S_n^{-\frac{1}{2}} \Big[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2 \Big] \to 0 \quad \text{in probability}.$$

The first term in the right-hand side of (2.5) converges to 0 in probability in view of $nA_n^{-1} \to 0$. From $\text{var}\Big[S_n^{-\frac{1}{2}} \sum\limits_{i=1}^{n} (x_i - \bar{x}_n)\delta_i \Big] = \sigma_1^2$ and $nA_n^{-1} \to 0$ we see that the second term in the right-hand side of (2.5) also converges to 0. Therefore

$$\sqrt{T_n S_n^{-1}}\, W_n \to 0 \quad \text{in probability}.$$

This together with $S_n^{-1} T_n \to 1$ a.s. gives

$$W_n \to 0 \quad \text{in probability}. \tag{2.6}$$

The definition of W_n implies

$$n = nT_n A_n^{-1} - \sqrt{T_n}\, W_n.$$

Substituting the relationship above into (2.4), we have

$$\begin{aligned} &T_n(\hat{\beta}_n - \beta) + T_n n A_n^{-1} \beta\sigma_1^2 - \sqrt{T_n}\, W_n \beta\sigma_1^2 \\ &= \sum_{i=1}^{n} (\xi_i - \bar{\xi}_n)\varepsilon_i - \beta \sum_{i=1}^{n} (x_i - \bar{x}_n)\delta_i - \beta \Big[\sum_{i=1}^{n} (\delta_i - \bar{\delta}_n)^2 - n\sigma_1^2 \Big]. \end{aligned} \tag{2.7}$$

Dividing both sides by $\sqrt{T_n}$ and noticing (2.6), we obtain

$$\sqrt{T_n}\,(\hat\beta_n-\beta+nA_n^{-1}\beta\sigma_1^2)=T_n^{-\frac{1}{2}}\sum_{i=1}^n(\xi_i-\bar\xi_n)\varepsilon_i-\beta T_n^{-\frac{1}{2}}\sum_{i=1}^n(x_i-\bar x_n)\delta_i$$
$$-\beta T_n^{-\frac{1}{2}}\Big[\sum_{i=1}^n(\delta_i-\bar\delta_n)^2-n\sigma_1^2\Big]+o_p(1). \qquad (2.8)$$

Because the third term in the right-hand side of (2.8) is $o_p(1)$ and $S_n^{-1}T_n\to 1$ a.s., we know that when $n\to\infty$, the limiting distribution of $\sqrt{S_n}\,(\hat\beta_n-\beta+nA_n^{-1}\beta\sigma_1^2)$ is the same as that of

$$\begin{aligned}V_n^*&\equiv S_n^{-\frac{1}{2}}\sum_{i=1}^n(\xi_i-\bar\xi_n)\varepsilon_i-\beta S_n^{-\frac{1}{2}}\sum_{i=1}^n(x_i-\bar x_n)\delta_i\\&=S_n^{-\frac{1}{2}}\sum_{i=1}^n[(x_i-\bar x_n)\varepsilon_i-\beta(x_i-\bar x_n)\delta_i+\delta_i\varepsilon_i]-nS_n^{-\frac{1}{2}}\bar\delta_n\bar\varepsilon_n.\end{aligned}$$

Because $\sqrt{n}\,\bar\delta_n$ and $\sqrt{n}\,\bar\varepsilon_n$ have the limiting distributions $N(0,\sigma_1^2)$ and $N(0,\sigma_2^2)$ respectively, we know $nS_n^{-\frac{1}{2}}\bar\delta_n\bar\varepsilon_n=o_p(1)$. Hence the limiting distribution of $\sqrt{S_n}\,(\hat\beta_n-\beta+nA_n^{-1}\beta\sigma_1^2)$ is the same as that of

$$V_n\equiv\sum_{i=1}^n S_n^{-\frac{1}{2}}[(x_i-\bar x_n)\varepsilon_i-\beta(x_i-\bar x_n)\delta_i+\delta_i\varepsilon_i]\equiv\sum_{i=1}^n t_{ni}.$$

Here $t_{n1},t_{n2},\cdots,t_{nn}$ are mutually independent and have zero mean. Further,

$$\begin{aligned}B_n^2&\equiv\mathrm{var}(V_n)=\sum_{i=1}^n S_n^{-1}[(x_i-\bar x_n)^2\sigma_2^2+(x_i-\bar x_n)^2\beta^2\sigma_1^2+\sigma_1^2\sigma_2^2]\\&=\sigma_2^2+\beta^2\sigma_1^2+nS_n^{-1}\sigma_1^2\sigma_2^2\to\sigma_2^2+\beta^2\sigma_1^2,\\C_n&\equiv\sum_{i=1}^n E|t_{ni}|^3\le\text{Const.}\cdot\sum_{i=1}^n[|x_i-\bar x_n|^3(d_2+|\beta|^3d_1)+d_1d_2]S_n^{-\frac{3}{2}},\end{aligned}$$

where $d_1=E|\delta_1|^3,\ d_2=E|\varepsilon_1|^3$. By the moment assumption imposed on δ_1 and ε_1, d_1 and d_2 are finite. Therefore

$$C_n\le\text{Const.}\cdot\Big\{\Big[\max_{1\le i\le n}|x_i-\bar x_n|S_n^{-\frac{1}{2}}\Big](d_2+|\beta|^3d_1)+nd_1d_2S_n^{-\frac{3}{2}}\Big\}.$$

(2.2) and $nS_n^{-1}\to 0$ imply $C_n\to 0$ and then $C_nB_n^{-\frac{3}{2}}\to 0$. Consequently V_n has limiting distribution $N(0,\sigma_2^2+\beta^2\sigma_1^2)$. The proof is completed.

One might notice that although the assumption $nS_n^{-1}\to 0$ implies $nA_n^{-1}\to 0$, the left-side of (2.3) cannot be replaced by $\sqrt{S_n}\,(\hat\beta_n-\beta)$. This is because

$$\sqrt{S_n}\,(\hat\beta_n-\beta+nA_n^{-1}\beta\sigma_1^2)=\sqrt{S_n}\,(\hat\beta_n-\beta)+\sqrt{S_n}\,nA_n^{-1}\beta\sigma_1^2,$$

while it is possible that $nA_n^{-1}\sqrt{S_n}$ does not converge to 0. If we assume σ_1^2 is known in model (A), we can replace $\hat\beta_n$ by $\tilde\beta_n=(1-nA_n^{-1}\sigma_1^2)\hat\beta_n$ as the estimate of β. Then

$$\sqrt{S_n}\,(\tilde\beta_n-\beta)\xrightarrow{L}N(0,\sigma_2^2+\beta^2\sigma_1^2).$$

3 Asymptotic Normality of $\hat{\alpha}_n$

From (1.2) we have

$$\hat{\alpha} - \alpha = (\beta - \hat{\beta}_n)\bar{x}_n + (\beta - \hat{\beta}_n)\bar{\delta}_n - \beta\bar{\delta}_n + \bar{\varepsilon}_n. \tag{3.1}$$

Again denote $\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2$ by T_n. Then

$$\begin{aligned} T_n(\hat{\alpha}_n - \alpha) = & -\bar{x}_n \sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)\varepsilon_i + \beta\bar{x}_n \sum_{i=1}^{n}(x_i - \bar{x}_n)\delta_i + \beta\bar{x}_n \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \\ & - \bar{\delta}_n \sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)\varepsilon_i + \beta\bar{\delta}_n \sum_{i=1}^{n}(x_i - \bar{x}_n)\delta_i + \beta\bar{\delta}_n \sum_{i=1}^{n}(\delta_i - \bar{\delta}_n)^2 \\ & - \beta T_n\bar{\delta}_n + T_n\bar{\varepsilon}_n. \end{aligned} \tag{3.2}$$

Theorem 3.1 *Under model* (A)*, suppose that δ_1 and ε_1 are independent, δ_1 has sixth order moment, and ε_1 has third order moment. Write*

$$A_n = S_n + n\sigma_1^2, \quad D_n^2 = (\sigma_2^2 + \beta^2\sigma_1^2)(\bar{x}_n^2 + n^{-1}S_n). \tag{3.3}$$

If (1.3) *and* (2.2) *hold, then*

$$D_n^{-1}\sqrt{S_n}\,(\hat{\alpha}_n - \alpha - n\bar{x}_n A_n^{-1}\beta\sigma_1^2) \xrightarrow{L} N(0,1). \tag{3.4}$$

We need the following preliminary fact.

Lemma 3.1 $S_n \to \infty$ *implies* $\bar{x}_n^2 S_n^{-1} \to 0$.

Consider two special cases: (1) $\{\bar{x}_n\}$ are bounded. (2) $|\bar{x}_n| \to \infty$. We need only to consider case (2). Given natural integer m, find n_0 sufficiently large such that when $n \geq n_0$,

$$|\bar{x}_n| \geq 2\max\{|x_1|, \cdots, |x_m|\}.$$

As $n \geq n_0$,

$$S_n \geq \sum_{i=1}^{m}(x_i - \bar{x}_n)^2 \geq \frac{m\bar{x}_n^2}{4},$$

that is $\bar{x}_n^2 S_n^{-1} \leq 4m^{-1}$. Hence

$$\bar{x}_n^2 S_n^{-1} \to 0.$$

Because we can draw a subsequence from arbitrary subsequence of $\{\bar{x}_n\}$ belonging to case (1) or case (2), the lemma is proved.

Now turn back to the proof of the theorem. From (3.2) we have

$$\begin{aligned} & \sqrt{T_n}\,(\hat{\alpha}_n - \alpha - n\bar{x}_n A_n^{-1}\beta\sigma_1^2) \\ = & \sqrt{T_n}\,(\hat{\alpha}_n - \alpha - n\bar{x}_n T_n^{-1}\beta\sigma_1^2) + \sqrt{T_n}\,n\bar{x}_n\beta\sigma_1^2(T_n^{-1} - A_n^{-1}) \\ = & \sqrt{T_n}\,n\bar{x}_n\beta\sigma_1^2(T_n^{-1} - A_n^{-1}) - \frac{\bar{x}_n}{\sqrt{T_n}} \sum_{i=1}^{n}(x_i - \bar{x}_n)\varepsilon_i - \frac{\bar{x}_n}{\sqrt{T_n}} \sum_{i=1}^{n}\delta_i\varepsilon_i + \frac{1}{\sqrt{T_n}}\,n\bar{x}_n\bar{\delta}_n\bar{\varepsilon}_n \end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta\bar{x}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(x_i-\bar{x}_n)\delta_i+\frac{\beta\bar{x}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(\delta_i^2-\sigma_1^2)-\frac{\beta\bar{x}_n}{\sqrt{T_n}}n\bar{\delta}_n^2-\frac{\bar{\delta}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(x_i-\bar{x}_n)\varepsilon_i\\
&-\frac{\bar{\delta}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)\varepsilon_i+\frac{\beta\bar{\delta}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(x_i-\bar{x}_n)\delta_i+\frac{\beta\bar{\delta}_n}{\sqrt{T_n}}\sum_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2-\sqrt{T_n}\,\beta\bar{\delta}_n+\sqrt{T_n}\,\bar{\varepsilon}_n\\
\equiv&\sum_{i=1}^{13}f_i. \qquad (3.5)
\end{aligned}
$$

Consider $f_i,\ 1\le i\le 13$ one by one. Firstly, $\mathrm{var}(n\bar{\delta}_n\bar{\varepsilon}_n)=\sigma_1^2\sigma_2^2<\infty$ and $\sqrt{n}\,\bar{\delta}_n\xrightarrow{L}N(0,1)$ together with $\bar{x}_n^2S_n^{-1}\to 0$ (see Lemma 3.1) and $S_n^{-1}T_n\to 1$ a.s. imply

$$f_4, f_7\to 0 \quad \text{in probability.}$$

Secondly $\mathrm{var}\Big(S_n^{-\frac{1}{2}}\sum\limits_{i=1}^{n}(x_i-\bar{x}_n)\varepsilon_i\Big)=\sigma_2^2<\infty$ and $\bar{\delta}_n\to 0$ a.s. imply

$$f_8\to 0 \ \text{ in probability.}$$

$f_{10}\to 0$ in probability is deduced similarly. Then $\mathrm{var}\Big(S_n^{-\frac{1}{2}}\sum\limits_{i=1}^{n}\delta_i\varepsilon_i\Big)=nS_n^{-1}\sigma_1^2\sigma_2^2\to 0$ implies

$$f_9\to 0 \quad \text{in probability,}$$

and $\mathrm{var}\Big(S_n^{-\frac{1}{2}}\sum\limits_{i=1}^{n}\delta_i^2\Big)=nS_n^{-1}\mathrm{var}(\delta_1^2)\to 0$ implies

$$f_{11}\to 0 \quad \text{in probability.}$$

Finally, by the definition of D_n,

$$D_n^{-1}f_1=-\beta\sigma_1^2\frac{\bar{x}_n}{D_n}\cdot\frac{S_n^2}{A_nT_n}\cdot\frac{n}{S_n}\cdot\frac{\sqrt{T_n}}{\sqrt{S_n}}\cdot\frac{2\sum\limits_{i=1}^{n}(x_i-\bar{x}_n)\delta_i+\sum\limits_{i=1}^{n}(\delta_i-\bar{\delta}_n)^2-n\sigma_1^2}{\sqrt{S_n}}.$$

Because $|\bar{x}_n|\le \text{Const.}\cdot D_n$ and $S_n^2A_n^{-1}T_n^{-1}\to 1$ a.s.,

$$D_n^{-1}f_1\to 0 \quad \text{in probability.}$$

Summing up the discussions above and observing that $S_n^{-1}T_n\to 1$ a.s., we know that the limiting distribution of $D_n^{-1}\sqrt{S_n}\,(\hat{\alpha}_n-\alpha-n\bar{x}_nA_n^{-1}\beta\sigma_1^2)$ is the same as that of

$$
\begin{aligned}
\widetilde{V}_n\equiv&\sum_{i=1}^{n}D_n^{-1}S_n^{-\frac{1}{2}}\{[\beta\bar{x}_n(x_i-\bar{x}_n)-\beta n^{-1}S_n]\delta_i+\beta\bar{x}_n(\delta_i^2-\sigma_1^2)\\
&-[\bar{x}_n(x_i-\bar{x}_n)-n^{-1}S_n]\varepsilon_i-\bar{x}_n\delta_n\varepsilon_n\}\\
\equiv&\sum_{i=1}^{n}\tilde{t}_{ni}. \qquad (3.6)
\end{aligned}
$$

Here $\tilde{t}_{n1}, \cdots, \tilde{t}_{nn}$ are mutually independent and have zero mean. Further,

$$\begin{aligned}
\widetilde{B}_n^2 &\equiv \operatorname{var}\Big(\sum_{i=1}^n \tilde{t}_{ni}\Big) = 1 + S_n^{-1}D_n^{-2}n\bar{x}_n^2[\sigma_1^2\sigma_2^2 + \beta^2\operatorname{var}(\delta_1^2)] - D_n^{-2}\beta^2\bar{x}_n E\delta_1^3 \to 1,\\
\widetilde{C}_n &\equiv \sum_{i=1}^n E|\tilde{t}_{ni}|^3 \le \text{Const.}\cdot D_n^{-3}S_n^{-\frac{3}{2}}\sum_{i=1}^n\Big[|\beta|^3|\bar{x}_n|^3|x_i-\bar{x}_n|^3 d_1 + \frac{|\beta|^3 S_n^3}{n^3}d_1\\
&\quad + |\beta|^3|\bar{x}_n|^3 d_3 + |\beta|^3|\bar{x}_n|^3\sigma_1^6 + |\bar{x}_n|^3|x_i-\bar{x}_n|^3 d_2 + \frac{S_n^3}{n^3}d_2 + |\bar{x}_n|^3 d_1 d_2\Big]\\
&\le \text{Const.}\cdot\Big\{\max_{1\le i\le n}|x_i-\bar{x}_n|S_n^{-\frac{1}{2}}D_n^{-3}|\bar{x}_n|^3(|\beta|^3 d_1 + d_2)\\
&\quad + D_n^{-3}S_n^{\frac{3}{2}}n^{-2}(|\beta|^3 d_1 + d_2) + nD_n^{-3}S_n^{-\frac{3}{2}}|\bar{x}_n|^3[|\beta|^3(d_3+\sigma_1^6) + d_1 d_2]\Big\},
\end{aligned}$$

where $d_1 = E|\delta_1|^3,\ d_2 = E|\varepsilon_1|^3,\ d_3 = E\delta_1^6$. By the assumption imposed upon δ_i and ε_i, $d_1,\ d_2$ and d_3 are all finite. Since $D_n \ge \text{Const.}\cdot\sqrt{n^{-1}S_n}$ and $D_n \ge \text{Const.}\cdot|\bar{x}_n|$, we have

$$D_n^{-3}S_n^{\frac{3}{2}}n^{-2} \le \text{Const.}\cdot n^{-\frac{1}{2}} \to 0, \quad nD_n^{-3}S_n^{-\frac{3}{2}}|\bar{x}_n|^3 \le \text{Const.}\cdot nS_n^{-\frac{3}{2}} \to 0.$$

This proves that $\widetilde{C}_n \to 0$ and $\widetilde{C}_n\widetilde{B}_n^{-\frac{3}{2}} \to 0$. Therefore V_n has the limiting distribution $N(0,1)$. The proof is completed.

The proof can be greatly simplified if measurement errors are assumed to have normal distribution:

$$(\delta_1, \varepsilon_1) \sim N(0, 0, \sigma_1^2, \sigma_2^2, 0).$$

From (3.1) we have

$$\begin{aligned}
&\sqrt{S_n}(\hat{\alpha}_n - \alpha - n\bar{x}_n A_n^{-1}\beta\sigma_1^2)\\
&= -\sqrt{S_n}\bar{x}_n(\hat{\beta}_n - \beta + nA_n^{-1}\beta\sigma_1^2) - \beta\sqrt{S_n}\bar{\delta}_n + \sqrt{S_n}\bar{\varepsilon}_n + \sqrt{S_n}\bar{\delta}_n(\beta - \hat{\beta}_n)\\
&\equiv J_{1n} + J_{2n} + J_{3n} + J_{4n}.
\end{aligned}$$

$J_{1n},\ J_{2n}$ and J_{3n} are mutually independent for the normal case. This is because J_{2n} and J_{3n} are mutually independent and J_{1n} is independent of (J_{2n}, J_{3n}). The latter assertion is because J_{1n} depends only on the following four variables

$$I_1 = \sum_{i=1}^n (x_i - \bar{x}_n)\delta_i, \quad I_2 = \sum_{i=1}^n (x_i - \bar{x}_n)\varepsilon_i, \quad I_3 = \sum_{i=1}^n (\delta_i - \bar{\delta}_n)^2, \quad I_4 = \sum_{i=1}^n (\delta_i - \bar{\delta}_n)(\varepsilon_i - \bar{\varepsilon}_n),$$

and the normal condition implies $\bar{\delta}_n,\ \bar{\varepsilon}_n,\ I_1,\ I_2$ and (I_3, I_4) are mutually independent. Further, we have

$$J_{2n} \sim N(0, n^{-1}S_n\beta^2\sigma_1^2), \quad J_{3n} \sim N(0, n^{-1}S_n\sigma_2^2),$$

and J_{1n} approaches asymptotically to normal distribution $N(0, \bar{x}_n^2(\sigma_2^2 + \beta^2\sigma_1^2))$. When (1.3) holds, $\hat{\beta}_n$ is consistent estimate of β (see [3]). Hence $J_{4n} = o_p(J_{2n})$. Therefore the limiting distribution of $D_n^{-1}\sqrt{S_n}\,(\hat{\alpha}_n - \alpha - n\bar{x}_n A_n^{-1}\beta\sigma_1^2)$ is $N(0,1)$.

Theorem 2.1 and Theorem 3.1 show a difference between EV model and classical regression model. Asymptotic normality in EV model does not point at $\hat{\alpha}_n - \alpha$ and $\hat{\beta}_n - \beta$ but at $\hat{\alpha}_n - \alpha_n$

and $\hat{\beta}_n - \beta_n$ with $\alpha_n \to \alpha$ and $\beta_n \to \beta$. This stems from the fact that in the ordinary regression model the LS estimates $\hat{\alpha}_n$ and $\hat{\beta}_n$ are unbiased, which is not necessary so in the EV case.

References

[1] Drygas, H., Weak and strong consistency of the least squares estimates in regression model, *Z. Wahrsch. verw. Gebiete*, **34**, 1976, 119–127.

[2] Lai, T. L., Robbins, H. and Wei, C. Z., Strong consistency of least squares estimates in multiple regression, *J. Multivariate Anal.*, **9**, 1979, 343–362.

[3] Anderson, T. W., Estimating linear statistical relationships, *Ann. Statist.*, **12**, 1984, 1–45.

[4] Cui, H. J., Asymptotic normality of M-estimates in the EV model, *Sys. Sci. and Math. Sci.*, **10**(3), 1997, 225–236.

[5] Fuller, W. A., Measurement Error Models, John Wiley & Sons, New York, 1987.

[6] Liu, J. X. and Chen, X. R., Consistency of LS estimator in simple linear EV regression models, *Acta Math. Sci.*, **25B**(1), 2005, 50–58.