Chin. Ann. Math.
27B(6), 2006, 683–700
DOI: 10.1007/s11401-005-0288-4

Chinese Annals of Mathematics, Series B

Attractors for a Three-Dimensional Thermo-Mechanical Model of Shape Memory Alloys*****

Pierluigi COLLI* Michel FRÉMOND**
Elisabetta ROCCA*** Ken SHIRAKAWA****

Abstract In this note, we consider a Frémond model of shape memory alloys. Let us imagine a piece of a shape memory alloy which is fixed on one part of its boundary, and assume that forcing terms, e.g., heat sources and external stress on the remaining part of its boundary, converge to some time-independent functions, in appropriate senses, as time goes to infinity. Under the above assumption, we shall discuss the asymptotic stability for the dynamical system from the viewpoint of the global attractor. More precisely, we generalize the paper [12] dealing with the one-dimensional case. First, we show the existence of the global attractor for the limiting autonomous dynamical system; then we characterize the asymptotic stability for the non-autonomous case by the limiting global attractor.

Keywords Shape memory, Thermomechanical model, Parabolic system of partial differential equations, Global attractor
2000 MR Subject Classification 35K55, 35B41, 74D10

1 Introduction

This paper is concerned with a mathematical model proposed by the second author (cf. [10, 15, 17]) to describe the thermomechanical evolution of a shape memory alloy. At a microscopic scale, such phenomenon has been ascribed to (solid-solid) phase transitions between different configurations of the metallic lattice, known as austenite and martensite from the metallurgical terminology.

Frémond's model is a macroscopic model which is constructed in terms of basic functionals like free energy and pseudo-potential of dissipation, and it turns out to be consistent with the fundamental laws of Thermodynamics. The model leads to the system of partial differential equations and related conditions (1.1)–(1.9) that is stated below. The balance equations for energy (cf. (1.1)) and momentum (cf. (1.3)) are coupled with the partial differential inclusion

Manuscript received July 8, 2005.
*Dipartimento di Matematica "F. Casorati", Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy. E-mail: pierluigi.colli@unipv.it
**Laboratoire Central des Ponts et Chaussées, 58 Boulevard Lefebvre, 75732 Paris Cedex 15, France. E-mail: fremond@lcpc.fr
***Dipartimento di Matematica "F. Enriques", Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy. E-mail: elisabetta.rocca@mat.unimi.it
****Department of Applied Mathematics, Faculty of Engineering, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan. E-mail: sirakawa@cs.kobe-u.ac.jp
*****Project supported by the MIUR-COFIN 2004 research program on "Mathematical Modelling and Analysis of Free Boundary Problems".

(1.2) governing the evolution of the pointwise phase variables χ_1, χ_2 (that are related to the volumetric fractions of austenite and martensites phases). The other unknown variables of the system are the absolute temperature ϑ and the displacement $\mathbf{u} = (u_1, u_2, u_3)$, which satisfies the quasistationary momentum balance equation (1.3).

An updated and minute presentation of the Frémond model in its generality is provided in [2, 3, 17] and [16, Chapter 13]. We also point out [2, 3] for recent existence and uniqueness results in the three-dimensional situation: there, the various nonlinear terms arising in the derivation of the model are accounted. For a list of related references as well as for a survey of previous mathematical work, we address the reader to [1, 9, 13, 24]. The large time behavior of solutions is investigated in [11] in connection with the convergence to steady-state solutions. The paper [11] deals with the one-dimensional case, as well as [12], which is concerned with global attractors and motivated the observation and remarks developed in this note.

Here, we consider a bounded, connected domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\Gamma := \partial\Omega$, which is split in two parts Γ_0 and Γ_1 (measurable sets with positive surface measures), and we study the following system, which renders the reduced Frémond model resulting from the "small perturbations" assumption.

$$(c_0\vartheta - L\chi_1)_t - \Delta\vartheta = f, \quad \text{a.e. in } Q := (0, +\infty) \times \Omega, \tag{1.1}$$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_t - \begin{pmatrix}\Delta\chi_1\\ \Delta\chi_2\end{pmatrix} + \partial I_K(\chi_1, \chi_2) \ni \begin{pmatrix} l(\vartheta^* - \vartheta)\\ -\alpha(\vartheta)\operatorname{div}\mathbf{u}\end{pmatrix}, \quad \text{a.e. in } Q, \tag{1.2}$$

$$\operatorname{div}(-\nu\Delta(\operatorname{div}\mathbf{u})\mathbb{I} + \lambda(\operatorname{div}\mathbf{u})\mathbb{I} + 2\mu\varepsilon(\mathbf{u}) + \alpha(\vartheta)\chi_2\mathbb{I}) + \mathbf{h} = 0, \quad \text{a.e. in } Q, \tag{1.3}$$

$$\partial_{\mathbf{n}}\vartheta + \gamma(\vartheta - \Pi) = 0, \quad \text{a.e. on } \Sigma := (0, +\infty) \times \Gamma, \tag{1.4}$$

$$\partial_{\mathbf{n}}\chi_i = 0, \; i = 1, 2, \quad \text{a.e. on } \Sigma, \tag{1.5}$$

$$\mathbf{u} = 0, \quad \text{a.e. on } \Sigma_0 := (0, +\infty) \times \Gamma_0, \tag{1.6}$$

$$((-\nu\Delta(\operatorname{div}\mathbf{u}) + \lambda\operatorname{div}\mathbf{u} + \alpha(\vartheta)\chi_2)\mathbb{I} + 2\mu\varepsilon(\mathbf{u}))\cdot\mathbf{n} = \mathbf{g}, \quad \text{a.e. on } \Sigma_1 := (0, +\infty) \times \Gamma_1, \tag{1.7}$$

$$\partial_{\mathbf{n}}(\operatorname{div}\mathbf{u}) = 0, \quad \text{a.e. on } \Sigma, \tag{1.8}$$

$$\vartheta(0) = \vartheta_0, \; \chi_i(0) = \chi_{i,0}, \quad i = 1, 2, \; \text{a.e. in } \Omega. \tag{1.9}$$

We notice that $\varepsilon_{ij}(\mathbf{u}) := \frac{\partial_{x_j}u_i + \partial_{x_i}u_j}{2}$, $i, j = 1, 2, 3$, are the components of the standard linearized strain tensor $\varepsilon(\mathbf{u})$, $\mathbb{I}$ denotes the identity matrix in $\mathbb{R}^3$, $\mathbf{n}$ stands for the outward normal vector to Γ. Concerning data, $f : Q \to \mathbb{R}$ represents a known source term, $\mathbf{h} : Q \to \mathbb{R}^3$ is a volume force, $\Pi : \Sigma \to \mathbb{R}$ denotes an energy flux coming from the exterior of the system, and $\mathbf{g} : \Sigma_1 \to \mathbb{R}^3$ yields the external contact force applied to Γ_1. Moreover, c_0, L, l, ϑ^*, ν, λ, μ, γ are positive coefficients with proper physical meaning; in particular, ϑ^* represents a critical temperature. The nonlinearity α acting on temperature values is a smooth non-negative decreasing function, vanishing on the interval $[\vartheta_c, +\infty)$ for a certain fixed temperature (the so-called Curie point) $\vartheta_c > \vartheta^*$ (see, for instance, [9, assumptions (2.12)–(2.13)]). Actually, among the properties of α, in our analysis we just use the fact that $\alpha \in W^{1,\infty}(\mathbb{R})$. As the Frémond model assumes a non-differentiable free energy, in (1.1)–(1.9) we find the maximal monotone graph ∂I_K, representing the subdifferential of the indicator function I_K of the plane triangle

$$K := \{[\xi, \eta] \in \mathbb{R}^2 \mid 0 \le \xi \le 1, \; |\eta| \le \xi\}. \tag{1.10}$$

The set K is convex and contains the admissible phase proportions. We also notice that $I_K(\chi_1, \chi_2) = 0$ if $[\chi_1, \chi_2] \in K$, $= +\infty$ otherwise. For definitions and basic properties of maximal monotone operators and subdifferentials of convex functions, we refer, for instance, to [4].

Let us comment now on the fact that, with the help of the usually considered boundary conditions (1.6)–(1.8) (see, e.g., [10]), from (1.3) it turns out that (cf. Lemma 2 in Section 2), at almost any time t, the displacement $\mathbf{u}$ can be completely determined in terms of the data $\mathbf{h}$, $\mathbf{g}$ and the other unknowns ϑ, χ_2. Thus, we may introduce the operator $F^t_{\mathbf{h},\mathbf{g}}(\vartheta, \chi_2)$, which maps the pair $[\vartheta(t), \chi_2(t)]$ (as well as $\mathbf{h}(t), \mathbf{g}(t)$) into $\operatorname{div} \mathbf{u}(t)$, where $\mathbf{u}(t)$ stands for the related solution of (1.3), (1.6)–(1.8). Then, you consider the following system, denoted by (SMA), whose unknowns are now the absolute temperature ϑ and the phase variables χ_1, χ_2.

$$\begin{cases} (c_0\vartheta - L\chi_1)_t - \Delta\vartheta = f, \quad \text{a.e. in } Q, \\ \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t - \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \partial I_K(\chi_1, \chi_2) \ni \begin{pmatrix} l(\vartheta^* - \vartheta) \\ \\ -\alpha(\vartheta) F^t_{\mathbf{h},\mathbf{g}}(\vartheta, \chi_2) \end{pmatrix}, \quad \text{a.e. in } Q, \\ \partial_{\mathbf{n}}\vartheta + \gamma(\vartheta - \Pi) = 0, \quad \text{a.e. on } \Sigma, \\ \partial_{\mathbf{n}}\chi_i = 0,\ i = 1, 2, \quad \text{a.e. on } \Sigma, \\ \vartheta(0) = \vartheta_0,\ \chi_i(0) = \chi_{i,0}, \quad i = 1, 2,\ \text{a.e. in } \Omega. \end{cases}$$

In this paper, we shall try to extend to the 3D case the results already presented in [12] for the 1D setting. Hence, we characterize the large time behavior according to the theory of dissipative dynamical systems. Let us underline that the main difference between the 3D and the 1D setting is the fact that in the 3D case we cannot write down explicitly the solution $\mathbf{u}$ of (1.3), (1.6)–(1.8), as it was instead done in [12] for the 1D case (even in the less regular framework in which $\nu = 0$, i.e., the fourth-order term is missing in the analog of (1.3)). Then, in the present setting we have to study carefully either the regularity properties of $\mathbf{u}$ with respect to the data $\mathbf{h}$ and $\mathbf{g}$, or the dependence of $\operatorname{div} \mathbf{u}$ on the functions ϑ and χ_2 in order to get the same type of result as in [12].

In fact, our aim is to discuss the large time behavior of solutions of (SMA) from the viewpoint of global attractors.

Definition 1.1 (Global Attractor) *Let H be a real Hilbert space, and $D_0 \subset H$ be a closed and convex set in H. Let $\{T(t) : D_0 \longrightarrow D_0, t \geq 0\}$ be a semigroup on D_0. Then, a set $\mathcal{A} \subset D_0$ is called a global attractor, if*

(A1) *A is nonempty, connected and compact in H*;

(A2) (invariance) *$T(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$*;

(A3) (attractiveness) *for any bounded set $B \subset D_0$, $\operatorname{dist}_H(T(t)B, \mathcal{A}) \to 0$ as $t \to +\infty$, where $\operatorname{dist}_H(\cdot, \cdot)$ is the Hausdorff semi-distance between two subsets in H, defined as $\operatorname{dist}_H(A, B) := \sup\limits_{a \in A} \inf\limits_{b \in B} |a - b|_H$ for any subsets A and $B \subset H$.*

For an autonomous system, the dynamical system associated with the solution forms a semigroup. But, system (SMA) is a non-autonomous system, so that you need a suitable approach to describe the asymptotic stability.

In this paper, the asymptotic stability for (SMA) is characterized in the framework of the general theory discussed in [19, 23]. Unfortunately, this theory is not directly applicable to our system (SMA) (cf. also [12] for other comments on this subject), but a similar characterization holds by virtue of the argumentation established by Chepyzhov and Vishik [6, 7] for attractors of non-autonomous systems.

We investigate system (SMA) within the theory of infinite-dimensional (dissipative) dynamical systems. We suppose that the time-dependent data $f(t,\cdot)$, $\mathbf{h}(t,\cdot)$, $\Pi(t,\cdot)$, $\mathbf{g}(t,\cdot)$ converge to time-independent functions $f^\infty(\cdot)$, $\mathbf{h}^\infty(\cdot)$, $\Pi^\infty(\cdot)$, $\mathbf{g}^\infty(\cdot)$, respectively, as t goes to $+\infty$. Then, we prove that the limiting autonomous dynamical system possesses a global attractor $\mathcal{A}_\infty$ which is related to the non-autonomous dynamical system (generating the process $E(t,s)$) in the ways that

(R1) $\mathcal{A}_\infty$ contains the ω-limit set $\omega_E(B)$ for any B bounded subset of the phase space;

(R2) $\mathcal{A}_\infty = \omega_E(B_E)$ for some suitable set B_E representing a uniform absorbing set for $E(t,s)$.

The above result is stated in Section 2 after reformulating the original problem and stating some existence, uniqueness and boundedness properties of the solution. Proofs come as plain consequences of a useful result proved in our Section 2 and of [12, Sections 3–5].

2 Statement of Main Results

First, we make precise assumption on data. In the sequel, for a Banach space X, we let $L^2_{\rm loc}(0,+\infty;X)$ stand for the set of all measurable functions v from $(0,+\infty)$ to X such that $v\in L^2(0,T;X)$ for all $T>0$ (let us omit the indication of X if $X=\mathbb{R}$). We assume that

(a1) $f\in L^2_{\rm loc}(0,+\infty;L^2(\Omega))$, $\mathbf{h}\in W^{1,2}_{\rm loc}(0,+\infty;(L^2(\Omega))^3)$, $\Pi\in W^{1,2}_{\rm loc}(0,+\infty;L^2(\Gamma))$, and $\mathbf{g}\in W^{1,2}_{\rm loc}(0,+\infty;(L^2(\Gamma_1))^3)$. Moreover, letting $\mathbf{h}_t$, Π_t, and $\mathbf{g}_t$ denote the time derivatives of $\mathbf{h}$, Π, and $\mathbf{g}$, the function $f^*\in L^2_{\rm loc}(0,+\infty)$,

$$\begin{aligned} f^*(t) := \{1 &+ |f(t)|^2_{L^2(\Omega)} + |\mathbf{h}(t)|^2_{(L^2(\Omega))^3} + |\mathbf{h}_t(t)|^2_{(L^2(\Omega))^3} + |\Pi(t)|^2_{L^2(\Gamma)} \\ &+ |\Pi_t(t)|^2_{L^2(\Gamma)} + |\mathbf{g}(t)|^2_{(L^2(\Gamma_1))^3} + |\mathbf{g}_t(t)|^2_{(L^2(\Gamma_1))^3}\}^{\frac12} \quad \text{for } t\ge 0, \end{aligned} \tag{2.1}$$

is bounded in the following sense

$$S(f^*) := \sup_{s\ge 0} |f^*(\cdot+s)|_{L^2(0,1)} < +\infty; \tag{2.2}$$

(a2) $c_0,\ L,\ l,\ \vartheta^*,\ \nu,\ \lambda,\ \mu,\ \gamma$ are all positive constants;

(a3) $\alpha\in W^{1,\infty}(\mathbb{R})$;

(a4) ∂I_K is the subdifferential of the indicator function I_K of the triangle K in (1.10).

Remark 2.1 Note that (a1) entails in particular that the three functions $\mathbf{h}$, Π, $\mathbf{g}$ belong to $L^\infty(0,+\infty;(L^2(\Omega))^3)$, $L^\infty(0,+\infty;L^2(\Gamma))$, and $L^\infty(0,+\infty;(L^2(\Gamma_1))^3)$, respectively. Indeed, take Π, for instance, and observe that, if $t\ge 0$ and n_t denotes the integral part of t, we have

$$|\Pi(t)|^2_{L^2(\Gamma)} \le \int_{n_t}^{n_t+1} |\Pi(s)|^2_{L^2(\Gamma)}ds + \int_{n_t}^{n_t+1} |\Pi_t(\tau)|^2_{L^2(\Gamma)}d\tau \le 2S(f^*)^2,$$

whence the boundedness of $|\Pi|_{L^\infty(0,+\infty;L^2(\Gamma))}$ follows.

Next, we specify some notation. Denote by $(\cdot\,,\cdot)$ the usual inner product in both $L^2(\Omega)$ and $(L^2(\Omega))^3$. Put $V = H^1(\Omega)$, with inner product

$$(u,v)_V := (\nabla u, \nabla v) + \gamma \int_\Gamma u_{|\Gamma} v_{|\Gamma} \quad \text{for any } u, v \in V,$$

where $u_{|\Gamma}$ stands here for the trace of u on Γ. If V^* is the dual space of V and we identify $L^2(\Omega)$ with its dual space, it is well known that

$$V \subset L^2(\Omega) \subset V^* \quad \text{with compact injections.} \tag{2.3}$$

Moreover, we define the Hilbert space

$$\mathbf{W} := \{\mathbf{v} \in (V)^3 : \, \mathbf{v}_{|\Gamma_0} = \mathbf{0}, \, \mathrm{div}\, \mathbf{v} \in V\} \tag{2.4}$$

endowed with the norm (cf. [9])

$$|\mathbf{v}|_{\mathbf{W}} := \Big(\nu \int_\Omega |\nabla(\mathrm{div}\, \mathbf{v})|^2 + \sum_{i=1}^{3} \int_\Omega |\nabla v_i|^2\Big)^{\frac{1}{2}}, \quad \mathbf{v} \in \mathbf{W}. \tag{2.5}$$

Let us mention that $\nu > 0$ is the coefficient appearing in the fourth order term of (1.3). Now, we need to make precise the operator $F^t_{\mathbf{h},\mathbf{g}}$, already used to set our system (SMA) in the Introduction, and plug it in a suitable functional setting. In particular, we are going to introduce a variational formulation of problem (1.3), (1.6)–(1.8) in the following lemma: we also recall known properties of the solution and include the proof for the reader's convenience. Before stating our helpful tool, we have to note that the operator $F^t_{\mathbf{h},\mathbf{g}}$ actually acts on the two variables ϑ and χ_2, and we are interested in treating the case in which, for every time t, $\vartheta(t)$ and $\chi_2(t)$ are both in $L^2(\Omega)$, and χ_2 is the second component of the pair $[\chi_1, \chi_2]$ that lies in K almost everywhere. Hence, thanks to (1.10) we are allowed to consider only the values

$$[\vartheta(t), \chi_2(t)] \in \mathcal{D}_F := \{[v_1, v_2] \in (L^2(\Omega))^2 : |v_2| \le 1, \text{ a.e. in } \Omega\}. \tag{2.6}$$

Thus, we are ready to state the following

Lemma 2.1 *Let* $\mathbf{h}$, $\mathbf{g}$ *be as in* (a1) *and let* ϑ, $\chi_2 \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$ *satisfy* (2.6) *for almost every* $t > 0$. *Define the bilinear form*

$$a(\mathbf{v}_1, \mathbf{v}_2) := \int_\Omega \Big[\nu \nabla(\mathrm{div}\, \mathbf{v}_1) \cdot \nabla(\mathrm{div}\, \mathbf{v}_2) + \lambda\, \mathrm{div}\, \mathbf{v}_1 \,\mathrm{div}\, \mathbf{v}_2 + 2\mu \sum_{i,j=1}^{3} \varepsilon_{ij}(\mathbf{v}_1)\varepsilon_{ij}(\mathbf{v}_2)\Big],$$

where $\varepsilon_{ij}(\mathbf{v}) = \frac{\partial_{x_j} v_i + \partial_{x_i} v_j}{2}$, $i, j = 1, 2, 3$, *and* $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{v}_1$, $\mathbf{v}_2 \in \mathbf{W}$. *If* (a3) *holds, then for almost every* $t \in (0, +\infty)$, *there exists a unique* $\mathbf{u}(t) \in \mathbf{W}$ *such that*

$$a(\mathbf{u}(t), \mathbf{v}) + \int_\Omega \alpha(\vartheta(t))\chi_2(t)\, \mathrm{div}\, \mathbf{v} = \int_\Omega \mathbf{h}(t) \cdot \mathbf{v} + \int_{\Gamma_1} \mathbf{g}(t) \cdot \mathbf{v}_{|\Gamma_1} \quad \text{for all } \mathbf{v} \in \mathbf{W}. \tag{2.7}$$

Moreover, if we term $F^t_{\mathbf{h},\mathbf{g}}$ *the operator*

$$F^t_{\mathbf{h},\mathbf{g}} : [\vartheta(t), \chi_2(t)] \in \mathcal{D}_F \mapsto \mathrm{div}\, \mathbf{u}(t) \in L^2(\Omega), \quad \textit{a.e. } t \in (0, +\infty),$$

where $\mathbf{u}(t)$ *denotes the unique solution of* (2.7)*, then for almost every* $t \in (0,+\infty)$ *there holds*

$$|F^t_{\mathbf{h},\mathbf{g}}(\vartheta,\chi_2)(t)|_{L^\infty(\Omega)} \le \bar{c}\,\{1+|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}+|\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\} \tag{2.8}$$

for some constant $\bar{c}$ *depending only on* Ω *and* $|\alpha|_{L^\infty(\mathbb{R})}$*. In addition, if*

$$\vartheta,\ \chi_2 \in W^{1,2}(\delta,T;L^2(\Omega)) \quad \textit{for some } \ 0\le\delta<T<+\infty,$$

then we have that

$$F^t_{\mathbf{h},\mathbf{g}}(\vartheta,\chi_2) \in W^{1,2}(\delta,T;L^2(\Omega))$$

and its time derivative $(F^t_{\mathbf{h},\mathbf{g}}(\vartheta,\chi_2))_t$ *fulfills*

$$|(F^t_{\mathbf{h},\mathbf{g}}(\vartheta,\chi_2))_t(t)|_{L^2(\Omega)} \le c_d\{|\mathbf{h}_t(t)|_{(L^2(\Omega))^3}+|\mathbf{g}_t(t)|_{(L^2(\Gamma_1))^3}+|\vartheta_t(t)|_{L^2(\Omega)}+|(\chi_2)_t(t)|_{L^2(\Omega)}\} \tag{2.9}$$

for a.e. $t\in(\delta,T)$ *and for some constant* c_d *depending only on* Ω *and* $|\alpha|_{W^{1,\infty}(\mathbb{R})}$*. Finally, there is a positive constant* $\tilde{c}$*, having the same dependencies as* c_d*, such that for almost every* $t\in(0,+\infty)$ *and for two arbitrary pairs* $[\tilde{\vartheta},\widetilde{\chi}_2]$*,* $[\vartheta,\chi_2]$ *that belong to* $L^2_{\rm loc}(0,+\infty;(L^2(\Omega))^2)$ *and obey* (2.6)*, we have the following estimate*

$$|F^t_{\mathbf{h},\mathbf{g}}(\tilde{\vartheta},\widetilde{\chi}_2)(t)-F^t_{\mathbf{h},\mathbf{g}}(\vartheta,\chi_2)(t)|^2_{L^2(\Omega)} \le \tilde{c}\{|\tilde{\vartheta}(t)-\vartheta(t)|^2_{L^2(\Omega)}+|\widetilde{\chi}_2(t)-\chi_2(t)|^2_{L^2(\Omega)}\}. \tag{2.10}$$

Proof Although it closely follows the proofs of [10, Lemma 1, p.44] and [8, Lemma 2], we prefer to detail it for the sake of clarity. By virtue of Korn's inequality (see, e.g., [14, p.115]), the bilinear form $a(\cdot,\cdot)$ is $\mathbf{W}$-elliptic, that is, coercive in $\mathbf{W}\times\mathbf{W}$. Then, existence and uniqueness of the solution $\mathbf{u}$ to (2.7) come out as a straightforward consequence of the Lax-Milgram lemma. Next, freeze a time $t>0$, test (2.7) by $\mathbf{v}=\mathbf{u}(t)$, and use (a3) and standard estimates in order to find that

$$|\mathbf{u}(t)|_{\mathbf{W}} \le c\{1+|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}+|\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\} \tag{2.11}$$

with c depending only on Ω and $|\alpha|_{L^\infty(\mathbb{R})}$. Then, it is not difficult to check that the solution $\mathbf{u}(t)=(u_1(t),u_2(t),u_3(t))$ satisfies

$$\begin{aligned}&\mu\Delta u_i(t)+\partial_{x_i}[(\lambda+\mu)\operatorname{div}\mathbf{u}(t)-\nu\Delta(\operatorname{div}\mathbf{u}(t))+\alpha(\vartheta(t)\chi_2(t))]\\&=h_i(t),\quad i=1,2,3,\ \text{in } \mathcal{D}'(\Omega),\end{aligned} \tag{2.12}$$

where $\mathbf{h}=(h_1,h_2,h_3)$. Hence, since from (2.11) it follows that $\Delta u_i(t)$, $i=1,2,3$, and $\Delta(\operatorname{div}\mathbf{u}(t))$ are uniformly bounded in $H^{-1}(\Omega)$, by comparison in (2.12) we deduce that

$$\begin{aligned}&\sum_{i=1}^{3}|\partial_{x_i}[(\lambda+\mu)\operatorname{div}\mathbf{u}(t)-\nu\Delta(\operatorname{div}\mathbf{u}(t))+\alpha(\vartheta(t)\chi_2(t))]\,|_{H^{-1}(\Omega)}\\&\quad+|(\lambda+\mu)\operatorname{div}\mathbf{u}(t)-\nu\Delta(\operatorname{div}\mathbf{u}(t))+\alpha(\vartheta(t)\chi_2(t))|_{H^{-1}(\Omega)}\\&\le c\big\{1+|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}+|\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\big\}.\end{aligned}$$

Therefore, we may use the Lions lemma (reported, e.g., in [25, Proposition 1.2, p.16]) and infer that $[(\lambda+\mu)\operatorname{div}\mathbf{u}(t)-\nu\Delta(\operatorname{div}\mathbf{u}(t))+\alpha(\vartheta(t),\chi_2(t))]$ is uniformly bounded in $L^2(\Omega)$. From these considerations it plainly results that

$$|-\Delta(\operatorname{div}\mathbf{u}(t))|_{L^2(\Omega)} \le c\{1+|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}+|\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\}. \tag{2.13}$$

We aim to point out that, while (1.6) is a simple consequence of $\mathbf{u}(t) \in \mathbf{W}$, boundary conditions (1.7) and (1.8) can now be recovered from (2.7) and (2.12), and they hold in a suitable sense that involves trace spaces. In particular, $\operatorname{div} \mathbf{u}(t)$ satisfies the homogeneous boundary condition (1.8). Hence, in view of (2.11) and (2.13), well-known regularity results for elliptic boundary value problems (see, e.g., [22]) yield

$$|\operatorname{div} \mathbf{u}(t)|_{H^2(\Omega)} \le c\{1 + |\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)} + |\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\},$$

whence (2.8) follows from the Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$. Now, to verify (2.9) it suffices to formally differentiate (2.7) with respect to time and take $\mathbf{v} = \mathbf{u}_t(t)$. Hence, as $|(\alpha(\vartheta)\chi_2)_t| = |\alpha'(\vartheta)\chi_2\vartheta_t + \alpha(\vartheta)(\chi_2)_t| \le c\big(|\vartheta_t| + |(\chi_2)_t|\big)$, a standard manipulation leads to (2.9). Finally, in order to show (2.10), we write (2.7) for $\tilde{\vartheta}(t), \widetilde{\chi}_2(t)$, subtract (2.7) written for $\vartheta(t), \chi_2(t)$, and choose $\mathbf{v} = \tilde{\mathbf{u}}(t) - \mathbf{u}(t)$ (where $\tilde{\mathbf{u}}(t)$ is the solution corresponding to $\tilde{\vartheta}(t), \widetilde{\chi}_2(t)$ and $\mathbf{u}(t)$ is the one corresponding to $\vartheta(t), \chi_2(t)$). Using (a3) and (2.6), you easily conclude the proof.

Now, we are ready to give the exact definition of solutions to (SMA).

Definition 2.1 (Definition of Solutions of (SMA)) *A triplet $[\vartheta, \chi_1, \chi_2]$ of functions $\vartheta : [0,+\infty) \longrightarrow L^2(\Omega)$ and $\chi_i : [0,+\infty) \longrightarrow L^2(\Omega)$, $i = 1, 2$, is called a solution of* (SMA) *if*

(s1) $\vartheta \in W^{1,2}_{\mathrm{loc}}((0,+\infty); V^*) \cap C([0,+\infty); L^2(\Omega)) \cap L^2_{\mathrm{loc}}(0,+\infty; V)$,
$\vartheta \in W^{1,2}_{\mathrm{loc}}(\delta,+\infty; L^2(\Omega))$ *for all* $\delta > 0$,
$\chi_i \in W^{1,2}_{\mathrm{loc}}(0,+\infty; L^2(\Omega)) \cap L^\infty_{\mathrm{loc}}(0,+\infty; V)$,

in particular, χ_i is weakly continuous from $[0,T]$ to V for all $T > 0$, $i = 1, 2$;

(s2) $\vartheta(0,\cdot) = \vartheta_0$ *in* $L^2(\Omega)$, *and*

$$((c_0\vartheta - L\chi_1)_t(t), z) + (\vartheta(t) - \vartheta_\Pi(t), z)_V = (f(t), z) \quad \text{for any } z \in V \text{ and a.e. } t > 0, \tag{2.14}$$

where $\vartheta_\Pi \in W^{1,2}_{\mathrm{loc}}(0,+\infty; V)$ is the unique solution of

$$(\vartheta_\Pi(t), z)_V = \gamma \int_\Gamma \Pi(t) z_{|\Gamma} \quad \text{for any } z \in V \text{ and a.e. } t > 0; \tag{2.15}$$

(s3) $\chi_i(0,\cdot) = \chi_{i,0}$ *in V, and there exists a pair $[\xi_1, \xi_2]$ of two functions $\xi_i \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$, $i = 1, 2$, such that*

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \partial I_K(\chi_1, \chi_2), \quad \text{a.e. in } Q, \tag{2.16}$$

$$\begin{aligned} &\sum_{i=1}^{2} \{((\chi_i)_t(t), z_i) + (\nabla\chi_i(t), \nabla z_i) + (\xi_i(t), z_i)\} \\ &= l(\vartheta^* - \vartheta(t), z_1) - (\alpha(\vartheta(t)) F^t_{\mathbf{h},\mathbf{g}}(\vartheta, \chi_2), z_2) \quad \text{for any } z_i \in V \text{ and a.e. } t > 0. \end{aligned} \tag{2.17}$$

For any $t \ge 0$, let φ^t be a proper, l.s.c. and convex function on $L^2(\Omega)$, defined by

$$\varphi^t(u) := \begin{cases} \dfrac{1}{2c_0}|u - \vartheta_\Pi(t)|_V^2, & \text{if } u \in V, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.18}$$

In addition, consider the proper, l.s.c. and convex function ψ on the product space $L^2(\Omega) \times L^2(\Omega)$ specified by

$$\psi(v_1, v_2) := \begin{cases} \dfrac{1}{2}\displaystyle\sum_{i=1}^{2}\int_0^1 |\nabla v_i(x)|^2 \, dx, & \text{if } v_i \in V,\ i = 1, 2,\ \text{and } [v_1, v_2] \in K,\ \text{a.e. in } \Omega, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.19}$$

For $t \geq 0$ and $u \in L^2(\Omega)$, let G_u^t be the operator

$$G_u^t(v_1, v_2) := [l(u - \vartheta^*), \alpha(u) F_{\mathbf{h},\mathbf{g}}(u, v_2)], \quad [v_1, v_2] \in L^2(\Omega) \times L^2(\Omega), \tag{2.20}$$

from the product space $L^2(\Omega) \times L^2(\Omega)$ into itself.

Remark 2.2 For the convex functions $\varphi^t(\,\cdot\,)$ and $\psi(\,\cdot\,,\cdot\,)$ and the operator $G_u^t(\,\cdot\,,\cdot\,)$ defined above, we easily see the following items:

(i) for any $t \geq 0$, $u^* \in \partial\varphi^t(u)$ if and only if $u \in H^2(\Omega)$, $-\Delta u = c_0 u^*$ in $L^2(\Omega)$ and $\partial_{\mathbf{n}} u + \gamma(u - \Pi(t)) = 0$, a.e. on Γ;

(ii) $\begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} \in \partial\psi(v_1, v_2)$ if and only if $v_i \in H^2(\Omega)$, $i = 1, 2$, and there exists a pair $[\zeta_1, \zeta_2]$ of functions $\zeta_i \in L^2(\Omega)$, $i = 1, 2$, such that $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \partial I_K(v_1, v_2)$, a.e. in Ω, $v_i^* = -\Delta v_i + \zeta_i$ in $L^2(\Omega)$ and $\partial_{\mathbf{n}} v_i = 0$, a.e. on Γ, $i = 1, 2$.

Moreover, we need to prove some properties holding for the functional G defined in (2.20).

Lemma 2.2 *There exists a positive constant L_G depending only on the norms $|\alpha|_{W^{1,\infty}(\mathbb{R})}$, $|\mathbf{g}|_{L^\infty(0,+\infty,(L^2(\Gamma_1))^3)}$, and $|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}$, such that*

$$\begin{aligned} &|G_{\tilde{u}}^t(\tilde{v}_1, \tilde{v}_2) - G_u^t(v_1, v_2)|^2_{L^2(\Omega)\times L^2(\Omega)} \leq L_G(|\tilde{u} - u|^2_{L^2(\Omega)} + |\tilde{v}_2 - v_2|^2_{L^2(\Omega)}) \\ &\qquad \textit{for any } \tilde{u}, u \in L^2(\Omega) \textit{ and } [\tilde{v}_1, \tilde{v}_2], [v_1, v_2] \in L^2(\Omega) \times L^2(\Omega). \end{aligned} \tag{2.21}$$

Proof Let t be a positive time and $\tilde{u}(t)$, $u(t)$, $\tilde{v}_1(t)$, $v_1(t)$, $\tilde{v}_2(t)$, $v_2(t) \in L^2(\Omega)$. In order to prove (2.21), we need to obtain the following inequality

$$\begin{aligned} &\int_\Omega |\alpha(\tilde{u}(t)) F_{\mathbf{h},\mathbf{g}}^t(\tilde{u}, \tilde{v}_2) - \alpha(u(t)) F_{\mathbf{h},\mathbf{g}}^t(u, v_2)|^2 \, dx \\ &\leq c(|\tilde{u}(t) - u(t)|^2_{L^2(\Omega)} + |\tilde{v}_2(t) - v_2(t)|^2_{L^2(\Omega)}) \end{aligned} \tag{2.22}$$

for some positive constant c depending only on $|\alpha|_{W^{1,\infty}(\mathbb{R})}$, $|\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}$, and on $|\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)}$. A first simple computation leads to

$$\begin{aligned} &\int_\Omega |\alpha(\tilde{u}(t)) F_{\mathbf{h},\mathbf{g}}^t(\tilde{u}, \tilde{v}_2) - \alpha(u(t)) F_{\mathbf{h},\mathbf{g}}^t(u, v_2)|^2 \, dx \\ &\leq 2 \int_\Omega |F_{\mathbf{h},\mathbf{g}}^t(\tilde{u}, \tilde{v}_2)|^2 |\alpha(\tilde{u}(t)) - \alpha(u(t))|^2 \, dx \\ &\quad + 2 \int_\Omega |\alpha(u(t))|^2 \cdot |F_{\mathbf{h},\mathbf{g}}^t(\tilde{u}, \tilde{v}_2) - F_{\mathbf{h},\mathbf{g}}^t(u, v_2)|^2 \, dx. \end{aligned} \tag{2.23}$$

Using now the definition of $F_{\mathbf{h},\mathbf{g}}^t$, the regularity (a3) of α and estimates (2.8), (2.10), (2.23) gives automatically the desired estimate (2.22), which leads (thanks to the definition of G_u^t) exactly to inequality (2.21). This concludes the proof of the lemma.

On the basis of Remark 2.2(i)–(ii), and Lemma 2.2, variational inequalities (equalities) (2.14) and (2.16)–(2.17) can be reformulated as the following evolution equations, of the form

$$\vartheta_t(t) + \partial\varphi^t(\vartheta(t)) \ni \frac{L}{c_0}(\chi_1)_t(t) + \frac{1}{c_0}f(t) \quad \text{in } L^2(\Omega), \tag{2.24}$$

$$\frac{d}{dt}[\chi_1(t), \chi_2(t)] + \partial\psi(\chi_1(t), \chi_2(t)) + G^t_{\vartheta(t)}(\chi_1(t), \chi_2(t)) \ni [0,0] \quad \text{in } L^2(\Omega) \times L^2(\Omega), \tag{2.25}$$

for a.e. $t > 0$, respectively.

Now, for the sake of simplicity, we define a Hilbert space W by setting

$$W := L^2(\Omega) \times V \times V$$

with the norm $|[u, v_1, v_2]|_W := \left(|u|^2_{L^2(\Omega)} + \sum_{i=1}^{2} |v_i|^2_V\right)^{\frac{1}{2}}$, $[u, v_1, v_2] \in W$, and put

$$D := \{[u, v_1, v_2] \in W \mid [v_1, v_2] \in K, \text{ a.e. in } \Omega\}.$$

As is easily seen, W and D indicate the range of solutions at any time $t \geq 0$ and the domain of initial values, respectively.

Our first theorem is concerned with the existence and uniqueness of solutions of (SMA).

Theorem 2.1 (Existence and Uniqueness) *Assume conditions* (a1)–(a4)*. Then, for any* $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$, (SMA) *admits a unique solution* $[\vartheta, \chi_1, \chi_2]$.

Our second theorem is concerned with the boundedness of solutions.

Theorem 2.2 (Boundedness) *Under conditions* (a1)–(a4)*, let* $[\vartheta, \chi_1, \chi_2]$ *be any solution of* (SMA) *with the initial value* $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$*. Then the following estimates hold for the solution* $[\vartheta, \chi_1, \chi_2]$:

(i) *there exists a positive constant* N_0*, independent of* f^* *and* $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$*, such that*

$$\begin{aligned} &|[\vartheta(t), \chi_1(t), \chi_2(t)]|^2_W + |\vartheta|^2_{L^2(t,t+1;V)} + \sum_{i=1}^{2} |\chi_i|^2_{W^{1,2}(t,t+1;L^2(\Omega))} \\ &\leq N_0\{|[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|^2_W + S(f^*)^2\} \quad \text{for any } t \geq 0; \end{aligned} \tag{2.26}$$

(ii) *for any (small)* $\delta > 0$*, there exists a positive constant* N_δ*, depending only on* δ*, such that*

$$\begin{aligned} &|\vartheta|^2_{W^{1,2}(t,t+1;L^2(\Omega))} + |\vartheta(t)|^2_V + \sum_{i=1}^{2}\{|\chi_i|^2_{W^{1,2}(t,t+1;V)} + |(\chi_i)_t(t)|^2_{L^2(\Omega)}\} \\ &\leq N_\delta(1 + |\mathbf{h}|^2_{L^\infty(0,+\infty;(L^2(\Omega))^3)} + |\mathbf{g}|^2_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}) \\ &\quad \cdot \{|[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|^2_W + S(f^*)^2\} \quad \text{for any } t \geq \delta. \end{aligned}$$

The following statement is an easy consequence of Theorem 2.2.

Corollary 2.1 *Under the same assumptions as in Theorem* 2.2*, let* $[\xi_1, \xi_2]$ *be a pair of functions satisfying* (2.16) *and* (2.17)*. Then, we have that*

(iii) *there is a positive constant M_0, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that*

$$\sum_{i=1}^{2} \Big\{ |\chi_i|^2_{L^2(t,t+1;H^2(\Omega))} + |\xi_i|^2_{L^2(t,t+1;L^2(\Omega))} \Big\}$$
$$\le M_0 \{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|^2_W + S(f^*)^2 \} \quad \text{for any } t \ge 0;$$

(iv) *for any $\delta > 0$, there is a positive constant M_δ, depending only on δ, such that*

$$\sum_{i=1}^{2} \{ |\chi_i(t)|^2_{H^2(\Omega)} + |\xi_i(t)|^2_{L^2(\Omega)} \} \le M_\delta (1 + |\mathbf{h}|^2_{L^\infty(0,+\infty;(L^2(\Omega))^3)} + |\mathbf{g}|^2_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)})$$
$$\cdot \{ |[\vartheta_0, \chi_{1,0}, \chi_{2,0}]|^2_W + S(f^*)^2 \} \quad \text{for any } t \ge \delta.$$

Proof In fact, since (2.17) is equivalent to

$$\begin{cases} -\Delta\chi_1(t) + \xi_1(t) = l(\vartheta^* - \vartheta(t)) - (\chi_1)_t(t), & \text{in } L^2(\Omega), \\ -\Delta\chi_2(t) + \xi_2(t) = -\alpha(\vartheta) F^t_{\mathbf{h},\mathbf{g}}(\vartheta, \chi_2) - (\chi_2)_t(t), & \text{in } L^2(\Omega), \end{cases}$$

and $(\chi_1(t), \chi_2(t)) \in K$ for a.e. $t > 0$, from (1.10), $\alpha \in L^\infty(\mathbb{R})$, (2.7)–(2.8), and a known regularity result (cf., e.g., [5]) we infer that that there exists a positive constant M_1, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, satisfying

$$\sum_{i=1}^{2} \{ |\chi_i(t)|^2_{H^2(\Omega)} + |\xi_i(t)|^2_{L^2(\Omega)} \}$$
$$\le M_1 \Big\{ 1 + |\mathbf{h}(t)|^2_{(L^2(\Omega))^3} + |\mathbf{g}(t)|^2_{(L^2(\Gamma_1))^3} + |\vartheta(t)|^2_{L^2(\Omega)} + \sum_{i=1}^{2} |(\chi_i)_t(t)|^2_{L^2(\Omega)} \Big\} \quad \text{for a.e. } t > 0.$$

Thus, by (a1) and Theorem 2.2(i)–(ii), we obtain assertions (iii) and (iv).

For any $s \ge 0$, we denote by $f^s \in L^2_{\text{loc}}(0,+\infty; L^2(\Omega))$, $\mathbf{h}^s \in L^2_{\text{loc}}(0,+\infty; (L^2(\Omega))^3)$, $\Pi^s \in W^{1,2}_{\text{loc}}(0,+\infty; L^2(\Gamma))$, and $\mathbf{g}^s \in W^{1,2}_{\text{loc}}(0,+\infty; (L^2(\Gamma_1))^3)$ forcing terms translated by s, more precisely, for $t > 0$,

$$\begin{aligned} f^s(t) &:= f(t+s) \quad \text{in } L^2(\Omega), \quad & \mathbf{h}^s(t) &:= \mathbf{h}(t+s) \quad \text{in } (L^2(\Omega))^3, \\ \Pi^s(t) &:= \Pi(t+s) \quad \text{in } L^2(\Gamma), \quad & \mathbf{g}^s(t) &:= \mathbf{g}(t+s) \quad \text{in } (L^2(\Gamma_1))^3. \end{aligned}$$

Let (SMA)^s ($s \ge 0$) be the system (SMA) with translated forcing terms f^s, $\mathbf{h}^s$, $\mathbf{g}^s$, and Π^s, namely,

$$\begin{cases} (c_0\vartheta - L\chi_1)_t - \Delta\vartheta = f^s, \quad \text{a.e. in } Q, \\ \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t - \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \partial I_K(\chi_1, \chi_2) \ni \begin{pmatrix} l(\vartheta^* - \vartheta) \\ -\alpha(\vartheta) F^t_{\mathbf{h}^s,\mathbf{g}^s}(\vartheta, \chi_2) \end{pmatrix}, \quad \text{a.e. in } Q, \\ -\partial_{\mathbf{n}}\vartheta + \gamma(\vartheta - \Pi^s) = 0, \quad \text{a.e. on } \Sigma, \\ \partial_{\mathbf{n}}\chi_i = 0, \ i = 1,2, \quad \text{a.e. on } \Sigma, \\ \vartheta(0) = \vartheta_0, \ \chi_i(0) = \chi_{i,0}, \quad i = 1,2, \ \text{a.e. in } \Omega. \end{cases}$$

Then, on account of Theorems 2.1 and 2.2, we define a solution operator $E(t,s) : D \longrightarrow D$ $(0 \le s \le t < +\infty)$, which assigns to any fixed triplet $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ the element $[\vartheta(t-s), \chi_1(t-$

$s), \chi_2(t-s)]$ given by the solution of $(\mathrm{SMA})^s$ at time $t-s$. Then, the family $\{E(t,s) \mid 0 \le s \le t < +\infty\}$ satisfies the evolution properties:

(E1) $E(s,s) = I$ (identity) for any $s \ge 0$;

(E2) $E(t,s) = E(t,t_0) \circ E(t_0,s)$ for any $0 \le s \le t_0 \le t < +\infty$.

Next, let $f^\infty \in L^2(\Omega)$, $\mathbf{h}^\infty \in (L^2(\Omega))^3$, $\Pi^\infty \in L^2(\Gamma)$, and $\mathbf{g}^\infty \in (L^2(\Gamma_1))^3$ be constant functions in time such that, for $s \to +\infty$,

$$\begin{aligned} &f^s \to f^\infty \quad \text{in } L^2_{\mathrm{loc}}(0,+\infty;L^2(\Omega)), \qquad &&\mathbf{h}^s \to \mathbf{h}^\infty \quad \text{in } L^2_{\mathrm{loc}}(0,+\infty;(L^2(\Omega))^3), \\ &\Pi^s \to \Pi^\infty \quad \text{in } W^{1,2}_{\mathrm{loc}}(0,+\infty;L^2(\Gamma)), \qquad &&\mathbf{g}^s \to \mathbf{g}^\infty \quad \text{in } W^{1,2}_{\mathrm{loc}}(0,+\infty;(L^2(\Gamma_1))^3). \end{aligned} \tag{2.27}$$

Then, we can consider the following system, denoted by $(\mathrm{SMA})^\infty$, as the limiting system:

$$\begin{cases} (c_0\vartheta - L\chi_1)_t - \Delta\vartheta = f^\infty, \quad \text{a.e. in } Q, \\ \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t - \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix} l(\vartheta^* - \vartheta) \\ -\alpha(\vartheta) F^t_{\mathbf{h}^\infty,\mathbf{g}^\infty}(\vartheta,\chi_2) \end{pmatrix}, \quad \text{a.e. in } Q, \\ -\partial_{\mathbf{n}}\vartheta + \gamma(\vartheta - \Pi^\infty) = 0, \quad \text{a.e. on } \Sigma, \\ \partial_{\mathbf{n}}\chi_i = 0,\ i = 1,2, \ \text{a.e. on } \Sigma, \\ \vartheta(0) = \vartheta_0,\ \chi_i(0) = \chi_{i,0}, \quad i = 1,2, \ \text{a.e. in } \Omega. \end{cases}$$

The well-posedness of problem $(\mathrm{SMA})^\infty$ can be inferred as a special case of that of (SMA). It is clear that for $(\mathrm{SMA})^\infty$ we can also define a solution operator $S(t) : D \longrightarrow D$ $(t \ge 0)$, which assigns to any initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ the triplet $[\vartheta(t), \chi_1(t), \chi_2(t)]$ specified by the solution of $(\mathrm{SMA})^\infty$ at time t. Here, the family $\{S(t),\ t \ge 0\}$ forms a semigroup on D, since $(\mathrm{SMA})^\infty$ is autonomous. Referring the reader to the later Proposition 3.1 (which is inspired by [12, Proposition 4.1] and continues to hold thanks to Lemma 2.2), we notice that both $\{E(t,s)\}$ and $\{S(t)\}$ satisfy appropriate continuity properties.

Definition 2.2 *Let $\{E(t,s)\}$, $\{S(t)\}$ be the dynamical systems introduced above.*

(I) (ω-Limit Sets) *For any subset B of D,*

(i) *the set* $\omega_E(B) := \bigcap_{\tau \ge 0} \overline{\bigcup_{\substack{t \ge \tau \\ s \ge 0}} E(t+s,s)B}^W$ *is called ω-limit set of B for $\{E(t,s)\}$;*

(ii) *the set* $\omega_S(B) := \bigcap_{\tau \ge 0} \overline{\bigcup_{t \ge \tau} S(t)B}^W$ *is called ω-limit set of B for $\{S(t)\}$.*

(II) (Absorbing Sets)

(i) (Uniform Absorbing Set) *A subset $B_E \subset D$ is called a uniform absorbing set for $\{E(t,s)\}$, if for any bounded subset $B \subset D$, there exists a finite time $t_B \ge 0$, depending only on B, such that $E(t+s,s)B \subset B_E$ for all $t \ge t_B$ and $s \ge 0$;*

(ii) (Absorbing Set) *A subset $B_S \subset D$ is called an absorbing set for $\{S(t)\}$, if for any bounded subset $B \subset D$, there exists a finite time $t_B \ge 0$, depending only on B, such that $S(t)B \subset B_S$ for all $t \ge t_B$.*

The main objective of this paper is to investigate the asymptotic stability for the dynamical system associated with (SMA) from the viewpoint of attractors. Our final result is concerned with a characterization of the asymptotic stability for $\{E(t,s)\}$ in terms of the global attractor (cf. Definition 1.1) for the limiting semigroup system $\{S(t)\}$.

Theorem 2.3 *Assume all conditions* (a1)–(a4) *and* (2.27). *Let* $\{E(t,s)\}$ *and* $\{S(t)\}$ *be dynamical systems as the above. Then, the following three statements are fulfilled.*

(i) *There exists a global attractor* $\mathcal{A}_\infty$ *for the semigroup* $\{S(t)\}$.

(ii) $\omega_E(B) \subset \mathcal{A}_\infty$ *for any bounded subset* B.

(iii) *There exist a uniform absorbing set* B_E *for* $\{E(t,s)\}$ *and an absorbing set* B_S *for* $\{S(t)\}$. *Moreover,* B_E *and* B_S *fulfill* $\omega_E(B_E) = \omega_S(B_S) = \mathcal{A}_\infty$.

Let us conclude by noting that we only sketch here the proofs of our main results (i.e. Theorems 2.1–2.3) because the proofs are strictly closed with the ones detailed in [12, Sections 2–5], except for the term F appearing in equation (2.17) of (SMA). However, let us note that this term may be treated using our Lemmas 2.1 and 2.2 proved in this section.

3 Proofs of Main Results

In this section, let us briefly present the proofs of our main theorems, which are Theorems 2.1–2.3. Since the demonstration technique is quite similar to that as in [12], we will show only the rough sketch of them, and refer to [12] for more details.

Let us put

$$D_0 := \{[u, v_1, v_2] \in D \mid u \in V\},$$

and first show the existence in the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D_0$.

Proof of the Existence in the Case of $[\boldsymbol{\vartheta_0, \chi_{1,0}, \chi_{2,0}}] \in \boldsymbol{D_0}$ Note that the system (SMA) can be reformulated as the system (2.24)–(2.25) of two abstract evolution equations governed by subdifferentials of appropriate convex functions. This kind of system was also treated in [21] to show a local existence result for some phase transition models. Indeed, by a similar argument as in [21, Section 3], we can also prove the local existence of solutions of (SMA) (see [12, Section 4] for details).

The global existence is shown by a contradiction argument. Let us assume that

$$T^* := \sup\{T > 0 \mid \text{(SMA) has a solution on } [0, T]\} < +\infty. \tag{3.1}$$

Then, by the local existence result, we immediately deduce that $T^* > 0$. Besides, since there holds some regularizing effect for the solutions, it turns out that the triplet $[\vartheta(T^*), \chi_1(T^*), \chi_2(T^*)]$ lies in D_0. Hence, taking $[\vartheta(T^*), \chi_1(T^*), \chi_2(T^*)]$ as the new initial value, we can extend the solution $[\vartheta, \chi_1, \chi_2]$ in time beyond T^*. This contradicts (3.1).

Next, we consider the case of $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$. In this case, the existence is shown on the basis of the continuous dependence of solutions with respect to the initial values. So, one first proves the uniqueness and the boundedness, because these properties are needed in the proof of the continuous dependence.

Proof of the Uniqueness Let $[\vartheta, \chi_1, \chi_2]$ and $[\tilde{\vartheta}, \widetilde{\chi}_1, \widetilde{\chi}_2]$ be two solutions of (SMA) with two initial values $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$ and $[\tilde{\vartheta}_0, \widetilde{\chi}_{1,0}, \widetilde{\chi}_{2,0}]$, respectively. Here, let us subtract (2.14) written for $[\tilde{\vartheta}, \widetilde{\chi}_1, \widetilde{\chi}_2]$ from the one for $[\vartheta, \chi_1, \chi_2]$, as well as (2.17), choose the appropriate test functions, and take the sum of results. Then, by the monotonicity of ∂I_K, we find positive constants ν_0 and ν_1, independent of forcing terms and initial values, such that

$$\frac{d}{dt}J_0(t) \le \nu_0\Big\{J_0(t) + \Big(|\vartheta_0 - \tilde{\vartheta}_0|^2_{L^2(\Omega)} + \sum_{i=1}^{2}|\chi_{i,0} - \widetilde{\chi}_{i,0}|^2_{L^2(\Omega)}\Big)\Big\}, \quad \text{a.e. } t > 0,$$

for the function

$$J_0(t) := \int_0^t |(\vartheta - \tilde{\vartheta})(\tau)|^2_{L^2(\Omega)}\, d\tau + \sum_{i=1}^{2}\int_0^t |(\chi_i - \widetilde{\chi}_i)(\tau)|^2_{L^2(\Omega)}\, d\tau + \nu_1\Big|\int_0^t(\vartheta - \tilde{\vartheta})(\tau)\, d\tau\Big|^2_V, \quad t \ge 0.$$

Now, applying Gronwall's lemma to J_0, we deduce the uniqueness.

Proof of Theorem 2.2(i) For any solution $[\vartheta, \chi_1, \chi_2]$ of (SMA) with the initial value $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$, we define

$$J(t) := \frac{c_0}{2L}|\vartheta(t) - \vartheta_\Pi(t)|^2_{L^2(\Omega)} + \frac{1}{2l}\sum_{i=1}^{2}|\chi_i(t)|^2_V, \quad t \ge 0. \tag{3.2}$$

Let us put $z = \frac{1}{L}(\vartheta(t) - \vartheta_\Pi(t))$ in (2.14), $z_i = \frac{1}{l}\chi_i(t)$ in (2.17), and (formally) $z_i = \frac{1}{l}(\chi_i)_t(t)$ in (2.17), $i = 1, 2$, then take the sum of results. Hence, applying condition (a1) and Young's inequality, we find positive constants ν_2, ν_3, ν_4 and N_1, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\frac{d}{dt}J(t) + \nu_2 J(t) \le \nu_3|f^*(t)|^2 \quad \text{for a.e. } t \ge 0, \tag{3.3}$$

$$\begin{aligned}&\frac{d}{dt}J(t) + \nu_4\Big\{|\vartheta(t) - \vartheta_\Pi(t)|^2_V + \sum_{i=1}^{2}\Big(|\nabla\chi_i(t)|^2_{(L^2(\Omega))^3} + |(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big)\Big\}\\ &\le N_1|f^*(t)|^2 \quad \text{for a.e. } t \ge 0.\end{aligned} \tag{3.4}$$

Now, applying Gronwall's lemma to (3.3) and integrating both sides of (3.4) over $[t, t+1]$, we obtain Theorem 2.2(i).

Proof of Theorem 2.2(ii) Let us argue formally and choose $z = \frac{1}{L}(\vartheta_t(t) - (\vartheta_\Pi)_t(t))$ in (2.14), $z_i = \frac{1}{l}(\chi_i)_t(t)$ in $\frac{d}{dt}$(2.17), $i = 1, 2$. Then, taking the sum of results and applying (2.8)–(2.9), condition (a1), and Young's inequality, we find a positive constant N_2, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\begin{aligned}&\frac{c_0}{4L}|\vartheta_t(t)|^2_{L^2(\Omega)} + \frac{1}{l}\sum_{i=1}^{2}|(\nabla\chi_i)_t|^2_{L^2(\Omega)} + \frac{d}{dt}\Big\{\frac{1}{2L}|\vartheta(t) - \vartheta_\Pi(t)|^2_V + \frac{1}{2l}\sum_{i=1}^{2}|(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big\}\\ &\le N_2\{1 + |\mathbf{h}|^2_{L^\infty(0,+\infty;(L^2(\Omega))^3)} + |\mathbf{g}|^2_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\}\\ &\quad \cdot \Big\{\sum_{i=1}^{2}|(\chi_i)_t(t)|^2_{L^2(\Omega)} + |f^*(t)|^2\Big\} \quad \text{for a.e. } t \ge 0.\end{aligned}$$

Here, letting $0 \le s \le t \le s+2$, we multiply both sides by $(t-s)$ and integrate them over $[s, t]$. Then, by (3.4), we find a positive constant N_3, independent of f^* and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}]$, such that

$$\begin{aligned}&\int_s^t (\tau - s)\Big(|\vartheta_t(\tau)|^2_{L^2(\Omega)} + \sum_{i=1}^{2}|(\nabla\chi_i)_t(\tau)|^2_{L^2(\Omega)}\Big)d\tau + (t-s)\Big(|\vartheta(t) - \gamma(t)|^2_V + \sum_{i=1}^{2}|(\chi_i)_t(t)|^2_{L^2(\Omega)}\Big)\\ &\le N_3\{1 + |\mathbf{h}|_{L^\infty(0,+\infty;(L^2(\Omega))^3)} + |\mathbf{g}|_{L^\infty(0,+\infty;(L^2(\Gamma_1))^3)}\}\Big(J(s) + \int_s^t |f^*(\tau)|^2\, d\tau\Big)\end{aligned} \tag{3.5}$$

for all s and t satisfying $0 \le s \le t \le s+2$. Now, combining condition (a1), (3.2), (3.5) and Theorem 2.2(i), we conclude Theorem 2.2(ii).

Remark 3.1 For solutions of systems $(\mathrm{SMA})^s$ with $s > 0$, we also obtain the same estimates as in Theorem 2.2 just as in the above argument. Then, we notice that constants N_0 and N_δ can be chosen independent of s, since forcing terms f^s, $\mathbf{h}^s$, $\mathbf{g}^s$, and Π^s are estimated by $S(f^*)$ uniformly with respect to s.

Now, we are on the stage to prove the existence of solutions in the case $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$. The keypoint of the proof is to show the continuous dependence of solutions, stated as follows.

Proposition 3.1 *Let $\{f_n\} \subset L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$, $\{\mathbf{h}_n\} \subset W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Omega))^3)$, $\{\mathbf{g}_n\} \subset W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Gamma_1))^3)$ and $\{\Pi_n\} \subset W^{1,2}_{\mathrm{loc}}(0,+\infty; L^2(\Gamma))$. For any $n \in \mathbb{N}$, let $[\vartheta_{0,n}, \chi_{1,0,n}, \chi_{2,0,n}] \in D$, and let $[\vartheta_n, \chi_{1,n}, \chi_{2,n}]$ be the solution of* (SMA) *corresponding to forcing terms f_n, $\mathbf{h}_n$, $\mathbf{g}_n$, Π_n and the initial value $[\vartheta_{0,n}, \chi_{1,0,n}, \chi_{2,0,n}]$. If there exist functions $f \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$, $\mathbf{h} \in W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Omega))^3)$, $\mathbf{g} \in W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Gamma_1))^3)$, $\Pi \in W^{1,2}_{\mathrm{loc}}(0,+\infty; L^2(\Gamma))$ and $[\vartheta_0, \chi_{1,0}, \chi_{2,0}] \in D$ such that*

$$\begin{cases} f_n \to f & \textit{in } \ L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega)), \\ \mathbf{h}_n \to \mathbf{h} & \textit{in } \ W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Omega))^3), \\ \mathbf{g}_n \to \mathbf{g} & \textit{in } \ W^{1,2}_{\mathrm{loc}}(0,+\infty; (L^2(\Gamma_1))^3), \\ \Pi_n \to \Pi & \textit{in } \ W^{1,2}_{\mathrm{loc}}(0,+\infty; L^2(\Gamma)), \\ [\vartheta_{0,n}, \chi_{1,0,n}, \chi_{2,0,n}] \to [\vartheta_0, \chi_{1,0}, \chi_{2,0}] \quad \textit{in } \ W \ \textit{as } n \to +\infty, \end{cases} \tag{3.6}$$

then there exists a triplet $[\vartheta, \chi_1, \chi_2] \in L^2_{\mathrm{loc}}(0,+\infty; W)$ such that

$$[\vartheta_n, \chi_{1,n}, \chi_{2,n}] \to [\vartheta, \chi_1, \chi_2] \quad \textit{in } \ C(J, W) \ \textit{as } n \to +\infty \tag{3.7}$$

for any compact interval $J \subset (0,+\infty)$. Moreover, the triplet $[\vartheta, \chi_1, \chi_2]$ is a solution of (SMA).

Proof For any $n \in \mathbb{N}$, let $[\xi_{1,n}, \xi_{2,n}]$ be the pair of functions $\xi_{i,n} \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$, $i = 1,2$, as in (2.16) and (2.17) with $[\vartheta, \chi_1, \chi_2] = [\vartheta_n, \chi_{1,n}, \chi_{2,n}]$. Then, by (3.6), Theorem 2.2 and Corollary 2.1(iii), we find subsequences $\{\vartheta_{n_k}\} \subset \{\vartheta_n\}$, $\{\chi_{i,n_k}\} \subset \{\chi_{i,n}\}$, $\{\xi_{i,n_k}\} \subset \{\xi_{i,n}\}$ and functions $\vartheta, \chi_i, \xi_i \in L^2_{\mathrm{loc}}(0,+\infty; L^2(\Omega))$, $i = 1,2$, satisfying

$$\begin{cases} \vartheta_{n_k} \to \vartheta & \text{weakly in } W^{1,2}(J_0; V^*) \cap L^2(J_0, V) \\ & \text{and strongly in } L^2(J_0, L^2(\Omega)), \\ \chi_{i,n_k} \to \chi_i & \text{weakly in } W^{1,2}(J_0; L^2(\Omega)) \cap L^2(J_0; H^2(\Omega)) \\ & \text{and strongly in } C(J_0; L^2(\Omega)) \cap L^2(J_0; H^1(\Omega)), \\ \xi_{i,n_k} \to \xi_i & \text{weakly in } L^2(J_0; L^2(\Omega)), \end{cases} \tag{3.8}$$

so it consequently follows from (2.10) and (3.6) that

$$F^{(\cdot)}_{\mathbf{h}_{n_k}, \mathbf{g}_{n_k}}(\vartheta_{n_k}, \chi_{2,n_k}) \to F^{(\cdot)}_{\mathbf{h},\mathbf{g}}(\vartheta, \chi_2) \quad \text{in } L^2(J_0; L^2(\Omega)) \text{ as } k \to +\infty,$$

for any compact interval $J_0 \subset [0,+\infty)$. Therefore, due to the demi-closedness of the graph ∂I_K as well, we infer that $[\chi_1, \chi_2]$ and $[\xi_1, \xi_2]$ satisfy (2.16) and (2.17).

Let now J be a compact interval contained in $(0,+\infty)$. For any $n\in\mathbb{N}$, we denote by $\vartheta_{\Pi_n}\in W^{1,2}_{\mathrm{loc}}(0,+\infty;V)$ the function specified by (2.15) as ϑ_Π, but in terms of the boundary data Π_n instead of Π. Analogously, we introduce the convex function φ_n^t using definition (2.18), with ϑ_Π replaced by ϑ_{Π_n}. Then, since ϑ_n solves

$$(\vartheta_n)_t(t)+\partial\varphi_n^t(\vartheta_n(t))\ni\frac{L}{c_0}(\chi_n)_t(t)+\frac{1}{c_0}f_n(t)\quad\text{in } L^2(\Omega),\ \text{a.e. } t>0,$$

for any $n\in\mathbb{N}$, it follows from (3.6), (3.8) and [20, Theorem 2.7.1] that $\vartheta_{n_k}\to\vartheta$ in $C(J;L^2(\Omega))$ as $k\to+\infty$ and ϑ satisfies (2.14). Thus, the limit triplet $[\vartheta,\chi_1,\chi_2]$ is a solution to (SMA). Moreover, by virtue of the uniqueness of the solution (3.8) and Theorem 2.2(ii), we conclude that (3.7) holds.

Proof of the Existence in the Case of $[\boldsymbol{\vartheta_0,\chi_{1,0},\chi_{2,0}}]\in \boldsymbol{D}$ For any $[\vartheta_0,\chi_{1,0},\chi_{2,0}]\in D$, let us choose a sequence $\{[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}]\}\subset D_0$ such that

$$[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}]\to[\vartheta_0,\chi_{1,0},\chi_{2,0}]\quad\text{in } W \text{ as } n\to+\infty.$$

For any $n\in\mathbb{N}$, let $[\vartheta_n,\chi_{1,n},\chi_{2,n}]$ be the solution of (SMA) with the initial value $[\vartheta_{0,n},\chi_{1,0,n},\chi_{2,0,n}]\in D_0$. Then, by Proposition 3.1, we find a solution $\{\vartheta,\chi_1,\chi_2\}$ of (SMA) with the initial value $[\vartheta_0,\chi_{1,0},\chi_{2,0}]\in D$ as the limit of the sequence $\{\vartheta_n,\chi_{1,n},\chi_{2,n}\}$ of solutions in the sense of (3.7).

Finally, we show our third theorem. The keypoint of the proof is to find so-called compact (uniform) absorbing sets, stated in the following lemma.

Lemma 3.1 *These exists a compact subset $B^*\subset D$ which satisfies the following absorbing property $(*)$:*

$(*)$ *for any bounded subset $B\subset D$, there is a finite time t_B, depending only on B, such that $E(t+s,s)B\cup S(t)B\subset B^*$ for any $s\ge 0$ and $t\ge t_B$.*

Proof For any $0\le s\le+\infty$, let us denote by $\vartheta_{\Pi^s}\in W^{1,2}_{\mathrm{loc}}(0,+\infty;V)$ the unique solution of

$$(\vartheta_{\Pi^s}(t),z)_V=\gamma\int_\Gamma \Pi^s(t)z_{|\Gamma}\quad\text{for any } z\in V.$$

Here, we note that ϑ_{Π^∞} is time-independent, since Π^∞ is constant. Also, let us denote by $[\vartheta^s,\chi_1^s,\chi_2^s]$ the solution of $(\mathrm{SMA})^s$, and put

$$J^s(t):=\frac{c_0}{2L}|\vartheta^s(t)-\vartheta_{\Pi^s}(t)|^2_{L^2(\Omega)}+\frac{1}{2l}\sum_{i=1}^{2}|\chi_i^s(t)|^2_{H^1(\Omega)}\quad\text{for all } 0\le s\le+\infty \text{ and } 0\le t<+\infty.$$

Let S^∞ be a positive constant defined as

$$S^\infty:=(1+|f^\infty|^2_{L^2(\Omega)}+|\mathbf{h}^\infty|^2_{(L^2(\Omega))^3}+|\mathbf{g}^\infty|^2_{(L^2(\Gamma_1))^3}+|\Pi^\infty|^2_{L^2(\Gamma)})^{\frac12}.$$

Then, by a similar way to obtain (3.3), we find two positive constants ν_0^* and ν_1^*, independent of s, such that

$$\frac{d}{dt}J^s(t)+\nu_0^*J^s(t)\le\begin{cases}\nu_1^*|f^*(t)|^2, & \text{if } 0\le s<+\infty,\\ \nu_1^*(S^\infty)^2, & \text{if } s=+\infty,\end{cases}\quad\text{for a.e. } t>0.$$

So, applying conditions (a1), (2.27) and Gronwall's lemma to the above inequality, we also find a positive constant ν_2^*, independent of s, such that

$$|[\vartheta^s(t),\chi_1^s(t),\chi_2^s(t)]|_W^2 \le \nu_2^* e^{-\nu_0^* t}|[\vartheta_0,\chi_{1,0},\chi_{2,0}]|_W^2 + \nu_0^*\nu_1^*\nu_2^*(S^\infty)^2$$
$$\text{for any } 0 \le s \le +\infty \text{ and } 0 \le t < +\infty. \tag{3.9}$$

Let us put $B_0^* := \{[u,v_1,v_2] \in D \mid |[u,v_1,v_2]|_W \le 1 + S^\infty\sqrt{\nu_0^*\nu_1^*\nu_2^*}\,\}$. Then, it is not so difficult to check that the set given by

$$B^* := \overline{\mathrm{conv}}\Big(\bigcup_{s\ge 0} E(s+1,s)B_0^* \cup S(1)B_0^*\Big)$$

is, as required, to be a compact set, where $\overline{\mathrm{conv}}\,(\,\cdot\,)$ is the closed convex hull of any subset in W.

Proof of Theorem 2.3(i) By Lemma 3.1, the construction of the global attractor $\mathcal{A}_\infty$ for $\{S(t)\}$ is a direct application of the general theory in [18, 26]. Accordingly, the global attractor $\mathcal{A}_\infty$ is given as the ω-limit set of the absorbing set B^*, namely,

$$\mathcal{A}_\infty := \omega_S(B^*) = \bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau} S(t)B^*}^{W}.$$

In the rest, let B be any bounded subset in D, let ε be any positive number, and let T be any finite time. Then, by (2.27) and Proposition 3.1, there exists a positive number $t_0 = t_0(B,\varepsilon,T) \ge \frac{1}{2\varepsilon}$, depending only on B and ε, such that

$$\sup_{\tau\in[\frac{T}{2},T]}|E(\tau+t+s,s)z - S(\tau)\circ E(t+s,s)z|_W \le \frac{\varepsilon}{2}$$
$$\text{for any } s\ge 0,\ z\in B \text{ and } t \ge t_0(B,\varepsilon,T). \tag{3.10}$$

Let $\mathcal{A}_\infty$ be the global attractor for $\{S(t)\}$. Then, by the attractiveness of $\mathcal{A}_\infty$, we find a finite time $\tau_0 = \tau_0(B,\varepsilon) \ge \frac{1}{2\varepsilon}$, depending only on B and ε, such that

$$\mathrm{dist}_W(S(\tau)\circ E(t+s,s)B,\mathcal{A}_\infty) \le \frac{\varepsilon}{2} \quad \text{for any } s\ge 0,\ t\ge 0 \text{ and } \tau\ge\tau_0.$$

So, by (3.10), we have

$$\sup_{\tau\in[\frac{\tau_0}{2},\tau_0]}|E(\tau+t+s,s)z - S(\tau)\circ E(t+s,s)z|_W \le \frac{\varepsilon}{2}$$
$$\text{for any } s\ge 0,\ z\in B \text{ and } t\ge t_0(B,\varepsilon,\tau_0).$$

Thus, putting $t_1(B,\varepsilon) := t_0(B,\varepsilon,\tau_0) = t_0(B,\varepsilon,\tau_0(B,\varepsilon))$, we obtain that

$$\begin{aligned}
&\mathrm{dist}_W(E(\tau_0+t+s,s)z,\mathcal{A}_\infty)\\
&\le |E(\tau_0+t+s,s)z - S(\tau_0)\circ E(t+s,s)z|_W + \mathrm{dist}_W(S(\tau_0)\circ E(t+s,s)z,\mathcal{A}_\infty)\\
&\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \quad \text{for all } s\ge 0,\ z\in B \text{ and } t\ge t_1(B,\varepsilon).
\end{aligned} \tag{3.11}$$

Proof of Theorem 2.3(ii) Let us assume that $z_\infty \in \omega_E(B)$. Then, we find sequences $\{t_n\}$, $\{s_n\}\subset[0,+\infty)$ and $\{z_n\}\subset B$ such that $t_n \ge \tau_0\big(B,\frac{1}{n}\big) + t_1\big(B,\frac{1}{n}\big) \ge n$ for any $n\in\mathbb{N}$, and $E(t_n+s_n,s_n)z_n \to z_\infty$ in W as $n\to+\infty$. Here, by (3.11) it turns out that

$$\mathrm{dist}_W(E(t_n+s_n,s_n)z_n,\mathcal{A}_\infty) \le \frac{1}{n} \quad \text{for any } n\in\mathbb{N},$$

whence

$$\begin{aligned}\mathrm{dist}_W(z_\infty, \mathcal{A}_\infty) &\le |E(t_n+s_n, s_n)z_n - z_\infty|_W + \mathrm{dist}_W(E(t_n+s_n,s_n)z_n, \mathcal{A}_\infty)\\ &\le |E(t_n+s_n, s_n)z_n - z_\infty|_W + \frac{1}{n} \to 0 \quad \text{as } n \to +\infty.\end{aligned}$$

It implies that $z_\infty \in \mathcal{A}_\infty$, since $\mathcal{A}_\infty$ is compact in W.

Proof of Theorem 2.3(iii) We shall show that putting $B_E := B^*$ gives a set with the required property, namely $\omega_E(B^*) = \mathcal{A}_\infty$. By the assertion of Theorem 2.3(ii), we immediately have $\omega_E(B^*) \subset \mathcal{A}_\infty$. So, it is enough to show the converse inclusion. By Lemma 3.1, we find a finite time $t_{B^*} \ge 0$, depending only on B^*, such that

$$E(t+s, s)B^* \subset B^* \quad \text{for any } s \ge 0 \text{ and } t \ge t_{B^*}. \tag{3.12}$$

Let us assume that $z_\infty \in \mathcal{A}_\infty = \omega_S(B^*)$, equivalently there are sequences $\{t_n\} \subset (0, +\infty)$, $\{z_n\} \subset B^*$ such that

$$t_n \ge t_{B^*} + n \quad \text{for any } n \in \mathbb{N}, \quad \text{and} \quad S(t_n)z_n \to z_\infty \quad \text{in } W \text{ as } n \to +\infty.$$

Now, it follows from Proposition 3.1 that for any $n \in \mathbb{N}$,

$$\sup_{\tau \in [t_n - t_{B^*}, t_n]} |E(\tau+s, s)z_n - S(\tau)z_n|_W \to 0 \quad \text{as } s \to +\infty.$$

So, for any $n \in \mathbb{N}$, we find $s_n \ge n$ such that

$$|E(t_n+s_n, s_n)z_n - S(t_n)z_n|_W \le \frac{1}{n}. \tag{3.13}$$

On account of (3.12) and (3.13), we infer $E(t_{B^*}+s_n, s_n)z_n \in B^*$ for any $n \in \mathbb{N}$, and

$$E(t_n+s_n, t_{B^*}+s_n) \circ E(t_{B^*}+s_n, s_n)z_n = E(t_n+s_n, s_n)z_n \to z_\infty \quad \text{in } W \text{ as } n \to +\infty,$$

which implies $z_\infty \in \omega_E(B^*)$.

References

[1] Bonetti, E., Global solution to a Frémond model for shape memory alloys with thermal memory, *Nonlinear Anal.*, **46**, 2001, 535–565.

[2] Bonetti, E., Global solvability of a dissipative Frémond model for shape memory alloys. I. Mathematical formulation and uniqueness, *Quart. Appl. Math.*, **61**, 2003, 759–781.

[3] Bonetti, E., Global solvability of a dissipative Frémond model for shape memory alloys. II. Existence, *Quart. Appl. Math.*, **62**, 2004, 53–76.

[4] Brezis, H., Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, North Holland Math. Studies, Vol. 5, North-Holland, Amsterdam, 1973.

[5] Brézis, H., Crandall, M. G. and Pazy, A., Perturbations of nonlinear maximal monotone sets in Banach space, *Comm. Pure Appl. Math.*, **23**, 1970, 123–144.

[6] Chepyzhov, V. V. and Vishik, M. I., Attractors of non-autonomous dynamical systems and their dimension, *J. Math. Pures Appl.* (9), **73**, 1994, 279–333.

[7] Chepyzhov, V. V. and Vishik, M. I., Attractors for equations of mathematical physics, Amer. Math. Soc. Colloq. Publ., Vol. 49, A. M. S., Providence, RI, 2002.

[8] Colli, P., An evolution problem related to shape memory alloys, Mathematical Models for Phase Change Problems, J.-F. Rodrigues (ed.), Internat. Ser. Numer. Math., Vol. 88, Birkhäuser, Basel, 1989, 75–88.

[9] Colli, P., Global existence for the three-dimensional Frémond model of shape memory alloys, *Nonlinear Anal.*, **24**, 1995, 1565–1579.

[10] Colli, P., Frémond, M. and Visintin, A., Thermo-mechanical evolution of shape memory alloys, *Quart. Appl. Math.*, **48**, 1990, 31–47.

[11] Colli, P., Laurençot, Ph. and Stefanelli, U., Long-time behavior for the full one-dimensional Frémond model for shape memory alloys, *Contin. Mech. Thermodyn.*, **12**, 2000, 423–433.

[12] Colli, P. and Shirakawa, K., Attractors for the one-dimensional Frémond model of shape memory alloys, *Asymptot. Anal.*, **40**, 2004, 109–135.

[13] Colli, P. and Sprekels, J., Positivity of temperature in the general Frémond model for shape memory alloys, *Contin. Mech. Thermodyn.*, **5**, 1993, 255–264.

[14] Duvaut, G. and Lions, J.-L., Inequalities in Mechanics and Physics, Springer-Verlag, Berlin-New York, 1976.

[15] Frémond, M., Shape memory alloys. A thermomechanical model, Free Boundary Problems: Theory and Applications, K. H. Hoffmann and J. Sprekels (eds.), Pitman Res. Notes Math. Ser., Vol. 185, Longman, London, 1990, 295–306.

[16] Frémond, M., Non-smooth Thermomechanics, Springer-Verlag, Berlin, 2002.

[17] Frémond, M. and Miyazaki, S., Shape Memory Alloys, CISM Courses and Lectures, Vol. 351, Springer, Vienna, 1996.

[18] Hale, J. K., Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs, Vol. 25, A. M. S., Providence, RI, 1988.

[19] Ito, A., Yamazaki, N. and Kenmochi, N., Attractors of nonlinear evolution systems generated by time-dependent subdifferentials in Hilbert spaces, Dynamical Systems and Differential Equations, W. Chen and S. Hu (eds.), Vol. I, Discrete Contin. Dynam. Systems, Added Volume I, 1998, 327–349.

[20] Kenmochi, N., Solvability of nonlinear evolution equations with time-dependent constraints and applications, *Bull. Fac. Ed., Chiba Univ.*, **30**, 1981, 1–87.

[21] Kenmochi, N. and Niezgódka, M., Evolution systems of nonlinear variational inequalities arising from phase change problems, *Nonlinear Anal.*, **22**, 1994, 1163–1180.

[22] Nečas, J., Les méthodes directes en théorie des équations elliptiques, Academia, Prague, 1967.

[23] Shirakawa, K., Ito, A., Yamazaki, N. and Kenmochi, N., Asymptotic stability for evolution equations governed by subdifferentials, Recent Developments in Domain Decomposition Methods and Flow Problems, H. Fujita, H. Koshigoe, M. Mori, et al (eds.), GAKUTO Intern. Ser. Math. Sci. Appl., Vol. 11, Gakkōtosho, Tokyo, 1998, 287–310.

[24] Stefanelli, U., Analysis of a variable time-step discretization of the three-dimensional Frémond model for shape memory alloys, *Math. Comp.*, **71**, 2002, 1431–1453.

[25] Temam, R., Problèmes mathématiques en plasticité, Methodes Math. Inform., Vol. 12, Gauthier-Villars, Montrouge, 1983.

[26] Temam, R., Infinite-dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, Vol. 68, Springer-Verlag, New York, 1988.