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The Double Ringel-Hall Algebras of Valued Quivers***

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Abstract This paper is devoted to the study of the structure of the double Ringel-Hall algebra $\mathcal{D}(\Lambda)$ for an infinite dimensional hereditary algebra Λ, which is given by a valued quiver Γ over a finite field, and also to the study of the relations of $\mathcal{D}(\Lambda)$-modules with representations of valued quiver Γ.

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1 Introduction

The Ringel-Hall algebra of a finite dimensional hereditary algebra Λ together with its torus algebra can be endowed with a Hopf algebra structure [7, 19]. The double composition algebra of the Ringel-Hall algebra is the quantized enveloping algebra of the corresponding Kac-Moody algebra [7, 15, 14]. In [17, 4], it was shown that the Drinfeld double $\mathcal{D}(\Lambda)$ of a Ringel-Hall algebra of any finite dimensional hereditary algebra Λ is the quantized enveloping algebra of a generalized Kac-Moody algebra.

In this paper we consider the situation for the infinite dimensional hereditary algebra Λ, which is the tensor algebra of a k-species $\mathcal{S}$ of a valued quiver Γ of any type. For the Ringel-Hall algebra of Λ, the double composition algebra $\mathcal{C}(\Lambda)$ is the quantized enveloping algebra of a generalized Kac-Moody algebra $\mathfrak{g}$ defined by a Borcherds-Cartan matrix A obtained from Γ. Also by decomposing the double Ringel-Hall algebra $\mathcal{D}(\Lambda)$, we can show that $\mathcal{D}(\Lambda)$ itself is also the quantized enveloping algebra of a (bigger) generalized Kac-Moody algebra. Moreover, we obtain the Weyl-Kac character formula for the irreducible highest weight module with dominant highest weight, and then we prove the Kac theorem for infinite dimensional hereditary algebras by applying the Ringel-Hall algebra approach (see [5]). And as a corollary, we get that for any indecomposable representation, there exists a nilpotent indecomposable representation such that they have the same dimension vector.

2 Preliminaries

2.1 Valued quiver

According to Dlab-Ringel [6], a valued graph Γ is a graph together with positive integer ε_i for each vertex i and a pair of nonnegative integers $(d_{ij}^{\rho}, d_{ji}^{\rho})$ for each edge ρ between i and j

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such that $d_{ij}^{\rho}\varepsilon_j = d_{ji}^{\rho}\varepsilon_i$. A valued quiver is an oriented valued graph. We do not exclude loops or multiple arrows in a valued quiver. We denoted by Γ_0 the set of vertices and by Γ_1 the set of arrows. We assume that the valued graph is connected.

Let $k = \mathbb{F}_q$ be a finite field of q elements, $\mathbb{F} = \overline{\mathbb{F}}_q$ be the fixed algebraic closure of $\mathbb{F}_q$.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a valued quiver, and let $\mathcal{S} = (F_i, ({}_iM_j^{\rho}, {}_jM_i^{\rho}))$ be a k-species of the valued quiver Γ. A k-representation $V = (V_i, {}_j\varphi_i^{\rho})$ of $\mathcal{S}$ consists of an F_i-vector space V_i for each $i \in \Gamma_0$ and an F_j-linear map

$$ {}_j\varphi_i^{\rho}: \quad V_i \otimes_{F_i} {}_iM_j^{\rho} \to V_j $$

for each $\rho : i \to j$ in Γ_1. $\underline{\dim}V = (\dim_{F_i} V_i)_{i\in\Gamma_0} \in \mathbb{N}^{\Gamma_0}$ is the dimension vector of V. Denote by rep-$\mathcal{S}$ the category of k-representations of $\mathcal{S}$. Let Λ be the tensor algebra of $\mathcal{S}$. Then the category rep-$\mathcal{S}$ is equivalent to the category mod-Λ of finite dimensional left Λ-modules. If Γ has loops or oriented cycles, the tensor algebra Λ is an infinite dimensional hereditary algebra [20].

Now we assume that the valued quiver $\Gamma = (\Gamma_0, \Gamma_1)$ is finite, and let $\mathcal{S}$ be a k-species of Γ. Given $V, W \in$ rep-$\mathcal{S}$, we define Euler form

$$ \langle \underline{\dim}V, \underline{\dim}W \rangle = \sum_{i\in\Gamma_0} \varepsilon_i a_i b_i - \sum_{\rho: i\to j} d_{ij}^{\rho}\varepsilon_j a_i b_j, $$

and symmetric Euler form

$$ (\underline{\dim}V, \underline{\dim}W) = \langle \underline{\dim}V, \underline{\dim}W \rangle + \langle \underline{\dim}W, \underline{\dim}V \rangle, $$

where $\underline{\dim}V = (a_1, a_2, \cdots)$, $\underline{\dim}W = (b_1, b_2, \cdots)$ are in $\mathbb{N}^{\Gamma_0}$. These two bilinear forms are well defined on $\mathbb{Z}^{\Gamma_0}$. By [16], it is known that

$$ \langle \underline{\dim}V, \underline{\dim}W \rangle = \dim_k \operatorname{Hom}_{\Lambda}(V, W) - \dim_k \operatorname{Ext}_{\Lambda}^1(V, W). $$

2.2 Borcherds-Cartan matrix and Borcherds datum

Let I be a finite index set. A real matrix $A = (a_{ij})_{i,j\in I}$ is called a Borcherds-Cartan matrix if it satisfies (1) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$, (2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$, (3) $a_{ij} = 0$ if and only if $a_{ji} = 0$ for all $i \neq j$. If there is a diagonal matrix $D = \operatorname{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DA is symmetric, then we say that A is symmetrizable. Let $I^{\mathrm{re}} = \{i \in I \mid a_{ii} = 2\}$ and $I^{\mathrm{im}} = \{i \in I \mid a_{ii} \leq 0\}$.

If a symmetric bilinear form $\bullet : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ and a set of positive integers $d = \{d_i\}_{i\in I}$ satisfy (1) $\frac{i\bullet i}{2d_i} \in \{1, 0, -1, -2, \cdots\}$ for all $i \in I$, (2) $\frac{i\bullet j}{d_i} \in \{0, -1, -2, \cdots\}$ for all $i \neq j$ in I, then we call $(I, \bullet, d)$ a Borcherds datum.

By [18], Any symmetrizable Borcherds-Cartan matrix A with integer entries and even diagonal entries is associated with a Borcherds datum, which is called the Borcherds datum of A. And there exists a k-species $\mathcal{S}$ of a valued quiver Γ such that the symmetric Euler form of $\mathcal{S}$ is the Borcherds datum of A. Moreover, we can get a Borcherds-Cartan matrix $A_{\Gamma} = (a_{ij})_{i,j\in\Gamma_0}$ from a valued quiver of any type. A_{Γ} is symmetrizable with $D = \operatorname{diag}(\varepsilon_i)$, and it has integer entries and even diagonal entries. So in this paper, we assume that Borcherds-Cartan matrix A is symmetrizable with integer entries and even diagonal entries.

In [18], by using the Frobenius morphism and σ-quiver theory (see [3]), we get the following results.

Theorem 2.1 (See [18]) *Let $M_\sigma(\alpha, q)$ (resp., $I_\sigma(\alpha, q)$) be the number of the isomorphic classes of the representations (resp., indecomposable representations) of valued quiver Γ of dimension vector α over the finite field $\mathbb{F}_q$. Then it is a polynomial in q with rational coefficients and is independent of the orientation of the valued quiver.*

2.3 Simple representations and nilpotent representations

For a valued quiver Γ without oriented cycles, the set $\{S(i) \mid i \in \Gamma_0\}$ is the complete set of simple representations which we call as the standard simple representations.

Under the correspondence between the valued quiver Γ and the Borcherds-Cartan matrix A, we set $\Gamma_0^{\mathrm{re}} = \{i \in \Gamma_0 \mid a_{ii} = 2\}$ and $\Gamma_0^{\mathrm{im}} = \{i \in \Gamma_0 \mid a_{ii} \leq 0\}$. Then $\{S(i) \mid i \in \Gamma_0^{\mathrm{re}}\}$ is the set of the standard real simple representations, and $\{S(i) \mid i \in \Gamma_0^{\mathrm{im}}\}$ is the set of the standard imaginary simple representations. Set $\underline{\dim} S(i) = e_i$ ($i \in \Gamma_0$). For a non-standard simple representation V, we have the following proposition.

Proposition 2.1 *Let $V = (V_i, \ {}_j\varphi_i^\rho)$ be a non-standard simple representation of the valued quiver Γ. Then the dimension vector $\underline{\dim} V = \alpha$ of V satisfies:*

(1) *The support* $\operatorname{supp}\alpha$ *of α is connected.*

(2) *Under the Euler form $\langle -, - \rangle$, we have $\langle \alpha, e_i \rangle \leq 0$ and $\langle e_i, \alpha \rangle \leq 0$ for all $i \in \Gamma_0$.*

Let $\Gamma = (\Gamma_0, \Gamma_1)$ and $\mathcal{S} = (F_i, \ {}_iM_j)$ be as above, and let F_{ij} be the extension field of k of degree $d_{ij}^\rho \varepsilon_j$. A k-representation $V = (V_i, \ \psi_{ij})$ of $\mathcal{S}$ is equivalent to a set of F_i-vector space V_i for each $i \in \Gamma_0$, together with F_{ij}-linear map

$$\psi_{ij} : \quad V_i \otimes_{F_i} {}_iM_j \to V_j \otimes_{F_j} {}_jM_i$$

for each arrow $\rho : i \to j$ in Γ_1.

A k-representation $V = (V_i, \ \psi_{ij})$ is called nilpotent if collection of linear maps

$$\{\psi_{ij} : V_i \otimes_{F_i} {}_iM_j \to V_j \otimes_{F_j} {}_jM_i \mid \text{ for all } \rho : i \to j \text{ in } \Gamma_1\}$$

satisfy the following conditions: for any $i_1 \in \Gamma_0$, there exists $r \in \mathbb{N}$ such that $\psi_{i_{r-1}i_r} \cdots \psi_{i_2i_3}\psi_{i_1i_2} = 0$ for any $i_2, \cdots, i_r \in \Gamma_0$.

Let Nrep-$\mathcal{S}$ be the full subcategory of rep-$\mathcal{S}$ containing all nilpotent representations of $\mathcal{S}$. Then this category is closed under taking extensions and finite number of direct sums. It is an Abelian category. The following is easy.

Lemma 2.1 *A k-representation $V = (V_i, \ \psi_{ij})$ is nilpotent if and only if its composition factors belong to the set $\{S(i) \mid i \in \Gamma_0^{\mathrm{re}}\}$.*

2.4 Borcherds class

Now let $C = (I, (-, -), d)$ be a Borcherds datum. We choose R to be a commutative integral domain of characteristic zero and choose v to be an invertible element of R.

Let $\mathcal{T}$ be the torus of C defined by the generators $\{K_\alpha \mid \alpha \in \mathbb{Z}[I]\}$. We say that a skew-Hopf pairing (A^+, A^-, φ) (see [19] for the details) belongs to the Borcherds datum C or (A^+, A^-, φ) is a member of the Borcherds class $\mathcal{L}(C)$ if the following conditions are satisfied:

(A^+1) $A^+ = \bigoplus\limits_{\nu \in \mathbb{N}[I]} A_\nu^+$ is an $\mathbb{N}[I]$-graded associated R-algebra generated by $x_i^+ \in A_i^+$ ($i \in I$) and by $A_0^+ = \mathcal{T}$, such that $K_\alpha x_i^+ = v^{(\alpha, i)} x_i^+ K_\alpha$ for all $i \in I, \ \alpha \in \mathbb{Z}[I]$.

(A$^-$1) $A^- = \bigoplus_{\nu\in\mathbb{N}[I]} A^-_\nu$ is an $\mathbb{N}[I]$-graded associated R-algebra generated by $x_i^- \in A_i^-$ $(i \in I)$ and by $A_0^- = \mathcal{T}$, such that $K_\alpha x_i^- = v^{-(\alpha,i)} x_i^- K_\alpha$ for all $i \in I,\ \alpha \in \mathbb{Z}[I]$.

(A2) $\Delta_{A^+}(x_i^+) = x_i^+ \otimes 1 + K_i \otimes x_i^+, \quad \Delta_{A^+}(K_\alpha) = K_\alpha \otimes K_\alpha.$
$\Delta_{A^-}(x_i^-) = x_i^- \otimes K_{-i} + 1 \otimes x_i^-, \quad \Delta_{A^-}(K_\alpha) = K_\alpha \otimes K_\alpha.$

(A3) $\varphi(x_i^+, x_j^-) = 0$ for $i \neq j$ in I. $\varphi(x_i^+, x_i^-) \neq 0$ for $i \in I$.
$\varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}, \quad \varphi(x_i^+, K_\alpha) = 0 = \varphi(K_\alpha, x_i^-)$ for $i \in I,\ \alpha, \beta \in \mathbb{Z}[I]$.

If restricted form $\varphi : \mathfrak{a}^+ \times \mathfrak{a}^- \to R$ is non-degenerate, then we say that (A^+, A^-, φ) is a restricted non-degenerate member of $\mathcal{L}(C)$, where $\mathfrak{a}^+$ (resp., $\mathfrak{a}^-$) is the subalgebra of A^+ (resp., A^-) generated by x_i^+ (resp., x_i^-) $(i \in I)$.

Two skew-Hopf pairings (A^+, A^-, φ) and (B^+, B^-, φ) in $\mathcal{L}(C)$ are said to be canonically isomorphic if there are Hopf algebra isomorphisms $f : A^+ \to B^+$ and $f : A^- \to B^-$ such that $f(x_i^\pm) = y_i^\pm$ for all $i \in I$ and f preserves $\mathcal{T} = A_0^\pm = B_0^\pm$ elementwise, where $x_i^\pm$ (resp., $y_i^\pm$) $(i \in I)$ are the generators of $A^\pm$ (resp., $B^\pm$).

Analogously to [7] (see also [19]), we have the following

Proposition 2.2 *Let $C = (I, (-,-), d)$ be a Borcherds datum. Then any two restricted non-degenerate skew-Hopf pairings in $\mathcal{L}(C)$ are canonically isomorphic.*

2.5 Quantum generalized Kac-Moody algebra

In this part, we assume that R is a field of characteristic 0, and v in R is not a root of unity.

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable Borcherds-Cartan matrix with integer entries and even diagonal entries with the symmetrizer $D = \mathrm{diag}\{s_i \mid i \in I\}$, and let $\mathfrak{g} = \mathfrak{g}(A)$ be the generalized Kac-Moody algebra associated with A generated by the elements h_i, d_i $(i \in I)$, e_i, f_i $(i \in I)$ with the relations as in [12, 13]. Then relative definitions, such as Cartan subalgebra $\mathfrak{h}$, $\mathbb{Z}$-lattice $P^\vee$, weight lattice P, simple reflection $\{r_i \mid i \in I^{\mathrm{re}}\}$ and Weyl group W etc., can be defined as in [12, 13].

The quantum generalized Kac-Moody algebra $U = U_v(\mathfrak{g})$ associated with a symmetrizable Borcherds-Cartan matrix A is an associative algebra with 1 over R generated by the elements e_i, f_i $(i \in I)$ and K_α $(\alpha \in \mathbb{Z}[I])$ with the defining relations:

(R1) $K_0 = 1, \quad K_\alpha K_\beta = K_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{Z}[I]$,

(R2) $K_\alpha e_i = v^{(\alpha,\alpha_i)} e_i K_\alpha, \quad K_\alpha f_i = v^{-(\alpha,\alpha_i)} f_i K_\alpha, \quad \alpha \in \mathbb{Z}[I],\ i \in I$,

(R3) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_{-i}}{v^{s_i} - v^{-s_i}}$,

(R4) $$\sum_{s+t=1-a_{ij}} (-1)^s e_i^{(s)} e_j e_i^{(t)} = 0 \quad \text{if } a_{ii} = 2,\ i \neq j,$$
$$\sum_{s+t=1-a_{ij}} (-1)^s f_i^{(s)} f_j f_i^{(t)} = 0 \quad \text{if } a_{ii} = 2,\ i \neq j,$$
where $e_i^{(n)} = e_i^n/[n]_i!$ and $f_i^{(n)} = f_i^n/[n]_i!$,

(R5) $e_i e_j - e_j e_i = 0 = f_i f_j - f_j f_i$ if $a_{ij} = 0$.

The algebra U has a Hopf algebra structure with comultiplication Δ, counit ε and antipode

S being given by

$$\begin{aligned}
&\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \varepsilon(K_\alpha) = 1, \quad S(K_\alpha) = K_{-\alpha},\\
&\Delta(e_i) = e_i \otimes 1 + K_i \otimes e_i, \quad \varepsilon(e_i) = 0, \quad S(e_i) = -K_i^{-1} e_i,\\
&\Delta(f_i) = f_i \otimes K_{-i} + 1 \otimes f_i, \quad \varepsilon(f_i) = 0, \quad S(f_i) = -f_i K_i, \quad S^{-1}(f_i) = -K_i f_i
\end{aligned}$$

for $\alpha \in \mathbb{Z}[I], i \in I$.

We denote by $U^{\geq 0}$ (resp., $U^{\leq 0}$) the subalgebra of U generated by K_α ($\alpha \in \mathbb{Z}[I]$) and e_i (resp., f_i) for $i \in I$. For $\beta \in Q_+$, set

$$U^\pm_{\pm\beta} = \{x \in U^\pm \mid K_\alpha x K_{-\alpha} = v^{\pm\beta(h)} x \text{ for all } \alpha \in \mathbb{Z}[I]\}.$$

By [13], there exists a bilinear form ψ on $U^{\geq 0} \times U^{\leq 0}$ which is given by $\psi(x, y) = \xi(y)(x)$ for $x \in U^{\geq 0}, y \in U^{\leq 0}$, where $\xi : U^{\leq 0} \to (U^{\geq 0})^*$ defined by $\xi(K^\alpha) = \phi_\alpha,\ \xi(f_i) = -\frac{1}{v^{s_i} - v^{-s_i}} \psi_i$ for $\alpha \in \mathbb{Z}[I],\ i \in I$ is an algebra homomorphism and the linear functionals $\phi_\alpha, \psi_i \in (U^{\geq 0})^*$ are given by

$$\begin{cases}
\phi_\alpha(xK_\beta) = \varepsilon(x) v^{-(\alpha,\beta)} & \text{for } x \in U^+,\ \beta \in \mathbb{Z}[I],\\
\psi_i(xK_\alpha) = 0 & \text{for } x \in U^+_\beta,\ \beta \in \mathbb{Z}[I] \setminus \{\alpha_i\},\\
\psi_i(e_i K_\alpha) = 1.
\end{cases}$$

And by the properties ψ (see [13]), we can show that $(U^{\geq 0}, U^{\leq 0}, \psi)$ is a restricted non-degenerate member of $\mathcal{L}(C_A)$ where $C_A = (I, (-,-), d)$ is the Borcherds datum of A.

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma U-module and $L(\lambda)$ be the corresponding irreducible quotient module. Let T denote the set of all imaginary simple roots α_i ($i \in I^{\text{im}}$). We have

Proposition 2.3 (See [1, 13]) *For $\lambda \in \mathfrak{h}^*$, if $(\lambda, \alpha_i) \geq 0$ for $i \in I$, and $(\lambda, \alpha_i) \in \mathbb{Z}$ for $i \in I^{\text{re}}$, then we have*

$$\operatorname{ch} M(\lambda) = \frac{e(\lambda)}{\prod\limits_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\dim \mathfrak{g}_\alpha}} = e(\lambda) \sum_{\beta \in Q_+} (\dim U^-_{-\beta}) e(-\beta),$$

$$\operatorname{ch} L(\lambda) = \frac{\sum\limits_{w \in W, F \subset T} (-1)^{\ell(w) + |F|} e(w(\lambda + \rho + s(F)) - \rho)}{\prod\limits_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\dim \mathfrak{g}_\alpha}},$$

where Δ_+ denotes the set of positive roots of $\mathfrak{g}$, $\mathfrak{g}_\alpha$ denotes the root space, and F runs over all finite subsets of T such that $(\lambda, \alpha_i) = 0$ for $\alpha_i \in F$ and $(\alpha_i, \alpha_j) = 0$ for $\alpha_i, \alpha_j \in F$ with $i \neq j$. $|F|$ denotes the number of elements in F and $s(F)$ denotes the sum of elements in F.

3 Double Ringel-Hall Algebra and Its Decomposition

In this part, we assume that $k = \mathbb{F}_q$ is a finite field of q elements, and set $v = \sqrt{q}$. Let Γ be any valued quiver, and $\mathcal{S}$ be a k-species of Γ with tensor algebra Λ.

3.1 Ringel-Hall algebra and its Drinfeld double

We denote by $\mathcal{P}$ the set of isomorphic classes of finite k-representations of $\mathcal{S}$, and denote by $I \subset \mathcal{P}$ the set of isomorphic classes of simple k-representations of $\mathcal{S}$. Set $\mathcal{P}_1 = \mathcal{P} \setminus \{0\}$. For each $\alpha \in \mathcal{P}$, let V_α be the representative in the isoclass α. And in particular, we denote by V_i the representative in the isoclass $i \in I$.

By Subsection 2.3, we have $\Gamma_0 \subseteq I$. Set $I_1 = I \setminus \Gamma_0$. The representative V_α in the isoclass α is a representation of $\mathcal{S}$ with $\underline{\dim} V_\alpha \in \mathbb{N}\Gamma_0$. For Euler form $\langle -, - \rangle$ and symmetric Euler form $(-,-)$, we define

$$\langle \alpha, \beta \rangle = \langle \underline{\dim} V_\alpha, \underline{\dim} V_\beta \rangle, \quad (\alpha, \beta) = (\underline{\dim} V_\alpha, \underline{\dim} V_\beta) \quad \text{for all } \alpha, \beta \in \mathcal{P}.$$

In particular, $\langle i, j \rangle = \langle \underline{\dim} V_i, \underline{\dim} V_j \rangle$ $(i, j \in I)$. Set $\underline{\dim} V_i = \alpha_i$ for $i \in I$ (in particular, $\alpha_i = e_i$ for $i \in \Gamma_0$), and set $\underline{\dim} V_\alpha = d_\alpha$ for $\alpha \in \mathcal{P} \setminus I$.

For any $\alpha, \beta, \lambda \in \mathcal{P}$, let $g_{\alpha\beta}^\lambda$ be the number of submodules M of V_λ such that $M \cong V_\beta$ and $V_\lambda / M \cong V_\alpha$. And more generally, if $\alpha_1, \cdots, \alpha_t, \lambda \in \mathcal{P}$, let $g_{\alpha_1 \cdots \alpha_t}^\lambda$ be the number of the filtrations $0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = V_\lambda$ such that $M_{i-1}/M_i \cong V_{\alpha_i}$ for all $1 \leq i \leq t$. For each $\lambda \in \mathcal{P}$, set $a_\lambda = |\operatorname{Aut}_k(V_\lambda)|$.

We recall the definition of the Ringel-Hall algebra of Λ and its Drinfeld double (see [4, 19]). Let R be a subfield of the real number field $\mathbb{R}$ containing v, and $\mathcal{H}^+(\Lambda)$ be an R-vector space with basis $\{K_\alpha u_\lambda^+ \mid \alpha \in \mathbb{Z}\Gamma_0, \lambda \in \mathcal{P}\}$. In the following sense, $\mathcal{H}^+(\Lambda)$ becomes a Hopf algebra:

(1) Multiplication (see [15]):

$$\begin{aligned} u_\alpha^+ u_\beta^+ &= \sum_{\lambda \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^\lambda u_\lambda^+ && \text{for all } \alpha, \beta \in \mathcal{P}, \\ K_\alpha u_\lambda^+ &= v^{(\alpha, \lambda)} u_\lambda^+ K_\alpha && \text{for all } \lambda \in \mathcal{P},\ \alpha \in \mathbb{Z}\Gamma_0, \\ K_\alpha K_\beta &= K_{\alpha+\beta} && \text{for all } \alpha, \beta \in \mathbb{Z}\Gamma_0 \end{aligned}$$

with unit $1 = u_0^+ = K_0$.

(2) Comultiplication (see [7]):

$$\begin{aligned} \Delta(u_\lambda^+) &= \sum_{\alpha, \beta \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^\lambda \frac{a_\alpha a_\beta}{a_\lambda} u_\alpha^+ K_{d_\beta} \otimes u_\beta^+ && \text{for all } \lambda \in \mathcal{P}, \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha && \text{for all } \alpha \in \mathbb{Z}\Gamma_0 \end{aligned}$$

with counit: $\varepsilon(u_\lambda^+) = 0$ for $0 \neq \lambda \in \mathcal{P}$, and $\varepsilon(K_\alpha) = 1$ for $\alpha \in \mathbb{Z}\Gamma_0$.

(3) Antipode (see [19]):

$$S(u_\lambda^+) = \delta_{\lambda 0} + \sum_{m \geq 1} (-1)^m \sum_{\substack{\pi \in \mathcal{P}, \\ \lambda_1, \cdots, \lambda_m \in \mathcal{P}_\infty}} v^{2 \sum_{i<j} \langle \lambda_i, \lambda_j \rangle} \frac{a_{\lambda_1} \cdots a_{\lambda_m}}{a_\lambda} g_{\lambda_1 \cdots \lambda_m}^\lambda g_{\lambda_1 \cdots \lambda_m}^\pi K_{-d_\lambda} u_\pi^+, \quad \lambda \in \mathcal{P},$$

$$S(K_\alpha) = K_{-\alpha} \quad \text{for all } \alpha \in \mathbb{Z}\Gamma_0.$$

We call Hopf algebra $\mathcal{H}^+(\Lambda)$ the (extended twisted) Ringel-Hall algebra of Λ. The subspace $\mathfrak{h}^+(\Lambda)$ of $\mathcal{H}^+(\Lambda)$ generated by $\{u_\lambda^+ \mid \lambda \in \mathcal{P}\}$ is an associative subalgebra. For simple representation V_i, $i \in I$, we have $\Delta(u_i^+) = u_i^+ \otimes 1 + K_{\alpha_i} \otimes u_i^+$, and $S(u_i^+) = -K_{-\alpha_i} u_i^+$. $\mathfrak{h}^+(\Lambda)$ and $\mathcal{H}^+(\Lambda)$ are all $\mathbb{N}\Gamma_0$-graded.

Dually, we can define a Hopf algebra $\mathcal{H}^-(\Lambda)$ and its subalgebra $\mathfrak{h}^-(\Lambda)$.

By [19], there exists a bilinear map $\varphi : \mathcal{H}^+(\Lambda) \times \mathcal{H}^-(\Lambda) \to R$, defined by

$$\varphi(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^+) = v^{-(\alpha, \alpha') - (\beta, \alpha') + (\alpha, \beta')} \frac{|V_\beta|}{a_\beta} \delta_{\beta\beta'}.$$

And in fact φ is a skew Hopf pairing. So there exists a Hopf algebra structure on $\mathcal{H}^+(\Lambda) \otimes \mathcal{H}^-(\Lambda)$. Therefore, we can define Drinfeld double of $(\mathcal{H}^+(\Lambda), \mathcal{H}^-(\Lambda), \varphi)$. The ideal generated by $\{K_\alpha \times K_{-\alpha} - 1 \mid \alpha \in \mathbb{Z}\Gamma_0\}$ is a Hopf ideal. The quotient by modular this Hopf ideal is a Hopf algebra, which is called the double Ringel-Hall algebra of Λ. We denote it by $\mathcal{D}(\Lambda)$. Let $\mathcal{T}$ be the torus algebra generated by $\{K_\alpha \mid \alpha \in \mathbb{Z}\Gamma_0\}$. Then we have triangular decomposition $\mathcal{D}(\Lambda) = \mathfrak{h}^-(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{h}^+(\Lambda)$.

The subalgebra $\mathcal{C}(\Lambda)$ of $\mathcal{D}(\Lambda)$ generated by $\{u_i^\pm, K_i^\pm \mid i \in \Gamma_0\}$ is called the double composition algebra of Λ. It is a Hopf subalgebra and admits a triangular decomposition $\mathcal{C}(\Lambda) = \mathfrak{c}^-(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{c}^+(\Lambda)$, where $\mathfrak{c}^+(\Lambda)$ (resp., $\mathfrak{c}^-(\Lambda)$) is the composition algebra, which is generated by $\{u_i^+ \mid i \in \Gamma_0\}$ (resp., $\{u_i^- \mid i \in \Gamma_0\}$), and is still $\mathbb{N}\Gamma_0$-graded.

Furthermore, $\mathcal{D}(\Lambda)$ admits an operator ω which is defined by

$$\begin{aligned} &\omega(u_\lambda^+) = u_\lambda^-, \quad \omega(u_\lambda^-) = u_\lambda^+ \quad \text{for } \lambda \in \mathcal{P}, \\ &\omega(K_\alpha) = -K_{-\alpha} \quad \text{for } \alpha \in \mathbb{Z}\Gamma_0 \end{aligned}$$

and $\varphi(x, y) = \varphi(\omega(y), \omega(x))$ for $x \in \mathfrak{h}^+(\Lambda),\ y \in \mathfrak{h}^-(\Lambda)$.

3.2 The structure of double Ringel-Hall algebra

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a valued quiver of any type and $\mathcal{S}$ be a k-species of Γ. Let A_Γ be the corresponding Borcherds Cartan matrix and let C_Γ be the Borcherds datum of A_Γ. Let Λ be the tensor algebra of $\mathcal{S}$. Following an idea of Sevenhant and Van den Bergh [17], we consider the structure of the double Ringel-Hall algebra $\mathcal{D}(\Lambda)$ (see also [4]).

Let $\mathcal{F}$ be the fundamental set of $\mathbb{N}\Gamma_0$, which by definition is the set

$$\{0 \neq \alpha \in \mathbb{N}\Gamma_0 \mid (\alpha, e_i) \leq 0 \text{ for all } i \in \Gamma_0, \text{ and } \operatorname{supp}\alpha \text{ is connected }\}.$$

Set

$$\mathcal{F}_0 = \mathcal{F} \setminus \Big(\bigcup_{i \in I \setminus \Gamma_0^{\mathrm{re}}} e_i \Big).$$

For $\alpha, \beta \in \mathbb{Z}\Gamma_0$, we define $\alpha \leq \beta$ by $\beta - \alpha \in \mathbb{N}\Gamma_0$.

For each $\theta \in \mathbb{N}\Gamma_0$, we define

$$\Xi_\theta^+ = \sum_{\substack{\mu+\nu=\theta \\ \mu \neq \theta \neq \nu}} \mathfrak{h}^+(\Lambda)_\mu \mathfrak{h}^+(\Lambda)_\nu \quad \text{and} \quad \Xi_\theta^- = \sum_{\substack{\mu+\nu=\theta \\ \mu \neq \theta \neq \nu}} \mathfrak{h}^-(\Lambda)_\mu \mathfrak{h}^-(\Lambda)_\nu.$$

It is easy to see $\Xi_\theta^- = \omega(\Xi_\theta^+)$.

For each $\theta \in \mathbb{N}\Gamma_0$, we also define

$$L_\theta^+ = \{x^+ \in \mathfrak{h}^+(\Lambda)_\theta \mid \varphi(x^+, \Xi_\theta^-) = 0\} \quad \text{and} \quad L_\theta^- = \{y^- \in \mathfrak{h}^-(\Lambda)_\theta \mid \varphi(\Xi_\theta^+, y^-) = 0\}.$$

So obviously, we have $L_\theta^- = \omega(L_\theta^+)$.

Lemma 3.1 (See [4]) (1) *The elements in each $L_\theta^\pm$ are primitive, that is for each $x^+ \in L_\theta^+$, we have $\Delta(x^+) = x^+ \otimes 1 + K_\theta \otimes x^+$ and $S(x^+) = -K_{-\theta}x^+$. For each $y^- \in L_\theta^-$, we have $\Delta(y^-) = y^- \otimes K_{-\theta} + 1 \otimes y^-$ and $S(y^-) = -y^- K_\theta$.*

(2) *For $x^+ \in L_\theta^+,\ y^- \in L_\theta^-$, we have*

$$x^+ y^- - y^- x^+ = -\varphi(x^+, y^-)(K_\theta - K_{-\theta}).$$

(3) *If $L_\theta^\pm \neq 0$, then $\theta \in \mathcal{F}_0$. And θ is the dimension vector of an indecomposable representation.*

For each $i \in I$, we have

$$u_i^+ u_j^- - u_j^- u_i^+ = -\varphi(u_i^+, u_i^-)(K_{\alpha_i} - K_{-\alpha_i})\delta_{ij} = -\frac{|V_i|(v-v^{-1})}{a_i}\frac{K_{\alpha_i} - K_{-\alpha_i}}{v-v^{-1}}\delta_{ij}, \tag{3.1}$$

where $a_i = |\operatorname{Aut} V_i|$. Set $\chi_i = -\frac{a_i}{|V_i|(v-v^{-1})}$ and set $E_i(0) = u_i^+,\ F_i(0) = \chi_i u_i^-$. Then

$$E_i(0)F_j(0) - F_j(0)E_i(0) = \frac{K_{\alpha_i} - K_{-\alpha_i}}{v-v^{-1}}\delta_{ij}.$$

In particular, if $i \in \Gamma_0$, then $\chi_i = -v^{-\varepsilon_i}$, and

$$E_i(0)F_j(0) - F_j(0)E_i(0) = \frac{K_i - K_{-i}}{v^{\varepsilon_i} - v^{-\varepsilon_i}}\delta_{ij}. \tag{3.2}$$

For $\theta \in \mathcal{F}_0$, let $\eta_\theta = \dim_R L_\theta^\pm$.

Lemma 3.2 (See [4]) *There exists an R-basis $\{E_p(\theta) \mid 1 \le p \le \eta_\theta\}$ of L_θ^+ and nonzero elements $\chi_{\theta,p} \in R$, such that*

$$\varphi(E_p(\theta), \chi_{\theta,q}\omega(E_q(\theta))) = \frac{-1}{v-v^{-1}}\delta_{pq}.$$

If set $F_p(\theta) = \chi_{\theta,p}\omega(E_p(\theta))$, then by the above lemmas, we have

$$E_p(\theta)F_q(\pi) - F_q(\pi)E_p(\theta) = \frac{K_\theta - K_{-\theta}}{v-v^{-1}}\delta_{pq}\delta_{\theta\pi} \tag{3.3}$$

for $\theta, \pi \in \mathcal{F}_0,\ 1 \le p \le \eta_\theta,\ 1 \le q \le \eta_\pi$.

Now for each subset $\mathcal{E} \subseteq \mathcal{F}_0$, we denote by $\mathfrak{d}_{\mathcal{E}}^\pm(\Lambda)$ the subalgebra of $\mathfrak{h}^\pm(\Lambda)$ generated by $\mathfrak{c}^\pm(\Lambda)$ and $L_\theta^\pm$ with $\theta \in \mathcal{E}$. Then $\mathfrak{d}_{\mathcal{E}}^\pm(\Lambda)$ is $\mathbb{N}\Gamma_0$-graded and the restriction of φ on $\mathfrak{d}_{\mathcal{E}}^+(\Lambda) \times \mathfrak{d}_{\mathcal{E}}^-(\Lambda)$ is non-degenerate. Set

$$\mathcal{D}_{\mathcal{E}}(\Lambda) = \mathfrak{d}_{\mathcal{E}}^-(\Lambda) \otimes \mathcal{T} \otimes \mathfrak{d}_{\mathcal{E}}^+(\Lambda) \quad \text{for } \mathcal{E} \subseteq \mathcal{F}_0.$$

In particular, we have

$$\mathcal{D}_\emptyset(\Lambda) = \mathcal{C}(\Lambda) \quad \text{and} \quad \mathcal{D}_{\mathcal{F}_0}(\Lambda) = \mathcal{D}(\Lambda).$$

So for any $\mathcal{E} \subseteq \mathcal{F}_0$, we get a family of subalgebras $\mathfrak{d}_{\mathcal{E}}^\pm(\Lambda)$ of $\mathfrak{h}^\pm(\Lambda)$ and a family of subalgebras $\mathcal{D}_{\mathcal{E}}(\Lambda)$ of $\mathcal{D}(\Lambda)$.

3.3 Uniqueness of skew-Hopf pairing

Now we extend Borcherds datum $C = (I, (-,-), d)$ as in [4]. Choose non-zero element $\delta_j \in \mathbb{N}[I]$ for each $j \in J$, where J is an index set. And assume that the set $\{j' \in J \mid \delta_{j'} = \delta_j\}$ for each $j \in J$ is finite. Denote by $\widetilde{C}$ the datum $(I, (-,-), d, \{\delta_j \mid j \in J\})$. In particular, we set $\delta_i = i$ for $i \in I$. We call $\widetilde{C}$ an extended Borcherds datum.

Analogously to Subsection 2.4, given an extended Borcherds datum $\widetilde{C} = (I, (-,-), d, \{\delta_j \mid j \in J\})$, a skew-Hopf paring (A^+, A^-, φ) is said to belong to $\widetilde{C}$ or (A^+, A^-, φ) is a member of the extended Borcherds datum $\mathcal{L}(\widetilde{C})$ if $A^\pm = \bigoplus_{\nu \in \mathbb{N}[I]} A_\nu^\pm$ is an $\mathbb{N}[I]$-graded associated R-algebra generated by $x_i^\pm \in A_i^\pm$ $(i \in I \cup J)$ and by $A_0^\pm = \mathcal{T}$, such that the similar relations ($\mathrm{A}^\pm 1$), (A2) and (A3) in Subsection 2.4 are satisfied.

And then we have the similar definitions of restricted non-degenerate member of $\mathcal{L}(\widetilde{C})$ and of canonically isomorphic for two skew-Hopf pairings in $\mathcal{L}(\widetilde{C})$.

Proposition 3.1 *Let $\widetilde{C} = (I, (-,-), d, \{\delta_j \mid j \in J\})$ be an extended Borcherds datum. Then any two restricted non-degenerated skew-Hopf pairings in $\mathcal{L}(\widetilde{C})$ are canonically isomorphic.*

Example 3.1 For each $\mathcal{E} \subseteq \mathcal{F}_0$, we set $J_{\mathcal{E}} = \{j = (\theta, p) \mid \theta \in \mathcal{E}, 1 \leq p \leq \eta_\theta\}$. If $j = (\theta, p) \in J_{\mathcal{E}}$, we set $\delta_j = \theta$. Set $\widetilde{C}_{\mathcal{E}} = (\Gamma_0, (-,-), d, \{\delta_j \mid j \in J_{\mathcal{E}}\})$. Then it is easy to see that $(\mathcal{D}^+_{\mathcal{E}}(\Lambda), \mathcal{D}^-_{\mathcal{E}}(\Lambda), \varphi)$ is a restricted non-degenerate member of $\mathcal{L}(\widetilde{C}_{\mathcal{E}})$. And in particular, for $J_\emptyset$ (resp., $J = J_{\mathcal{F}_0}$), $(\mathcal{C}^+(\Lambda), \mathcal{C}^-(\Lambda), \varphi)$ (resp., $(\mathcal{H}^+(\Lambda), \mathcal{H}^-(\Lambda), \varphi)$) is a restricted non-degenerate member of $\mathcal{L}(\widetilde{C}_\emptyset) = \mathcal{L}(C_\Gamma)$ (resp., $\mathcal{L}(\widetilde{C}_{\mathcal{F}_0})$), where $\widetilde{C}_{\mathcal{F}_0} = (\Gamma_0, (-,-), d, \{\delta_j \mid j \in J\})$.

For a valued quiver Γ of any type, let Γ' be the valued quiver obtained from Γ by choosing another orientation. Let $\mathcal{S}'$ be a k-species of Γ' with tensor algebra Λ'. Then $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing ${}_iM^\rho_j$ by its k-dual whenever the orientation of $\rho : i - j$ in Γ' is different of it in Γ. Let $\mathcal{H}^\pm(\Lambda')$ be the Ringel-Hall algebra of Λ'. Then according to the fact that the number of the isoclasses of indecomposable representations of fixed dimension vector is independent of the orientation of Γ, analogously to [4], we have

Theorem 3.1 *For any subset $\mathcal{E} \subseteq \mathcal{F}_0$, there exists Hopf algebra isomorphism: $\Phi_{\mathcal{E}}$: $\mathcal{D}_{\mathcal{E}}(\Lambda) \to \mathcal{D}_{\mathcal{E}}(\Lambda')$ such that for $\mathcal{E} \subseteq \mathcal{G} \subseteq \mathcal{F}_0$, we have the following commutative diagram*

$$\begin{array}{ccccc}
\mathcal{D}_{\mathcal{E}}(\Lambda) & \subseteq & \mathcal{D}_{\mathcal{G}}(\Lambda) & \subseteq & \mathcal{D}_{\mathcal{F}_0}(\Lambda) \\
{\scriptstyle \Phi_m}\downarrow & & {\scriptstyle \Phi_{m+1}}\downarrow & & {\scriptstyle \Phi}\downarrow \\
\mathcal{D}_{\mathcal{E}}(\Lambda') & \subseteq & \mathcal{D}_{\mathcal{G}}(\Lambda') & \subseteq & \mathcal{D}_{\mathcal{F}_0}(\Lambda')
\end{array}$$

and in particular, Ringel-Hall algebra $\mathcal{H}(\Lambda)$ and $\mathcal{H}(\Lambda')$ are canonically isomorphic.

Remark 3.1 Ringel-Hall algebra $\mathcal{H}(\Lambda)$ is independent of the orientation of the valued quiver Γ.

3.4 Generic composition algebra

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be any valued quiver with positive integers $\{\varepsilon_i\}$ and a pair of nonnegative integers $(d^\rho_{ij}, d^\rho_{ji})$. Let $\mathcal{S}$ be a k-species of Γ. Assume that $C = (\Gamma_0, (-,-), d)$ is the Borcherds datum of Γ.

Let $\mathcal{K}$ be a set of finite field k, such that the set $\{|k| \mid k \in \mathcal{K}\}$ is infinite. Let R be a subfield of real number field $\mathbb{R}$ containing, for each $k \in \mathcal{K}$, an element v_k such that $v_k^2 = |k|$. For each finite field $k \in \mathcal{K}$, we have the (extended twisted) Ringel-Hall algebra $\mathcal{H}^+(\Lambda_k)$, where Λ_k is the tensor algebra of k-species $\mathcal{S}_k$ which is associated with $\mathcal{S}$, and then the composition algebra $\mathcal{C}^+(\Lambda_k)$ which is the R-algebra generated by the elements $u^+_i(k)$ $(i \in \Gamma_0)$ and $K_\alpha = K_\alpha(k)$ $(\alpha \in \mathbb{Z}\Gamma_0)$. Consider the direct product

$$\mathcal{H}^+(\widetilde{\Lambda}) = \prod_{k \in \mathcal{K}} \mathcal{H}^+(\Lambda_k).$$

v, v^{-1} and $\tilde{u}^+_i$ are elements of $\mathcal{H}^+(\widetilde{\Lambda})$ whose k-components are v_k, v_k^{-1} and $u^+_i(k)$ respectively. Denote by $\mathcal{C}^+(\widetilde{\Lambda})$ the subalgebra of $\mathcal{H}^+(\widetilde{\Lambda})$ generated by $\mathbb{Q}, v, v^{-1}$ and $\tilde{u}^+_i$ and $\widetilde{K}_\alpha = (K_\alpha)$. Since v is central in $\mathcal{C}^+(\widetilde{\Lambda})$ and there is no $p(T) \in \mathbb{Q}(T)$ such that $p(v) = 0$ unless $p(T) = 0$, we may regard $\mathcal{C}^+(\widetilde{\Lambda})$ as the $\mathbb{Q}[v, v^{-1}]$-algebra generated by $\tilde{u}^+_i$ $(i \in \Gamma_0)$ and $\widetilde{K}_\alpha$ $(\alpha \in \mathbb{Z}\Gamma_0)$. Denote by $\mathfrak{c}^+(\widetilde{\Lambda})$ the $\mathbb{Q}[v, v^{-1}]$-subalgebra of $\mathcal{C}^+(\widetilde{\Lambda})$ generated by $\tilde{u}^+_i$ $(i \in \Gamma_0)$. Define $\mathbb{Q}(v)$-algebra

$$\mathcal{C}^{*+}(\widetilde{\Lambda}) = \mathbb{Q}(v) \otimes_{\mathbb{Q}[v, v^{-1}]} \mathcal{C}^+(\widetilde{\Lambda})$$

with $u_i^{*+} = 1 \otimes \tilde{u}_i^+ \in \mathcal{C}^{*+}(\widetilde{\Lambda})$, called the generic composition algebra of the Borcherds datum C.

$$\mathfrak{c}^{*+}(\widetilde{\Lambda}) = \mathbb{Q}(v) \otimes_{\mathbb{Q}[v,v^{-1}]} \mathfrak{c}^{+}(\widetilde{\Lambda})$$

is a subalgebra of $\mathcal{C}^{*+}(\widetilde{\Lambda})$ generated by u_i^{*+} $(i \in \Gamma_0)$.

Dually we can construct $\mathcal{H}^-(\widetilde{\Lambda})$, $\mathcal{C}^-(\widetilde{\Lambda})$, $\mathfrak{c}^-(\widetilde{\Lambda})$, $\mathcal{C}^{*-}(\widetilde{\Lambda})$ and $\mathfrak{c}^{*-}(\widetilde{\Lambda})$.

The skew-Hopf pairing $\varphi_k : \mathcal{H}^+(\Lambda_k) \times \mathcal{H}^-(\Lambda_k) \to R_k$, for each $k \in \mathcal{K}$, induces an R-linear map

$$\tilde{\varphi}: \quad \mathcal{H}^+(\widetilde{\Lambda}) \times \mathcal{H}^-(\widetilde{\Lambda}) \to \prod_{k \in \mathcal{K}} R_k$$

which is given by $\tilde{\varphi}(x,y) = (\varphi_k(x_k, y_k))_{k \in \mathcal{K}}$ for $x = (x_k)_{k\in\mathcal{K}} \in \mathcal{H}^+(\widetilde{\Lambda})$ and $y = (y_k)_{k\in\mathcal{K}} \in \mathcal{H}^-(\widetilde{\Lambda})$, where $R_k = R$ for all $k \in \mathcal{K}$. And we have

$$\tilde{\varphi}(\tilde{u}_i^+, \tilde{u}_i^-) = (\varphi_k(u_i^+(k), u_i^-(k)))_{k\in\mathcal{K}} = \Big(\frac{v_k^{2\varepsilon_i}}{v_k^{2\varepsilon_i - 1}}\Big)_{k\in\mathcal{K}}.$$

For $x = (x_k)_{k\in\mathcal{K}} \in \mathcal{C}^+(\widetilde{\Lambda})_\mu$ and $y = (y_k)_{k\in\mathcal{K}} \in \mathcal{C}^-(\widetilde{\Lambda})_\mu$ where $\mu = \sum_{i\in\Gamma_0} \mu_i i$, there exists $M_{x,y}(v) \in \mathbb{Q}[v, v^{-1}]$ and positive integer $n(x,y)$, such that $\tilde{\varphi}(x,y) = M_{x,y}(v) \prod_{i\in\Gamma_0} \tilde{\varphi}(\tilde{u}_i^+, \tilde{u}_i^-)^{\mu_i}$ and $(v^{2\varepsilon_i} - 1)^{n(x,y)} \tilde{\varphi}(x,y) \in \mathbb{Q}[v, v^{-1}]$.

Hence, $\tilde{\varphi}$ induces a skew-Hopf pairing $\varphi : \mathcal{C}^{*+}(\widetilde{\Lambda}) \times \mathcal{C}^{*-}(\widetilde{\Lambda}) \to \mathbb{Q}(v)$ which is a member of $\mathcal{L}(C)$ over $\mathbb{Q}(v)$, and then we can get the reduced Drinfeld double $\mathcal{C}^*(\widetilde{\Lambda})$, called the double generic composition algebra. It admits a decomposition $\mathcal{C}^*(\widetilde{\Lambda}) = \mathfrak{c}^{*-}(\widetilde{\Lambda}) \otimes \mathcal{T} \otimes \mathfrak{c}^{*+}(\widetilde{\Lambda})$.

And for $m, n, i, j \in \Gamma_0$, we have

$$\begin{aligned}\varphi(\widetilde{K}_m u_i^{*+}, \widetilde{K}_n u_j^{*-}) &= (\varphi_k(K_m u_i^+(k), K_n u_j^+(k))_{k\in\mathcal{K}}) \\ &= \Big(v_k^{-(e_m,e_n)-(e_i,e_n)+(e_m,e_i)} \frac{v_k^{2\varepsilon_i}}{v_k^{2\varepsilon_i} - 1} \delta_{ij}\Big)_{k\in\mathcal{K}} \\ &= v^{-(e_m,e_n)-(e_i,e_n)+(e_m,e_i)} \frac{v^{2\varepsilon_i}}{v^{2\varepsilon_i} - 1} \delta_{ij} 1.\end{aligned}$$

In particular for $\mathbb{F} = \mathbb{Q}$, we have the quantum generalized Kac-Moody algebra $U = U_v(\mathfrak{g})$ over $\mathbb{Q}(v)$. For bilinear form $\psi : U^{\geq 0} \times U^{\leq 0} \to \mathbb{Q}(v)$, by formula (3.1), we have

$$\begin{aligned}\psi(K_m e_i, K_n(-v^{d_i})f_j) &= (-v^{\varepsilon_i})(-v^{-(e_m,e_n)-(e_i,e_n)+(e_m,e_i)}) \frac{1}{v^{\varepsilon_i} - v^{-\varepsilon_i}} \delta_{ij} \\ &= v^{-(e_m,e_n)-(e_i,e_n)+(e_m,e_i)} \frac{v^{2\varepsilon_i}}{v^{2\varepsilon_i} - 1} \delta_{ij}.\end{aligned}$$

So we have

Theorem 3.2 *Let Γ be any valued quiver and $\mathcal{S}$ be a k-species of Γ. $C_\Gamma = (\Gamma_0, (-,-), d)$ is the Borcherds datum of Γ and A is the Borcherds-Cartan matrix of Γ. Let $U = U_v(\mathfrak{g})$ be the quantum generalized Kac-Moody algebra of A over $\mathbb{Q}(v)$, and $\mathcal{C}^*(\widetilde{\Lambda})$ be the double generic composition algebra associated with C_Γ. Then the correspondence $u_i^{*+} \mapsto e_i$, $u_i^{*-} \mapsto -v^{\varepsilon_i} f_i$, $\widetilde{K}_i \mapsto K_i$ $(i \in \Gamma_0)$ induces a Hopf algebra isomorphism $\mathcal{C}^*(\widetilde{\Lambda}) \to U$.*

Remark 3.2 The theorem still holds if we give up the condition: "generic" and take v to be the square root of $|k|$ (see [14]).

3.5 Drinfeld double $\mathcal{D}'(\Lambda)$

Let $C = (\Gamma_0, (-,-), d)$ be the Borcherds datum of a valued quiver Γ (or a Borcherds-matrix), and $\widetilde{C} = (\Gamma_0, (-,-), d, \{\delta_j \mid j \in J\})$ be the extended Borcherds datum, where $J = J_{\mathcal{F}_0} = \{(\theta, p) \mid \theta \in \mathcal{F}_0, 1 \le p \le \eta_\theta\}$ and $\delta_{(\theta,p)} = \theta$ for $\theta \in \mathcal{F}_0$ and for all $1 \le p \le \eta_\theta$. We define $\widetilde{C}' = (\Gamma_0 \cup J, (-,-)', d')$, where $(i,j)' = (\delta_i, \delta_j)$ for all $i, j \in \Gamma_0 \cup J$ and $d' = (d_i)_{i \in \Gamma_0 \cup J}$ with $d'_i = d_i$ for $i \in \Gamma_0$ and $d'_i = 1$ for $i \in J$. Then we have the reduced Drinfeld double $\mathcal{D}'(\Lambda)$ of the restricted non-degenerate member of $\mathcal{L}(\widetilde{C}')$.

We extend the torus $\mathcal{T}$ of $C = (\Gamma_0, (-,-), d)$ to the torus $\mathcal{T}'$ of $\widetilde{C}' = (\Gamma_0 \cup J, (-,-)', d')$ and view $\mathcal{D}(\Lambda)$ as a $\mathbb{Z}[\Gamma_0 \cup J]$-graded Hopf algebra. Set $x_i = E_i(0),\ y_i = F_i(0)$ for $i \in \Gamma_0$, and set $x_j = E_p(\theta),\ y_j = F_p(\theta)$ for $j = (\theta, p) \in J$. Thus $\mathcal{D}'(\Lambda)$ admits a triangular decomposition

$$\mathcal{D}'(\Lambda) = \mathfrak{h}'^{-}(\Lambda) \otimes \mathcal{T}' \otimes \mathfrak{h}'^{+}(\Lambda),$$

where $\mathfrak{h}'^{+}(\Lambda)$ (resp., $\mathfrak{h}'^{-}(\Lambda)$) is generated by x_i (resp., y_i) for $i \in \Gamma_0 \cup J$. We also set $\mathcal{H}'^{+}(\Lambda) = \mathcal{T}' \otimes \mathfrak{h}'^{+}(\Lambda)$ and $\mathcal{H}'^{-}(\Lambda) = \mathcal{T}' \otimes \mathfrak{h}'^{-}(\Lambda)$. They are all naturally $\mathbb{N}[\Gamma_0 \cup J]$-graded.

Let $\varphi' : \mathcal{H}'^{+}(\Lambda) \times \mathcal{H}'^{-}(\Lambda) \to R$ be the restricted paring induced by φ, which satisfies that $\varphi'(x_i, y_j) = \delta_{ij}$ for all $i \in \Gamma_0 \cup J$.

Moreover, it is easy to see that there exists a Hopf algebra epimorphism $p : \mathcal{D}'(\Lambda) \to \mathcal{D}(\Lambda)$ such that $p(x_i) = x_i, p(y_i) = y_i$ and $p(K_i) = K_{\delta_i}$ for $i \in \Gamma_0 \cup J$.

For the Borcherds datum $\widetilde{C}' = (\Gamma_0 \cup J, (-,-)', d')$, there exists a Borcherds-Cartan matrix $A' = (a'_{ij})_{i,j \in \Gamma_0 \cup J}$ associated to $\widetilde{C}'$, such that $d'_i a'_{ij} = (\delta_i, \delta_j) = (i,j)'$ for all $i, j \in \Gamma_0 \cup J$, that is

$$a'_{ij} = \begin{cases} \frac{(\delta_i, \delta_j)}{d_i} & \text{for } i \in \Gamma_0, \\ (\delta_i, \delta_j) & \text{for } i \in J. \end{cases}$$

And A' is symmetrizable, with $D = \mathrm{diag}(d'_i \mid i \in \Gamma_0 \cup J)$. So we can define the generalized Kac-Moody algebra $\mathfrak{g}' = \mathfrak{g}(A')$ and its quantization $U' = U_v(\mathfrak{g}')$ as in Subsection 2.5. Let Δ' be the root system of $\mathfrak{g}'$. The symmetric bilinear form on the Cartan subalgebra $\mathfrak{h}'$of $\mathfrak{g}'$ is exactly the bilinear form $(-,-)'$ in the Borcherds datum $\widetilde{C}' = (\Gamma_0 \cup J, (-,-)', d')$.

Let W' be the Weyl group of $\mathfrak{g}'$ generated by the reflections $\{r'_i \mid i \in \Gamma_0^{\mathrm{re}}\}$ defined by $r'_i(\lambda) = \lambda - \frac{(\lambda, i)'}{d'_i} i$ for $\lambda \in \mathbb{Z}[\Gamma_0 \cup J]$.

Define a linear map $\delta : \mathbb{Z}[\Gamma_0 \cup J] \to \mathbb{Z}\Gamma_0$ by $\delta(i) = \delta_i$ for $i \in \Gamma_0 \cup J$. Then we have $\delta r'_i = r_i \delta$ for $r_i \in W$, the Weyl group of $\mathfrak{g} = \mathfrak{g}(A)$. It is easy to see that $r'_i \mapsto r_i, i \in \Gamma_0^{\mathrm{re}}$ induces a group isomorphism $W' \cong W$. So we have

Lemma 3.3 $\delta(\Delta') = \Delta$.

4 Representation Theory and Complete Reducibility

4.1 Category $\mathcal{O}$ and category $\widetilde{\mathcal{O}}$

Let Λ be the hereditary algebra defined as in the above section, and $\mathcal{D}(\Lambda)$ be the double Ringel-Hall algebra of Λ. Let X be the weight lattice of $C = (\Gamma_0, (-,-), d)$, i.e., $X = \{\lambda \in \mathbb{Z}\Gamma_0 \mid (\lambda, i) \in \mathbb{Z} \text{ for all } i \in \Gamma_0\}$. A $\mathcal{D}(\Lambda)$-module M is called a weight module if M admits a weight space decomposition $M = \bigoplus_{\lambda \in X} M_\lambda$, where $M_\lambda := \{m \in M \mid K_\alpha m = v^{(\alpha,\lambda)} m \text{ for all } \alpha \in \mathbb{Z}\Gamma_0\}$. We call $\mathrm{wt}(M) := \{\lambda \in X \mid M_\lambda \ne 0\}$ the set of *weights* of M.

For $\alpha = \sum k_i i \in \mathbb{Z}\Gamma_0$, set $\mathrm{tr}\,\alpha = \sum k_i$.

We denote by $\mathcal{O}$ the category consisting of weight modules M which satisfy: (1) every weight space is finite dimensional, (2) for every $x \in M$, there exists an $n_0 \geq 0$ such that $\mathfrak{h}^+(\Lambda)_\alpha x = 0$ for $\alpha \in \mathbb{Z}\Gamma_0$ whenever $\operatorname{tr}\alpha \geq n_0$.

A weight $\mathcal{D}(\Lambda)$-module V is called a highest weight module with *highest weight* $\lambda \in X$ if there exists a nonzero vector $v_\lambda \in V$ (called a highest weight vector), such that (1) $u_i^+ v_\lambda = 0$ for all $i \in \Gamma_0$, (2) $\mathcal{D}(\Lambda)v_\lambda = V$.

For $\lambda \in X$, denote by $J(\lambda)$ the left ideal of $\mathcal{D}(\Lambda)$ generated by $E_p(\theta)$ $(j = (\theta, p) \in \Gamma_0 \cup J)$ and $K_\alpha - v^{(\delta(\alpha),\lambda)}1$ $(\alpha \in \mathbb{Z}[\Gamma_0 \cup J])$, and set $V(\lambda) = \mathcal{D}(\Lambda)/J(\lambda)$. Then $V(\lambda)$ admits a left $\mathcal{D}(\Lambda)$-module structure under left multiplication. $V(\lambda)$ is called the Verma module. Then $L(\lambda) := V(\lambda)/N(\lambda)$ is an irreducible highest weight $\mathcal{D}(\Lambda)$-module with highest weight λ, where $N(\lambda)$ is the unique maximal submodule of $V(\lambda)$.

Set $X^+ = \{\lambda \in X \mid (\lambda, i) \geq 0 \text{ for all } i \in \Gamma_0\}$, the set of the dominant integral weights. We consider the structure of the irreducible highest weight $\mathcal{D}(\Lambda)$-module $L(\lambda)$ with $\lambda \in X^+$.

Proposition 4.1 *Let $\lambda \in X^+$ be a dominant integral weight, and μ be a weight of $L(\lambda)$. For each $j = (\theta, p) \in \Gamma_0^{\mathrm{im}} \cup J$, we have*

(a) $(\mu, \delta_j) = (\mu, \theta) \in \mathbb{Z}_{\geq 0}$,

(b) *if $(\mu, \delta_j) = 0$, then $L(\lambda)_{\mu - \delta_j} = 0$, and $F_p(\theta)(L(\lambda)_\mu) = 0$,*

(c) *if $(\mu, \delta_j) \leq -(\delta_j, \delta_j)$ and $(\delta_j, \delta_j) \neq 0$, then $E_p(\theta)(L(\lambda)_\mu) = 0$.*

Proof For $\mu \in \operatorname{wt}(L(\lambda))$, we may assume that $\mu = \lambda - \alpha$ where $\alpha = \sum_{k=1}^{s} j_k$ with $j_k \in \Gamma_0$. For any $j \in \Gamma_0^{\mathrm{im}} \cup J$, we have that $(j_k, \delta_j) \leq 0$ for each $1 \leq k \leq s$. Then $(\alpha, \delta_j) \leq 0$, and

$$(\mu, \delta_j) = (\lambda - \alpha, \delta_j) = (\lambda, \delta_j) - (\alpha, \delta_j) \geq (\lambda, \delta_j) \geq 0.$$

If $(\mu, \delta_j) = 0$, then $(\lambda, \delta_j) = (\alpha, \delta_j) = \left(\sum_{k=1}^{s} j_k, \delta_j\right) = 0$. So $(j_k, \delta_j) = 0$ for all $1 \leq k \leq s$, and $F_{j_k}(0)F_p(\theta) = F_p(\theta)F_{j_k}(0)$. Then $F_p(\theta)u_\alpha^- v_\lambda = u_\alpha^- F_p(\theta)v_\lambda = 0$. But $u_\alpha^- v_\lambda \in L(\lambda)_{\lambda-\alpha} = L(\lambda)_\mu$, so $F_p(\theta)(L(\lambda)_\mu) = 0$.

If $(\mu, \delta_j) \leq -(\delta_j, \delta_j) \neq 0$, then $(\mu + \delta_j, \delta_j) = (\mu, \delta_j) + (\delta_j, \delta_j) \leq 0$. If $(\mu + \delta_j, \delta_j) = 0$, then by (b), we have $L(\lambda)_\mu = 0$, a contradiction to the fact $\mu \in \operatorname{wt}(L(\lambda))$. And then by (a), we have that $E_p(\theta)(L(\lambda)_\mu) = 0$.

We define

Definition 4.1 *The category $\widetilde{\mathcal{O}}$ consists of $\mathcal{D}(\Lambda)$-module M satisfying the following properties:*

(1) *M belongs to the category $\mathcal{O}$,*

(2) *if $j \in \Gamma_0^{\mathrm{re}}$, then the action of u_j^- on M is locally nilpotent,*

(3) *if $j \in \Gamma_0^{\mathrm{im}} \cup J$, then $(\mu, \delta_j) \in \mathbb{Z}_{\geq 0}$ for all $\mu \in \operatorname{wt}(M)$,*

(4) *if $j \in \Gamma_0^{\mathrm{im}} \cup J$, and $(\mu, \delta_j) = 0$, then $F_p(\theta)M_\mu = 0$,*

(5) *if $j \in \Gamma_0^{\mathrm{im}} \cup J$, $(\mu, \delta_j) = -(\delta_j, \delta_j)$ and $(\delta_j, \delta_j) \neq 0$, then $E_p(\theta)M_\mu = 0$.*

Proposition 4.2 *Let $L(\lambda)$ be the irreducible highest weight $\mathcal{D}(\Lambda)$-module with highest weight $\lambda \in X$. Then $L(\lambda)$ belongs to the category $\widetilde{\mathcal{O}}$ if and only if $\lambda \in X^+$.*

Proof Assume that $L(\lambda)$ belongs to the category $\widetilde{\mathcal{O}}$. If $i \in \Gamma_0^{\mathrm{re}}$, the action of u_i^- on $L(\lambda)$ is locally nilpotent, so there exists a non-negative integer n_i such that $(u_i^-)^{n_i} \neq 0$, but $(u_i^-)^{n_i+1} = 0$. Then by $u_i^+ u_i^- = u_i^- u_i^+ + \frac{-|V_i|}{a_i}(K_i - K_{-i})$, we have

$$0 = u_i^+ (u_i^-)^{n_i+1} v_\lambda = \frac{-|V_i|}{a_i} \frac{1 - v^{-(i,i)(n_i+1)}}{1 - v^{-(i,i)}} \Big(v^{(i,\lambda)} - \frac{v^{-(i,\lambda)}}{v^{-n_i(i,i)}} \Big) (u_i^-)^{n_i} v_\lambda.$$

So $(i,\lambda) = \frac{(i,i)n_i}{2} \in \mathbb{Z}_{\geq 0}$.

If $i \in \Gamma_0^{\mathrm{im}}$, we have $(\lambda, i) \in \mathbb{Z}_{\geq 0}$ by the above definition.

If $\lambda \in X^+$, by Proposition 4.1, $L(\lambda)$ belongs to the category $\widetilde{\mathcal{O}}$.

A weight $\mathcal{D}(\Lambda)$-module M in the category $\widetilde{\mathcal{O}}$ is said to be integrable.

4.2 Category $\mathcal{O}'$ and category $\widetilde{\mathcal{O}}'$

Let $\mathcal{D}'(\Lambda)$ be the reduced Drinfeld double in Subsection 3.5 which is generated by x_j, y_j ($j \in \Gamma_0 \cup J$) with triangular decomposition $\mathcal{D}'(\Lambda) = \mathfrak{h}'^-(\Lambda) \otimes \mathcal{T}' \otimes \mathfrak{h}'^+(\Lambda)$.

Let $X' = \{\lambda \in \mathbb{Z}[\Gamma_0 \cup J] \mid (\lambda, i)' \in \mathbb{Z} \text{ for all } i \in \Gamma_0 \cup J\}$ be the weight lattice of $\widetilde{C}' = (\Gamma_0 \cup J, (-,-)', d')$.

Now for $\alpha = \sum k_j j \in \mathbb{Z}[\Gamma_0 \cup J]$, let $\bar{\alpha} = \sum k_j \delta_j \in \mathbb{Z}\Gamma_0$. We set $\operatorname{tr}\alpha = \operatorname{tr}\bar{\alpha}$.

We define $\mathcal{O}'$ to be the category consisting of weight $\mathcal{D}'(\Lambda)$-modules M which satisfy: for each $m \in M$, there exists $n_0 \geq 0$ such that $\mathfrak{h}'^+(\Lambda)_\alpha m = 0$ for $\alpha \in \mathbb{Z}[\Gamma_0 \cup J]$ with $\operatorname{tr}\alpha \geq n_0$.

For each $\lambda \in X'$, let $J'(\lambda)$ be the left ideal of $\mathcal{D}'(\Lambda)$ generated by x_i ($i \in \Gamma_0 \cup J$) and $K_\alpha - v^{(\lambda,\alpha)'}1$ ($\alpha \in \mathbb{Z}[\Gamma_0 \cup J]$). Set $V'(\lambda) = \mathcal{D}'(\Lambda)/J'(\lambda)$. Then $V'(\lambda)$ admits a left $\mathcal{D}'(\Lambda)$-module structure under left multiplication. We call $V'(\lambda)$ the Verma module. Using the triangular decomposition of $\mathcal{D}'(\Lambda)$, we have a bijection $\eta : \mathfrak{h}'^-(\Lambda) \to V'(\lambda);\ y \mapsto y + J'(\lambda)$. Under this bijection, $\mathfrak{h}'^-(\Lambda)$ admits a left $\mathcal{D}'(\Lambda)$-module structure. So η is in fact a module isomorphism. The module structure of $\mathfrak{h}'^-(\Lambda)$ is given by

$$K_\alpha \cdot y = v^{(\alpha,\lambda-\beta)'} y, \quad y_i \cdot y = y_i y, \quad x_i \cdot 1 = 0$$

for all $y \in \mathfrak{h}'^-(\Lambda)_\beta$, $\alpha \in \mathbb{Z}[\Gamma_0 \cup J]$ and $i \in \Gamma_0 \cup J$.

Let $L'(\lambda)$ be the irreducible highest weight module with highest weight λ.

We define $X'^+ = \{\lambda \in X' \mid (\lambda, i)' \geq 0 \text{ for all } i \in \Gamma_0 \cup J\}$, and set $J_\lambda = \{i \in \Gamma_0 \cup J \mid (\lambda, i)' = 0\}$.

Proposition 4.3 *Let $\lambda \in X'^+$ be a dominant integral weight.*

(1) *The highest weight vector v'_λ of $L'(\lambda)$ satisfies that*

$$\begin{cases} y_i^{\frac{(\lambda,i)'}{d'_i}+1} v'_\lambda = 0, & \text{if } i \in \Gamma_0^{\mathrm{re}}, \\ y_i v'_\lambda = 0, & \text{if } i \in J_\lambda. \end{cases} \tag{4.1}$$

(2) *Let (V, λ, v) be a highest weight $\mathcal{D}'(\Lambda)$-module with highest weight $\lambda \in X'^+$ and highest weight vector v, if v satisfies relation* (4.1)*, then V is isomorphic to $L'(\lambda)$.*

Similarly to Proposition 4.1, we have

Proposition 4.4 *Let μ be a weight of $L'(\lambda)$ with $\lambda \in X'^+$, and $i \in \Gamma_0^{\mathrm{im}} \cup J$. Then*

(1) $(\mu, i)' \in \mathbb{Z}_{\geq 0}$,

(2) *if $(\mu, i)' = 0$, then $L'(\lambda)_{\mu-i} = 0$ and $y_i(L'(\lambda)_\mu) = 0$,*

(3) *if $(\mu, i)' \leq -(i,i)'$ and $(i,i)' \neq 0$, then $x_i(L'(\lambda)_\mu) = 0$.*

By formula (3.3), for $i = (\theta, p) \in \Gamma_0 \cup J$, we have $x_i y_i - y_i x_i = \frac{K_\theta - K_{-\theta}}{v - v^{-1}}$. And for $n \in \mathbb{N}$, we have

$$x_i y_i^{n+1} v'_\lambda = \frac{1}{v - v^{-1}} \frac{1 - v^{-(\theta,i)'(n+1)}}{1 - v^{-(\theta,i)'}} \Big(v^{(\theta,\lambda)'} - v^{-(\theta,\lambda)'} \frac{1}{v^{-(\theta,i)'n}} \Big) y_i^n v'_\lambda. \tag{4.2}$$

Now we define the category $\widetilde{\mathcal{O}}'$ of integrable $\mathcal{D}'(\Lambda)$-module.

Definition 4.2 *The category $\widetilde{\mathcal{O}}'$ consists of $\mathcal{D}'(\Lambda)$-modules M satisfying the following properties:*

(1) *M lies in the category $\mathcal{O}'$,*
(2) *if $i \in \Gamma_0^{\mathrm{re}}$, the action of y_i on M is locally nilpotent,*
(3) *if $i \in \Gamma_0^{\mathrm{im}} \cup J$, then $(\mu, i)' \in \mathbb{Z}_{\geq 0}$ for all $\mu \in \mathrm{wt}(M)$,*
(4) *if $i \in \Gamma_0^{\mathrm{im}} \cup J$, and $(\mu, i)' = 0$, then $y_i M_\mu = 0$,*
(5) *if $i \in \Gamma_0^{\mathrm{im}} \cup J$, $(\mu, i)' = -(i, i)'$ and $(i, i)' \neq 0$, then $x_i M_\mu = 0$.*

Then we have

Proposition 4.5 *Let $L'(\lambda)$ be the irreducible highest weight $\mathcal{D}'(\Lambda)$-module with highest weight $\lambda \in X'$. Then $L'(\lambda)$ belongs to the category $\widetilde{\mathcal{O}}'$ if and only if $\lambda \in X'^{+}$.*

Proposition 4.6 (1) *If M is a highest weight $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$ with highest weight $\lambda \in X'$, then $\lambda \in X'^{+}$, and $M \cong L'(\lambda)$.*

(2) *Every irreducible $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$ is isomorphic to some $L'(\lambda)$ for some $\lambda \in X'^{+}$.*

Proof (1) Suppose M lies in the category $\widetilde{\mathcal{O}}'$ with highest weight vector v'_λ. If $i \in \Gamma_0^{\mathrm{re}}$, then by formula (4.2), setting $n_i = \frac{2(\theta,\lambda)'}{(\theta,i)'} = \frac{2(i,\lambda)'}{(i,i)}$, we have $y_i^{n_i+1} v'_\lambda = 0$.

If $i \in \Gamma_0^{\mathrm{im}} \cup J$, then by Definition 4.2, we have $(\lambda, i)' \in \mathbb{Z}_{\geq 0}$. And if $(\lambda, i)' = 0$, then $y_i M_\lambda = 0$, i.e., $y_i v'_\lambda = 0$. Hence $\lambda \in X'^{+}$. By Proposition 4.3, we have $M \cong L'(\lambda)$.

(2) Let V be an irreducible $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$. By the definition, V lies in the category $\mathcal{O}'$. Let $V = \bigoplus_{\mu \in X'} V_\mu$, where V_μ's are finite dimensional weight spaces. And for any $m \in V$, there exists an integer $n \geq 0$, such that $\mathfrak{h}'^{+}(\Lambda)_\alpha m = 0$ for $\alpha \in \mathbb{Z}[\Gamma_0 \cup J]$ with $\mathrm{tr}\,\alpha \geq n$. Now for weight space V_μ, let m_μ be any element in V_μ, so there exists n such that $x_\alpha m_\mu = 0$ for α with $\mathrm{tr}\,\alpha \geq n$. Let n_0 be the minimal integer satisfying this condition, i.e., there exists α_0 such that $x_{\alpha_0} m_\mu \neq 0$ with $\mathrm{tr}\,\alpha_0 < n_0$. Now assume that $\mathrm{tr}\,\alpha_0$ is the maximal ones. Then for any $i \in \Gamma_0 \cup J$, $x_i x_{\alpha_0} m_\mu = 0$, and $K_\beta x_{\alpha_0} m_\mu = v^{(\beta, \mu+\alpha_0)'} x_{\alpha_0} m_\mu$. So $V = \mathcal{D}'(\Lambda) x_{\alpha_0} m_\mu$ is a highest weight $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$. By (1), we have $V \cong L'(\lambda)$.

4.3 Complete reducibility

In the following part, we consider the complete reducibility of modules in the category $\widetilde{\mathcal{O}}$ and the category $\widetilde{\mathcal{O}}'$.

Define an anti-involution σ of $\mathcal{D}'(\Lambda)$ by $\sigma(x_i) = y_i, \sigma(y_i) = x_i, \sigma(K_i) = K_i$ for all i.

Let $M = \bigoplus_{\lambda \in X'} M_\mu$ be a $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$. We define its finite dual (see [9, 10]) to be the vector space

$$M^* := \bigoplus_{\lambda \in X'} M_\lambda^* \quad \text{where } M_\lambda^* := \mathrm{Hom}_R(M_\lambda, R).$$

Using σ, we can construct a $\mathcal{D}'(\Lambda)$-module structure on M^* by

$$(x \cdot \phi)(v) = \phi(\sigma(x) \cdot v) \quad \text{for } x \in \mathcal{D}'(\Lambda),\ \phi \in M^*,\ v \in M.$$

Let M be a $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$. A maximal weight of M is a weight $\lambda \in \mathrm{wt}(M)$ such that $\lambda + i$ is not a weight of M for any $i \in \Gamma_0 \cup J$. Then for a nonzero vector $v_\lambda \in M_\lambda$ and set $V = \mathcal{D}'(\Lambda) v_\lambda$, we have $\lambda \in X'^{+}$ and $V \cong L'(\lambda)$.

Take $\varphi_\lambda \in M_\lambda^*$ satisfying $\varphi_\lambda(v_\lambda) = 1$, and set $W = \mathcal{D}'(\Lambda)\varphi_\lambda$. Then $W \cong L'(\lambda)$.

Lemma 4.1 *Let M be a $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$ and V be the submodule of M generated by a nonzero vector v_λ of maximal weight λ. Then we have $M \cong V \oplus (M/V)$.*

Theorem 4.1 *Every $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$ is isomorphic to a direct sum of irreducible highest weight module $L'(\lambda)$ with $\lambda \in X'^{+}$.*

Proof By Proposition 4.3, it is enough to prove that any $\mathcal{D}'(\Lambda)$-module M in the category $\widetilde{\mathcal{O}}'$ is semisimple.

Firstly, if M is generated as a $\mathcal{D}'(\Lambda)$-module by a finite dimensional $\mathcal{H}'^{+}(\Lambda)$-module $\overline{M}$, we use induction on the dimension of $\overline{M}$ to prove that M is semisimple.

If $\overline{M} \neq 0$, we choose a nonzero vector v_λ of maximal weight λ of $\overline{M}$ in X'^{+}, and set $V = \mathcal{D}'(\Lambda)v_\lambda \cong L'(\lambda)$. By the above lemma, we have $M \cong V \oplus (M/V)$. $M/V = \mathcal{D}'(\Lambda)(\overline{M}/\overline{M} \cap V)$ and $\dim(\overline{M}/\overline{M} \cap V) < \dim \overline{M}$. So by induction hypothesis, M/V is semisimple. Hence M is semisimple.

Let M ba an arbitrary $\mathcal{D}'(\Lambda)$-module in the category $\widetilde{\mathcal{O}}'$. For any $m \in M$, $\mathcal{H}'^{+}(\Lambda)m$ is finite dimensional, and then $\mathcal{D}'(\Lambda)m$ is semisimple. And M is a sum of $\mathcal{D}'(\Lambda)$-module $\mathcal{D}'(\Lambda)m$, which is equivalent to the direct sum of $\mathcal{D}'(\Lambda)m$ by [2]. So M is completely reducible.

Proposition 4.7 *Let $L(\lambda)$ be the irreducible highest weight $\mathcal{D}(\Lambda)$-module with highest weight $\lambda \in X^{+}$ and highest weight vector v_λ. Then*

$$\begin{cases} (u_i^-)^{\frac{(\lambda,\delta_i)}{\varepsilon_i}+1} v_\lambda = 0, & \text{if } i \in \Gamma_0^{\mathrm{re}}. \\ (u_i^-) v_\lambda = 0, & \text{if } i \in J_\lambda \cap \Gamma_0. \\ F_p(\theta) v_\lambda = 0, & \text{if } i = (\theta, p) \in J_\lambda \cap (J \setminus \Gamma_0). \end{cases} \tag{4.3}$$

Theorem 4.2 *Every $\mathcal{D}(\Lambda)$-module in the category $\widetilde{\mathcal{O}}$ is isomorphic to a direct sum of irreducible highest weight module $L(\lambda)$ with $\lambda \in X^{+}$.*

Proof For any $\mathcal{D}(\Lambda)$-module M in the category $\widetilde{\mathcal{O}}$, M can be viewed as a $\mathcal{D}'(\Lambda)$-module denoted by $\overline{M}$. According to the definition of $\widetilde{\mathcal{O}}'$, we get that $\overline{M}$ is in $\widetilde{\mathcal{O}}'$. Then by Theorem 4.1, $\overline{M}$ is completely reducible. By the action of δ, we get M is the direct sum of $L(\lambda)$ $(\lambda \in X^{+})$.

4.4 Canonical isomorphism

By the choices of index $i \in \Gamma_0^{\mathrm{re}}$ and analogously to [4], we have the following remark.

Remark 4.1 The double Ringel-Hall algebra $\mathcal{D}(\Lambda)$ is generated by x_i, y_i $(i \in \Gamma_0 \cup J)$ and K_α $(\alpha \in \mathbb{Z}\Gamma_0)$ satisfying the following relations:

(1) $K_0 = 1, \quad K_i K_j = K_{i+j} \quad$ for $i, j \in \Gamma_0 \cup J$.

(2) $K_\alpha x_i = v^{(\alpha,\delta_i)} x_i K_\alpha, \quad K_\alpha y_i = v^{-(\alpha,\delta_i)} y_i K_\alpha \quad$ for $i \in \Gamma_0 \cup J$, $\alpha \in \mathbb{Z}\Gamma_0$.

(3) $x_i y_j - y_j x_i = \delta_{ij} \dfrac{K_{\delta_i} - K_{-\delta_i}}{v' - v'^{-1}}$, where $v' = v^{\varepsilon_i}$ if $i \in \Gamma_0$ and $v' = v$ if $i \in J$.

(4) $\displaystyle\sum_{s+t=1-a_{ij}} (-1)^s x_i^{(s)} x_j x_i^{(t)} = 0, \quad \sum_{s+t=1-a_{ij}} (-1)^s y_i^{(s)} y_j y_i^{(t)} = 0$ for $i \in \Gamma_0^{\mathrm{re}}$, $i \neq j$, where $x_i^{(n)} = x_i^n/[n]_i!$, $y_i^{(n)} = y_i^n/[n]_i!$, $a_{ij} = \frac{(\delta_i,\delta_j)}{\varepsilon_i}$.

(5) $x_i x_j - x_j x_i = 0 = y_i y_j - y_j y_i$ if $(\delta_i, \delta_j) = 0$.

Let U be the associative algebra over R generated by $\{e_i, f_i, K_i^{\pm} \mid i \in \Gamma_0 \cup J\}$, such that the generators satisfy the similar relations as in Remark 4.1. We define comultiplication Δ, counint

ε and antipode S as we did in Subsection 2.5. Then U is a quantum generalized Kac-Moody algebra.

Let $U^{\geq 0}$ and $U^{\leq 0}$ be the subalgebra of U as we defined in Subsection 2.5. We define $\varphi : U^{\geq 0} \times U^{\leq 0} \to R$ satisfying:

(1) $\varphi(1,1) = 1$,

(2) $\varphi(e_i, f_j) = \delta_{ij}(1 - v^{-(\delta_i,\delta_i)})^{-1}, \quad \varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}$,

(3) $\varphi(x, yy') = \varphi(\Delta(x), y \otimes y') \quad$ for all $x \in U^{\geq 0},\ y, y' \in U^{\leq 0}$,

(4) $\varphi(xx', y) = \varphi(x \otimes x', \Delta^{\rm opp}(y)) \quad$ for all $x, x' \in U^{\geq 0},\ y \in U^{\leq 0}$.

Then $(U^{\geq 0}, U^{\leq 0}, \varphi)$ is a member of $\mathcal{L}(\widetilde{C}')$.

For any $i \in \Gamma_0^{\rm re}$, we can define Lusztig symmetries T_i on U as in [4]. The map $\Pi : U \to \mathcal{D}'(\Lambda)$ defined by $\Pi(e_i) = x_i,\ \Pi(f_i) = y_i,\ \Pi(K_i) = K_i\ (\forall\, i \in \Gamma_0 \cup J)$ is a surjection. In fact it is an isomorphism. By the uniqueness of skew-Hopf pairing, it suffices to show that $(U^{\geq 0}, U^{\leq 0}, \varphi)$ is restricted non-degenerate in $\mathcal{L}(\widetilde{C}')$, i.e., to prove

$$\mathcal{I}^+ = \{x \in U^+ \mid \varphi(x, U^-) = 0\}, \quad \mathcal{I}^- = \{y \in U^- \mid \varphi(U^+, y) = 0\}$$

are zero in U^+ and U^- respectively. Using Lusztig symmetries and analogously to [4], we have

Lemma 4.2 *It holds that* $\mathcal{I}^+ = 0$ (*resp.,* $\mathcal{I}^- = 0$) *in* U^+ (*resp.,* U^-).

From this lemma, we get that the map $\Pi : U \to \mathcal{D}'(\Lambda)$ is an isomorphism. And we have

Theorem 4.3 *Double Ringel-Hall algebra* $\mathcal{D}(\Lambda)$ *is generated by* x_i, y_i $(i \in \Gamma_0 \cup J)$ *and* K_α $(\alpha \in \mathbb{Z}\Gamma_0)$ *with the generating relations* (1)–(5) *in Remark* 4.1.

As a corollary, we have

Corollary 4.1 *Double composition algebra* $\mathcal{C}(\Lambda)$ *is generated by* u_i^+, u_i^- $(i \in \Gamma_0)$ *and* K_α $(\alpha \in \mathbb{Z}\Gamma_0)$*, which satisfy the generating relations:*

(1) $K_0 = 1, \quad K_i K_j = K_{i+j} \quad$ *for* $i, j \in \Gamma_0$.

(2) $K_\alpha u_i^+ = v^{(\alpha,i)} u_i^+ K_\alpha, \quad K_\alpha u_i^- = v^{-(\alpha,i)} u_i^- K_\alpha \quad$ *for* $i \in \Gamma_0,\ \alpha \in \mathbb{Z}\Gamma_0$.

(3) $u_i^+ u_j^- - u_j^- u_i^+ = \delta_{ij} \dfrac{K_i - K_{-i}}{v^{\varepsilon_i} - v^{-\varepsilon_i}} \quad$ *for* $i, j \in \Gamma_0$.

(4) $\displaystyle\sum_{s+t=1-a_{ij}} (-1)^s (u_i^+)^{(s)} (u_j^+) (u_i^+)^{(t)} = 0, \quad \sum_{s+t=1-a_{ij}} (-1)^s (u_i^-)^{(s)} (u_i^-) (u_j^-)^{(t)} = 0$ *for* $i \in \Gamma_0^{\rm re}$ *and* $i \neq j$, *where* $(u_i^+)^{(n)} = (u_i^+)^n/[n]_i!,\ (u_i^-)^{(n)} = (u_i^-)^n/[n]_i!$.

(5) $u_i^+ u_j^+ - u_j^+ u_i^+ = 0 = u_i^- u_j^- - u_j^- u_i^- \quad$ *if* $(i, j) = 0$.

5 Weyl-Kac Character Formula and Kac Theorem

Now we consider the Weyl-Kac character formula of an irreducible highest weight $\mathcal{D}(\Lambda)$-module.

5.1 Character formula

Let Φ^+ be the set of the dimension vectors of all indecomposable representations of $\mathcal{S}$. Set $\Phi^- = -\Phi^+$ and $\Phi = \Phi^- \cup \Phi^+$. Let W be the Weyl group corresponding to the Borcherds datum $C = C_\Gamma$.

By Theorem 2.1, the number of isomorphism classes of indecomposable representations of Γ with a fixed dimension vector over a finite field is independent of the orientation of Γ. Therefore, we can apply the BGP-reflection functors σ_i for $i \in \Gamma_0^{\mathrm{re}}$ to the set Φ, to obtain the action of the fundamental reflections r_i on Φ by $r_i(\alpha) = \alpha - \frac{(\alpha,i)}{d_i} i$ (as we did in [5]). The same as in [5] we have the well-defined action of W on Φ.

Let M be a $\mathcal{D}(\Lambda)$-module in the category $\mathcal{O}$ and let $M = \bigoplus_{\lambda \in X} M_\lambda$. The formal character of M is defined by

$$\operatorname{ch} M = \sum_\lambda (\dim M_\lambda) e(\lambda).$$

For $\alpha \in \mathbb{N}\Gamma_0$, denote by $m(\alpha, q)$ the number of isomorphic classes of representations of $\mathcal{S}$ with dimension vector α over finite field $\mathbb{F}_q$, and by $I(\alpha, q)$ the number of isomorphic classes of indecomposable representations of $\mathcal{S}$ with dimension vector α in Φ^+.

For Verma module $V(\lambda) = \mathcal{D}(\Lambda)/J(\lambda) \cong \mathfrak{h}^-(\Lambda)$, if $V(\lambda) \neq 0$, $\dim V(\lambda)_\beta = m(\lambda - \beta, q)$ by $V(\lambda)_\beta \cong \mathfrak{h}^-(\Lambda)_{\lambda-\beta}$, so

$$\operatorname{ch} V(\lambda) = \sum_\beta m(\lambda - \beta, q) e(\beta) = \sum_{\alpha \in \mathbb{N}\Gamma_0} e(\lambda) m(\alpha, q) e(-\alpha) = e(\lambda) \sum_{\alpha \in \mathbb{N}\Gamma_0} m(\alpha, q) e(-\alpha).$$

Let $L = \{\alpha \in \mathcal{P} \mid V_\alpha \text{ is an indecomposable representation of } \mathcal{S}\}$. For $\alpha \in L$, let $u^-_{\alpha,i}$ ($1 \leq i \leq \dim \mathfrak{h}^-(\Lambda)_{\lambda-\alpha} = \tau_{\lambda-\alpha}$) be an R-basis of $\mathfrak{h}^-(\Lambda)_{\lambda-\alpha}$. Then

$$\prod_{\alpha \in L} (u^-_{\alpha,1})^{n_{11}} \cdots (u^-_{\alpha,\tau_{\lambda-\alpha}})^{n_{1\tau_{\lambda-\alpha}}}, \quad \text{where } \sum_{\alpha \in L} (n_{11} + \cdots + n_{1\tau_{\lambda-\alpha}})\alpha = \lambda - \beta,$$

form a basis of $\mathfrak{h}^-(\Lambda)_{\lambda-\beta} \cong V(\lambda)_\beta$; that is, $\{u^-_\alpha \mid \alpha \in L\}$ in a fixed order provides a universal PBW-basis of $\mathfrak{h}^-(\Lambda)$ (see [8]). So

$$\operatorname{ch} V(\lambda) = e(\lambda) \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-I(\alpha,q)}. \tag{5.1}$$

Let $\lambda \in X^+$ and $L(\lambda)$ be the corresponding irreducible $\mathcal{D}(\Lambda)$-module. Then we have $\operatorname{mult}_{L(\lambda)} \mu = \operatorname{mult}_{L(\lambda)} w(\mu)$ for each $w \in W$ and $\mu \in \operatorname{wt}(L(\lambda))$. If we define the action of Weyl group on formal exponential by $w(e(\lambda)) = e(w(\lambda))$ for $\lambda \in X, w \in W$, then we have $w(\operatorname{ch} L(\lambda)) = \operatorname{ch} L(\lambda)$ for $\lambda \in X^+, w \in W$.

Let $R = \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{I(\alpha,q)}$. For $w \in W$, set $\varepsilon(w) = (-1)^{\ell(w)}$. Furthermore, we have

$$w(e(\rho)R) = \varepsilon(w) e(\rho) R \quad \text{for } w \in W.$$

Firstly, for $\mathcal{D}'(\Lambda)$-module in the category $\mathcal{O}'$, we have the following

Lemma 5.1 *If V is a $\mathcal{D}'(\Lambda)$-module from the category $\mathcal{O}'$, then*

$$\operatorname{ch} V = \sum_{\lambda \in \operatorname{wt}(V)} [V : L'(\lambda)] \operatorname{ch} L'(\lambda),$$

where $[V : L'(\lambda)]$ is the multiplicity of $L'(\lambda)$ in V.

In particular, for $L'(\lambda')$ with $\lambda' \in X'^+$, we have

Lemma 5.2

$$\operatorname{ch} L'(\lambda') = \sum_{\substack{\mu' \leq \lambda' \\ (\lambda'+\rho, \lambda'+\rho)' = (\mu'+\rho, \mu'+\rho)'}} c_{\mu'} \operatorname{ch} V'(\mu'),$$

where $c_{\mu'} \in \mathbb{Z}$ and $c_{\lambda'} = 1$.

By sending $\sum\limits_{i\in\Gamma_0\cup J} a_i i$ to $\sum\limits_{i\in\Gamma_0\cup J} a_i\delta_i$, we defined a map $\delta : \mathbb{Z}[\Gamma_0 \cup J] \to \mathbb{Z}\Gamma_0$ in Subsection 3.5. For $\lambda \in X^+$, set $\tilde{\lambda} \in X'^+$ such that $(\tilde{\lambda}, i)' = (\lambda, \delta_i)$ for all $i \in \Gamma_0 \cup J$.

Lemma 5.3 *Let $\lambda \in X^+$ and M be a highest weight $\mathcal{D}(\Lambda)$-module with highest weight λ. Then*

$$\operatorname{ch} M = \sum_{\substack{\beta' \leq \tilde{\lambda} \\ (\tilde{\lambda}+\rho,\tilde{\lambda}+\rho)'=(\beta'+\rho,\beta'+\rho)'}} c_{\beta'} \operatorname{ch} V(\beta),$$

where β satisfies $\delta(\tilde{\lambda} - \beta') = \lambda - \beta$.

So from this lemma, we have

$$\operatorname{ch} L(\lambda) = \sum_{\substack{\beta' \leq \tilde{\lambda} \\ (\tilde{\lambda}+\rho,\tilde{\lambda}+\rho)'=(\beta'+\rho,\beta'+\rho)' \\ \delta(\tilde{\lambda}-\beta')=\lambda-\beta}} c_{\beta'} \operatorname{ch} V(\beta), \tag{5.2}$$

where $c_{\beta'} \in \mathbb{Z}$.

For convenience, we write $\delta(\rho)$ as ρ.

Our main result is

Theorem 5.1 *For $\lambda \in X^+$, we have*

$$\operatorname{ch} L(\lambda) = \frac{\sum\limits_{w\in W,T} (-1)^{\ell(w)+|T|} e(w(\lambda + \rho + \delta(S(T))) - \rho)}{\prod\limits_{\alpha\in\Phi^+} (1 - e(-\alpha))^{I(\alpha,q)}},$$

where T runs over all finite subsets of $(\Gamma_0 \cup J)^{\mathrm{im}}$ such that $(\lambda, \delta_i) = 0$ for $i \in T$ and $(\delta_i, \delta_j) = 0$ for $i \neq j$ in T. $|T|$ is the number of elements in T. $S(T)$ is the sum of elements in T.

Proof By formulae (5.1) and (5.2), it follows that

$$e(\rho)R \operatorname{ch} L(\lambda) = \sum_{\substack{\beta' \leq \tilde{\lambda} \\ (\tilde{\lambda}+\rho,\tilde{\lambda}+\rho)'=(\beta'+\rho,\beta'+\rho)' \\ \delta(\tilde{\lambda}-\beta')=\lambda-\beta}} c_{\beta'} e(\beta + \rho). \tag{5.3}$$

Set $\tilde{\lambda} - \beta' = \gamma'$, $\lambda - \beta = \gamma$. Then $\beta \leq \lambda$. By the fact $(\tilde{\lambda} + \rho, \tilde{\lambda} + \rho)' = (\beta' + \rho, \beta' + \rho)'$, we have

$$0 = 2(\tilde{\lambda}, \gamma')' + 2(\rho, \gamma')' - (\gamma', \gamma')' = 2(\lambda, \gamma) + 2(\rho, \gamma) - (\gamma, \gamma), \tag{5.4}$$

so

$$(\lambda + \rho, \lambda + \rho) - (\beta + \rho, \beta + \rho) = 0. \tag{5.5}$$

Conversely, by formula (5.5), we can get formula (5.4). So if β satisfies $\delta(\tilde{\lambda} - \beta') = \lambda - \beta$, $(\lambda + \rho, \lambda + \rho) = (\beta + \rho, \beta + \rho)$ and $(\tilde{\lambda} + \rho, \tilde{\lambda} + \rho)' = (\beta' + \rho, \beta' + \rho)'$ are equivalent.

Thus formula (5.3) is equivalent to

$$e(\rho)R\,\mathrm{ch}\,L(\lambda)=\sum_{\substack{\beta'\le\tilde\lambda\\(\lambda+\rho,\lambda+\rho)=(\beta+\rho,\beta+\rho)\\\delta(\tilde\lambda-\beta')=\lambda-\beta}}c_{\beta'}e(\beta+\rho),\tag{5.6}$$

and both sides of (5.6) are antisymmetric under W, i.e., for any $w\in W$, we have

$$w(e(\rho)R\,\mathrm{ch}\,L(\lambda))=\varepsilon(w)e(\rho)R\,\mathrm{ch}\,L(\lambda).$$

Let S_λ be the sum of the terms on the right of (5.6) for which $\beta+\rho$ satisfies $(\beta+\rho,\delta_i)\ge 0$ for all $i\in\Gamma_0^{\mathrm{re}}$. If $(\beta+\rho,\delta_i)\not\ge 0$, then $\sum\limits_{w\in W}\varepsilon(w)e(w(\beta+\rho))=0$. So

$$e(\rho)R\,\mathrm{ch}\,L(\lambda)=\sum_{w\in W}\varepsilon(w)w(S_\lambda).\tag{5.7}$$

Now if $\beta+\rho$ satisfies that $(\beta+\rho,\delta_i)\ge 0$ for all $i\in\Gamma_0^{\mathrm{re}}$, we write $\beta=\lambda-\sum\limits_{i\in\Gamma_0\cup J}a_i\delta_i$ with $a_i\in\mathbb{Z}_{>0}$. Because of the fact $(\beta+\rho,\beta+\rho)=(\lambda+\rho,\lambda+\rho)$, we have

$$\sum_i a_i(\delta_i,\lambda)+\sum_i a_i(\delta_i,\beta+2\rho)=0.\tag{5.8}$$

By the fact that $\lambda\in X^+$, we have $(\lambda,\delta_i)\ge 0$ for all i.

For $i\in\mathrm{supp}(\lambda-\beta)$, if i is a real index, we have $(\delta_i,\beta+2\rho)=(\delta_i,\beta+\rho)+(\delta_i,\rho)\ge(\delta_i,\beta+\rho)\ge 0$.

If i is an imaginary index, then

$$(\delta_i,\beta+2\rho)=(\delta_i,\lambda)-\sum_{j\ne i}a_j(\delta_i,\delta_j)+(1-a_i)(\delta_i,\delta_i)\ge 0.$$

So $(\lambda,\delta_i)=0=(\delta_i,\beta+2\rho)$. But $(\delta_i,\beta+2\rho)=(\delta_i,\beta+\rho)+\frac12(\delta_i,\delta_i)$, so $(\delta_i,\beta+\rho)=-\frac12(\delta_i,\delta_i)\ge 0$. Then i is an imaginary index.

Furthermore, we get that $(\delta_i,\lambda-\beta-\delta_i)=-(\delta_i,\beta+2\rho)=0$, and $\lambda-\beta-\delta_i=\sum\limits_{j\ne i}a_j\delta_j+(a_i-1)\delta_i$. So $(\delta_i,\delta_j)=0$ unless $i=j$ and $a_i=1$.

Then for any term $c_{\beta'}e(\beta+\rho)$ in S_λ, β is of the form $\lambda-\sum a_i\delta_i$, where $a_i\in\mathbb{Z}_{>0}$, all the i's are imaginary index and $(\delta_i,\lambda)=0$, and $(\delta_i,\delta_j)=0$ if $i\ne j$, and $(\delta_i,\delta_i)=0$ if $a_i\ne 1$. And furthermore, β of thus form naturally satisfies $(\beta,\delta_j)\ge 0$ for all $j\in\Gamma_0^{\mathrm{re}}$.

If $e\big(\lambda-\sum\limits_i a_i\delta_i+\rho\big)$ is a term of $\mathrm{ch}\,L(\lambda)$, then there exists j such that $(\lambda,\delta_j)\ne 0$. So the terms of the form $e\big(\lambda-\sum\limits_i a_i\delta_i+\rho\big)$ of the right side of formula (5.6) in S_λ are those coming from $e(\lambda+\rho)R=e(\lambda+\rho)\prod\limits_{\alpha\in\Phi^+}(1-e(-\alpha))^{I(\alpha,q)}$. If $(\delta_i,\delta_j)=0$ for $i\ne j$, then the coefficient of $e(\lambda+\rho-\sum a_i\delta_i)$ is to be 0 if $a_i>1$ for some i, and $(-1)^{\sum a_i}$ otherwise. So $S_\lambda=e(\lambda+\rho)\sum\limits_s\varepsilon(s)e(\delta(s))$, where the sum is taken over all s of simple roots. If we set $s=\sum\limits_{j=1}^m i_j$ $(i_j\in\Gamma_0^{\mathrm{im}}\cup J)$, then we have $\varepsilon(s)=(-1)^m$ if $i_j\ne i_k$ $(j\ne k)$, $(\delta_{i_j},\delta_{i_k})=0$ $(\forall\, j\ne k)$ and $(\lambda,\delta_{i_j})=0$ $(\forall\, j)$, and $\varepsilon(s)=0$ otherwise. Or equivalently, we have $S_\lambda=e(\lambda+\rho)\sum\limits_T(-1)^{|T|}e(\delta(S(T)))$, where T runs over all finite subsets of $\Gamma_0^{\mathrm{im}} \cup J$ such that $(\lambda, \delta_i) = 0$ for $i \in T$ and $(\delta_i, \delta_j) = 0$ for $i \neq j$ in T. $|T|$ is the number of elements in T. $S(T)$ is the sum of elements in T. So we have

$$\begin{aligned} e(\rho) R \operatorname{ch} L(\lambda) &= \sum_w \varepsilon(w) w \Big(e(\lambda+\rho) \sum_T (-1)^{|T|} e(\delta(S(T))) \Big) \\ &= \sum_{w \in W; T} \varepsilon(w) (-1)^{|T|} e(w(\lambda + \rho + \delta(S(T)))) \end{aligned}$$

and then we can get the formula in the theorem.

5.2 Kac theorem

Let Γ be any valued quiver, and let A_Γ be the matrix of Γ. Set Δ be the root system of A_Γ.

Let A' be the Borcherds-Cartan matrix defined by extended Borcherds datum $\widetilde{C}'$ and Δ' be the root system. Then by Subsection 3.5 and Lemma 3.5, we have $\delta(\Delta') = \Delta$.

By Subsection 4.4, $\mathcal{D}'(\Lambda)$ and U are canonically isomorphic, where U is the quantum generalized Kac-Moody algebra associated to A'. Then $\mathcal{D}'(\Lambda)$ is a quantum generalized Kac-Moody algebra.

For quantum algebra U, let U^- be the negative part of U which can be viewed as a highest weight U-module. Define character

$$\operatorname{ch} U^- = \sum_{\mu \in \mathbb{N}[\Gamma_0 \cup J]} \dim U^-_{-\mu} e(-\mu).$$

Then by Proposition 2.3 in Subsection 2.5, we have

$$\operatorname{ch} U^- = \prod_{\alpha \in \Delta'_+} (1 - e(-\alpha))^{-\operatorname{mult} \alpha}. \tag{5.9}$$

For the negative part $\mathfrak{h}'^-(\Lambda)$ of $\mathcal{D}'(\Lambda)$, $\mathfrak{h}'^-(\Lambda)$ can be viewed as a $\mathcal{D}'(\Lambda)$-module. Define character

$$\operatorname{ch} \mathfrak{h}'^-(\Lambda) = \sum_{\mu \in \mathbb{N}[\Gamma_0 \cup J]} \dim \mathfrak{h}'^-(\Lambda)_{-\mu} e(-\mu).$$

Since $\mathfrak{h}'^-(\Lambda)$ and U^- are isomorphic, we have $\operatorname{ch} \mathfrak{h}'^-(\Lambda) = \operatorname{ch} U^-$.

Now we have our second main result.

Theorem 5.2 (1) $\Phi^+ = \Delta_+$, *the set of dimension vectors of indecomposable representations of* Γ *is the positive root system of* Γ.

(2) *If* $\alpha \in \Delta_+^{\mathrm{re}}$, *then up to isomorphism, there exists unique indecomposable representation of* Γ *of dimension vector* α.

Proof For subalgebra $\mathfrak{h}^-(\Lambda)$ of $\mathcal{D}(\Lambda)$, by Subsection 5.1, we have

$$\operatorname{ch} \mathfrak{h}^-(\Lambda) = \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))^{-I(\alpha, q)}.$$

We know that the difference between $\mathcal{D}'(\Lambda)$ and $\mathcal{D}(\Lambda)$ only lies in the gradation. For subalgebras $\mathfrak{h}'^-(\Lambda)$ and $\mathfrak{h}^-(\Lambda)$, we have $\delta(\operatorname{ch} \mathfrak{h}'^-(\Lambda)) = \operatorname{ch} \mathfrak{h}^-(\Lambda) = \delta(\operatorname{ch} U^-)$. So by formula

(5.9) and by $\delta(\Delta') = \Delta$, we have

$$\begin{aligned}\operatorname{ch}\mathfrak{h}^-(\Lambda) &= \prod_{\alpha\in\Phi^+}(1-e(-\alpha))^{-I(\alpha,q)} = \delta\Big(\prod_{\alpha'\in\Delta'_+}(1-e(-\alpha'))^{-\operatorname{mult}\alpha'}\Big)\\ &= \prod_{\alpha\in\Delta_+}\prod_{\substack{\alpha'\\ \delta(\alpha')=\alpha}}(1-e(-\alpha))^{-\operatorname{mult}\alpha'} = \prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\sum_{\alpha',\delta(\alpha')=\alpha}-\operatorname{mult}\alpha'}.\end{aligned}$$

Then we have $\Phi^+ = \Delta_+$. And if α is a real root, there is only one α' such that $\delta(\alpha') = \alpha$, and $\operatorname{mult}\alpha' = 1$. So $I(\alpha, q) = 1$.

5.3 Situation of nilpotent representations

We denote by $\mathcal{P}_{\mathrm{N}}$ the set of isomorphic classes of nilpotent representations of $\mathcal{S}$ (a k-species of Γ), and by I_{N} the set of isomorphic classes of simple nilpotent representations of $\mathcal{S}$. Then according to Subsection 3.1, we can define the Rignel-Hall algebra $\mathcal{H}^{\pm}_{\mathrm{N}}(\Lambda)$ and the double Ringel-Hall algebra $\mathcal{D}_{\mathrm{N}}(\Lambda)$, which have basis indexed by $\mathcal{P}_{\mathrm{N}}$. We simply call them the nilpotent Ringel-Hall algebra and the nilpotent double Ringel-Hall algebra.

Let $\mathcal{C}_{\mathrm{N}}(\Lambda)$ be the double composition subalgebra of $\mathcal{D}_{\mathrm{N}}(\Lambda)$. Then we have $\mathcal{C}_{\mathrm{N}}(\Lambda) = \mathcal{C}(\Lambda)$. It is easy to see that $I_{\mathrm{N}} = \Gamma_0$ as index set by Lemma 2.1.

If $\mathcal{F}$ is the fundamental set of $\mathbb{N}\Gamma_0$, then we set $\mathcal{F}_{0_{\mathrm{N}}} = \mathcal{F} \setminus \Big(\bigcup_{i\in\Gamma_0^{\mathrm{im}}} e_i\Big)$.

For each subset $\mathcal{E} \subseteq \mathcal{F}_{0_{\mathrm{N}}}$, we can get subalgebras $\mathcal{D}_{\mathrm{N}_{\mathcal{E}}}(\Lambda)$ of $\mathcal{D}_{\mathrm{N}}(\Lambda)$ as similar as in Subsection 3.2. And $(\mathcal{H}^+_{\mathrm{N}}(\Lambda), \mathcal{H}^-_{\mathrm{N}}(\Lambda), \varphi)$ is a restricted non-degenerate member of $\mathcal{L}(\widetilde{C}_{\mathcal{F}_{0_{\mathrm{N}}}})$, where $\widetilde{C}_{\mathcal{F}_{0_{\mathrm{N}}}} = (\Gamma_0, (-,-), d, \{\delta_j \mid j \in J_{\mathrm{N}}\})$ with $J_{\mathrm{N}} = J_{\mathcal{F}_{0_{\mathrm{N}}}}$.

Since $\mathcal{H}^{\pm}_{\mathrm{N}}(\Lambda)$ and $\mathcal{D}_{\mathrm{N}}(\Lambda)$ are subalgebras of $\mathcal{H}^{\pm}(\Lambda)$ and $\mathcal{D}(\Lambda)$ respectively, we have the relations $\mathcal{C}(\Lambda) \subset \mathcal{D}_{\mathrm{N}}(\Lambda) \subset \mathcal{D}(\Lambda)$ as graded subalgebras. If we let Φ^+_{N} be the set of dimension vectors of the nilpotent indecomposable representations of $\mathcal{S}$ and let Φ^+ be the set of all indecomposable representations of $\mathcal{S}$, then we have $\Phi^+_{\mathrm{C}} \subseteq \Phi^+_{\mathrm{N}} \subseteq \Phi^+$, where $\Phi^+_{\mathrm{C}} = \Delta_+$ is the positive root system of Γ, because of the fact of Theorem 4.2 and its remark. By the Theorem 5.2, we have the following corollary.

Corollary 5.1 $\Phi^+_{\mathrm{C}} = \Phi^+_{\mathrm{N}} = \Phi^+$.

So by this corollary, we can apply BGP-reflection functors σ_i for all $i \in \Gamma_0^{\mathrm{re}}$ to the set $\Phi_{\mathrm{N}} := \Phi^+_{\mathrm{N}} \cup -\Phi^+_{\mathrm{N}}$ to obtain the action of the fundamental reflections r_i on Φ_{N}. Let $I_{\mathrm{N}}(\alpha, q)$ be the number of the isomorphic classes of nilpotent indecomposable representations of $\mathcal{S}$ with dimension vector α in Φ^+_{N}. Then if we let $L_{\mathrm{N}}(\lambda)$ be the irreducible highest weight $\mathcal{D}_{\mathrm{N}}(\Lambda)$-module with $\lambda \in X^+$, we have the following character formula.

Theorem 5.3 *For $\lambda \in X^+$, we have*

$$\operatorname{ch} L_{\mathrm{N}}(\lambda) = \frac{\sum_{w\in W,T}(-1)^{\ell(w)+|T|}e(w(\lambda+\rho+\delta(S(T)))-\rho)}{\prod_{\alpha\in\Phi^+}(1-e(-\alpha))^{I_{\mathrm{N}}(\alpha,q)}},$$

where T runs over all finite subsets of $(\Gamma_0 \cup J_{\mathrm{N}})^{\mathrm{im}}$ such that $(\lambda, \delta_i) = 0$ for $i \in T$ and $(\delta_i, \delta_j) = 0$ for $i \neq j$ in T. $|T|$ is the number of elements in T. $S(T)$ is the sum of elements in T.

Similarly as in Subsection 3.2, we can define $\Xi^{\pm}_{\theta}$ and $L^{\pm}_{\theta}$ for each $\theta \in \mathbb{N}\Gamma_0$. Then we can get a similar lemma as Lemma 3.1 with the third conclusion being given by:

Lemma 5.4 *If $L_\theta^\pm \neq 0$, then $\theta \in \mathcal{F}_{0_{\rm N}}$. And θ is the dimension vector of a nilpotent indecomposable representation.*

We also have the Kac theorem for the situation of nilpotent representations.

Theorem 5.4 (1) *$\Phi_{\rm N}^+ = \Delta_+$, the set of dimension vectors of nilpotent indecomposable representations of Γ is the positive root system of Γ.*

(2) *If $\alpha \in \Delta_+^{\rm re}$, then up to isomorphism, there exists unique nilpotent indecomposable representation of Γ of dimension vector α.*

We have the following corollary

Corollary 5.2 *For any indecomposable representation W of a valued quiver Γ, there exists a nilpotent indecomposable representation V of Γ such that $\underline{\dim} V = \underline{\dim} W$.*

Therefore any indecomposable representation with dimension vector being real root is nilpotent.

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