

# Global Asymptotics of Krawtchouk Polynomials — a Riemann-Hilbert Approach\*\*

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**Abstract** In this paper, we study the asymptotics of the Krawtchouk polynomials  $K_n^N(z; p, q)$  as the degree  $n$  becomes large. Asymptotic expansions are obtained when the ratio of the parameters  $\frac{n}{N}$  tends to a limit  $c \in (0, 1)$  as  $n \rightarrow \infty$ . The results are globally valid in one or two regions in the complex  $z$ -plane depending on the values of  $c$  and  $p$ ; in particular, they are valid in regions containing the interval on which these polynomials are orthogonal. Our method is based on the Riemann-Hilbert approach introduced by Deift and Zhou.

**Keywords** Global asymptotics, Krawtchouk polynomials, Parabolic cylinder functions, Airy functions, Riemann-Hilbert problems

**2000 MR Subject Classification** 41A60, 33C45

## 1 Introduction

Let  $p > 0$ ,  $q > 0$  and  $p + q = 1$ , and let  $N$  be a positive integer. By the binomial expansion, we have

$$(1 - pw)^{N-x}(1 + qw)^x = \sum_{n=0}^{\infty} K_n^N(x; p, q) w^n, \quad (1.1)$$

where

$$K_n^N(x; p, q) = \sum_{k=0}^n \binom{N-x}{n-k} \binom{x}{k} (-p)^{n-k} q^k. \quad (1.2)$$

For convenience, we put  $K_n^N(x) \equiv K_n^N(x; p, q)$ . It is easy to see that  $K_n^N(x)$  is a polynomial in  $x$  of degree  $n$ . These polynomials are known as the Krawtchouk polynomials, and they are orthogonal on the discrete set  $\{0, 1, 2, \dots, N\}$  with respect to the weight function

$$\rho(x) = \binom{N}{x} p^x q^{N-x}, \quad x = 0, 1, \dots, N. \quad (1.3)$$

More precisely, we have

$$\sum_{j=0}^N K_n^N(j) K_m^N(j) \binom{N}{j} p^j q^{N-j} = \binom{N}{n} p^n q^n \delta_{n,m}, \quad n, m = 0, 1, \dots, N. \quad (1.4)$$

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Manuscript received May 12, 2006.

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\*\*Project supported by the the Research Grants Council of the Hong Kong Special Administrative Region, China (No. CityU 102504).

For further properties of this type of polynomial, we refer to Szegő [22].

Formula (1.4) tells us that these polynomials are orthogonal on an unbounded interval as  $n \rightarrow \infty$ . To facilitate our future analysis, we wish to introduce a rescaling so that the polynomials become orthogonal on a bounded interval. Let  $X_N$  be a set defined by:

$$X_{N,j} := \{x_{N,j} \mid j = 0, 1, \dots, N-1\},$$

where

$$x_{N,j} = \frac{j + \frac{1}{2}}{N}.$$

The  $x_{N,j}$ 's are called nodes, and they all lie in the interval  $(0, 1)$ . Also, let

$$w_{N,j} := \frac{N^{N-1} \sqrt{pq}}{q^N \Gamma(N)} \binom{N-1}{j} p^j q^{N-1-j} \quad (1.5)$$

and

$$P_{N,n}(z) := K_n^{N-1} \left( Nz - \frac{1}{2} \right).$$

It can be easily verified that the polynomials  $P_{N,n}(x)$  are orthogonal on the nodes  $x_{N,j}$  with respect to the weight  $w_{N,j}$ ; that is

$$\sum_{j=0}^{N-1} P_{N,n}(x_{N,j}) P_{N,m}(x_{N,j}) w_{N,j} \begin{cases} = 0 & \text{for } n \neq m, \\ \neq 0 & \text{for } n = m. \end{cases} \quad (1.6)$$

As usual, we also define the monic polynomials

$$\pi_{N,n}(z) := \frac{n!}{N^n} P_{N,n}(z). \quad (1.7)$$

Rewriting the weight function (1.5) in the form

$$w_{N,j} = e^{-NW_N(x_{N,j})} \prod_{\substack{m=0 \\ m \neq j}}^{N-1} |x_{N,j} - x_{N,m}|^{-1}, \quad (1.8)$$

then a simple calculation shows

$$W_N(x) \equiv W(x) := \nu x, \quad (1.9)$$

where  $\nu = \log \frac{q}{p}$ . Without loss of generality, we can assume  $0 < p < q < 1$  so that  $\nu > 0$ . Also, let  $c_n = \frac{n}{N}$ , and assume that it tends to a limit  $c \in (0, 1)$  as  $n \rightarrow \infty$ . The case  $p = q = \frac{1}{2}$  is trivial and we will not consider it here.

It is well known that the properties of zeros of Krawtchouk polynomials are important in the study of the Hamming scheme of classical coding theory (see [4, 14, 16, 21]). Also, Lloyd's theorem (see [13, 16]) states that the existence of a perfect code in the Hamming metric corresponds to the Krawtchouk polynomials having integer zeros. Recently, there has been a considerable amount of interest in the asymptotics of the Krawtchouk polynomials, when the degree  $n$  grows to infinity (see [11, 15, 19]).

Since the Krawtchouk polynomials do not satisfy a differential equation, most of the results in the literature are obtained by using the steepest descent method or the saddle point method for integrals, which come from the generating function in (1.1) (see [11, 15, 19]). For more information about these classical integral methods, we refer to Wong [25]. Recently, Baik et al [2, 3] have studied the asymptotics of discrete orthogonal polynomials with respect to a general weight function by using the Riemann-Hilbert approach, introduced by Deift and Zhou [8] and further developed in [6, 7]. The starting point of this method is a theorem of Fokas, Its and Kitaev [10], which makes a beautiful connection between orthogonal polynomials and Riemann-Hilbert problems (RHP). However, the results in [3] are too general and, as a consequence, not very applicable. Moreover, the results are local in nature; that is, they have different asymptotic formulas valid in different regions.

The purpose of this paper is to study uniform asymptotic behavior of the polynomial  $P_{N,n}(z)$  as  $n \rightarrow \infty$ . After transforming the discrete RHP associated with this polynomial into a specific continuous one, we find that this problem is similar to some of the problems considered previously (see, e.g., [23, 24, 26]), and our method in [5] can be applied. More precisely, for  $0 < c < p$ , we present an infinite asymptotic expansion which is valid uniformly in the whole complex plane bounded away from  $(-\infty, 0]$  and  $[1, \infty)$ . This expansion involves parabolic cylinder functions. For the case  $p < c < \frac{1}{2}$ , since there exists a so-called hard edge (see [3, p. 27]), we need two expansions each valid in a different region; these regions overlap and together cover the whole complex plane bounded away from the two infinite lines on the real axis, mentioned in the former case. Since there is a kind of dual property between the cases  $c$  and  $1 - c$ , the result for the case  $\frac{1}{2} < c < 1$  is very similar to that for the case  $0 < c < \frac{1}{2}$ .

The presentation of this paper is arranged as follows: In Section 2, we review some preliminaries, including weak asymptotics of the zero distribution and the formulation of the first RHP. In Sections 3 and 4, we solve the RHP in two different cases:  $0 < c < p$  and  $p < c < \frac{1}{2}$ . The limiting case  $c = p$  is quite different, and the method used here is not applicable. We will study this exceptional case in a separate paper.

## 2 Preliminaries

### 2.1 Weak asymptotics

From the orthogonal properties in (1.4), the following proposition can be easily established (see also [3]).

**Proposition 2.1** *Each discrete polynomial  $P_{N,n}(z)$  has  $n$  simple real zeros. All zeros lie in the range  $x_{N,0} < z < x_{N,N-1}$  and no more than one zero lies in the closed interval  $[x_{N,j}, x_{N,j+1}]$  between any two consecutive nodes.*

It is known that the zero distribution of these kinds of discrete orthogonal polynomials is related to a constrained equilibrium problem for logarithmic potential with an external field  $\varphi(x)$ , which is given by the formula

$$\varphi(x) := W(x) + \int_a^b \log|x-y|\rho^0(y)dy \quad \text{for } x \in (a, b) \quad (2.1)$$

(see [12] and references therein). Here,  $\rho^0(y)$  is the density function of the nodes and is real analytic in a complex neighborhood of  $[a, b]$ . In our case,  $\rho^0(y) = 1$  and the external field  $\varphi(x)$  is given explicitly as

$$\varphi(x) = \nu x + \int_0^1 \log |x - y| dy = \nu x + x \log x + (1 - x) \log(1 - x) - 1. \quad (2.2)$$

Note that  $e^{-N\varphi(x)}$  is the limit of  $w_{N,j}$  in (1.8). The function  $\varphi(x)$  is real analytic in the interval  $(0, 1)$ , and the variational problem related to the Krawtchouk polynomials can be stated as follows:

With  $\varphi(x)$  and  $c$  given, the variational problem is to find a Borel measure  $\mu_c$  on  $[0, 1]$  which minimizes the following energy functional

$$E_c[\mu_c] := -c \int_0^1 \int_0^1 \log |x - y| \mu_c(x) dx \mu_c(y) dy + \int_0^1 \varphi(x) \mu_c(x) dx, \quad (2.3)$$

where  $\mu_c(x)$  satisfies the upper and lower constraints

$$0 \leq \mu_c(x) \leq \frac{1}{c} \quad (2.4)$$

and the normalization condition

$$\int_0^1 \mu_c(x) dx = 1. \quad (2.5)$$

The minimizer is called the equilibrium measure. From (2.4) one can see that there is an upper bound for the measure  $\mu_c$ , which does not appear in the continuous case. This fact is the key difference between discrete orthogonal polynomials and continuous ones. This can also be seen in Proposition 2.1, since the equilibrium measure reflects the distribution of the zeros of the orthogonal polynomials.

The equilibrium measure  $\mu_c(x)dx$  divides the interval  $[0, 1]$  into subintervals of three kinds: (1) achieving the lower constraint; (2) attaining the upper constraint; (3) not reaching the constraints. We call the open intervals of type (1) Voids (V), type (2) Saturated Regions (S), and type (3) Bands (B). These terminologies are taken from [3].

The theory about the existence of a unique minimizer measure under the constraint in an external field is well established (see [20]). Recently, Dragnev and Saff [9] have given the exact density function for the zeros of Krawtchouk polynomials in three different cases. More precisely, let

$$\alpha_c := (\sqrt{(1-c)p} - \sqrt{cq})^2, \quad \beta_c := (\sqrt{(1-c)p} + \sqrt{cq})^2. \quad (2.6)$$

They show that for  $0 < c < p$ , it is a V-B-V case, which means  $\mu_c(x)dx$  is supported on  $[\alpha_c, \beta_c] \subset [0, 1]$ , and the density function is given by

$$\mu_c(x) = \frac{1}{c\pi} \left[ \frac{\pi}{2} - \arctan \sqrt{\frac{\alpha_c(\beta_c - x)}{\beta_c(x - \alpha_c)}} - \arctan \sqrt{\frac{(1 - \beta_c)(x - \alpha_c)}{(1 - \alpha_c)(\beta_c - x)}} \right]. \quad (2.7)$$

For  $p \leq c < q$ , this is an S-B-V case; that is  $\mu_c(x) = \frac{1}{c}$  for  $x \in [0, \alpha_c]$ ,

$$\mu_c(x) = \frac{1}{c\pi} \left[ \frac{\pi}{2} + \arctan \sqrt{\frac{\alpha_c(\beta_c - x)}{\beta_c(x - \alpha_c)}} - \arctan \sqrt{\frac{(1 - \beta_c)(x - \alpha_c)}{(1 - \alpha_c)(\beta_c - x)}} \right] \quad (2.8)$$

for  $x \in [\alpha_c, \beta_c]$ , and  $\mu_c(x) = 0$  for  $x \in [\beta_c, 1]$ . For  $q \leq c < 1$ , this is an S-B-S case, i.e.,

$$\mu_c(x) = \frac{1}{c\pi} \left[ \frac{\pi}{2} + \arctan \sqrt{\frac{\alpha_c(\beta_c - x)}{\beta_c(x - \alpha_c)}} + \arctan \sqrt{\frac{(1 - \beta_c)(x - \alpha_c)}{(1 - \alpha_c)(\beta_c - x)}} \right] \quad (2.9)$$

for  $x \in [\alpha_c, \beta_c]$ , and  $\mu_c(x) = \frac{1}{c}$  for  $x \in [0, \alpha_c] \cup [\beta_c, 1]$ . From here, it can be seen that the zero density function  $\mu_c(x)$  satisfies a symmetry property in  $c$  and  $1 - c$ ; more precisely, we have

$$\mu_{1-c}(1 - x) = \frac{1}{1 - c} [1 - c\mu_c(x)] \quad \text{for } x \in [0, 1]. \quad (2.10)$$

Notice that there are two critical values  $c = p$  and  $c = q$ . In the two cases,  $\alpha_p$  and  $\beta_q$  coincide with the left endpoint 0 and the right endpoint 1, respectively (see (2.6)). Furthermore,  $\mu_c(x)$  does not reach the upper or the lower constraints at the points  $\alpha_p$  and  $\beta_q$ , which is different from other cases; more precisely,  $\mu_p(0) = \frac{1}{2p}$  and  $\mu_q(1) = \frac{1}{2q}$ . These two cases are special, and the method in [3] is not applicable. As mentioned in Section 1, we will study these special cases in a separate paper. For convenience, in situations with no confusion we will ignore the dependence of  $c$  and use the simpler notations  $\mu(x)$ ,  $\alpha$  and  $\beta$  instead of  $\mu_c(x)$ ,  $\alpha_c$  and  $\beta_c$ , respectively.

## 2.2 Riemann-Hilbert problem

Like the continuous orthogonal polynomials, it is easily verified that the discrete ones are also connected with RHP (see [3]). For instance, the discrete RHP for Krawtchouk polynomials  $P_{N,n}(z)$  can be stated as follows:

(Y<sub>a</sub>)  $Y(z)$  is analytic for  $z \in \mathbb{C} \setminus X_N$ ;

(Y<sub>b</sub>) at each  $x_{N,k} \in X_N$ , the second column of  $Y$  has a simple pole where the residue is

$$\operatorname{Res}_{z=x_{N,k}} Y(z; N, n) = \lim_{z \rightarrow x_{N,k}} Y(z; N, n) \begin{pmatrix} 0 & \frac{w_{N,k}}{2i} \\ 0 & 0 \end{pmatrix}; \quad (2.11)$$

(Y<sub>c</sub>) as  $z \rightarrow \infty$ ,

$$Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Using the theorem of Fokas, Its and Kitaev [10], it can be shown that the solution to the above RHP is

$$Y(z) = \begin{pmatrix} \pi_{N,n}(z) & \frac{1}{2i} \sum_{k=0}^{N-1} \frac{w_{N,k} \pi_{N,n}(x_{N,k})}{z - x_{N,k}} \\ \gamma_{N,n-1} P_{N,n-1}(z) & \frac{1}{2i} \sum_{k=0}^{N-1} \frac{\gamma_{N,n-1} w_{N,k} P_{N,n-1}(x_{N,k})}{z - x_{N,k}} \end{pmatrix}. \quad (2.12)$$

The proof of this result is very similar to that of the continuous case, just using the discrete orthogonal property instead of the continuous one (see [3]). Here,  $\pi_{N,n}(z)$  is the monic polynomial defined in (1.7).

If we can transform a discrete Riemann-Hilbert problem into a continuous one, then we can apply the techniques that we have developed in [5, 23]. Due to the sensitivity of parameter  $c$ , this transformation is different in different cases.

### 3 Case I $0 < c < p$

In this V-B-V case, the upper constraint is not active. To get a continuous RHP, we introduce the first transformation

$$R(z) := Y(z) \begin{pmatrix} 1 & \mp \frac{1}{2} e^{\mp i N \pi (1-z)} e^{-N \nu z} \frac{1}{\prod_{k=0}^{N-1} (z - x_{N,k})} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Omega_{\pm}. \quad (3.1)$$

For  $z \notin \Omega_{\pm}$ , we put  $R(z) := Y(z)$ . The regions  $\Omega_{\pm}$  and the contour  $\Sigma = (0, 1) \cup \Sigma_{\pm}$  are depicted in Figure 1. It is easy to verify that this transformation removes all the poles  $x_{N,k}$ , and makes  $R_+(x)$  and  $R_-(x)$  continuous on the interval  $(0, 1)$ . Therefore, we get the continuous RHP for  $R$  given below.

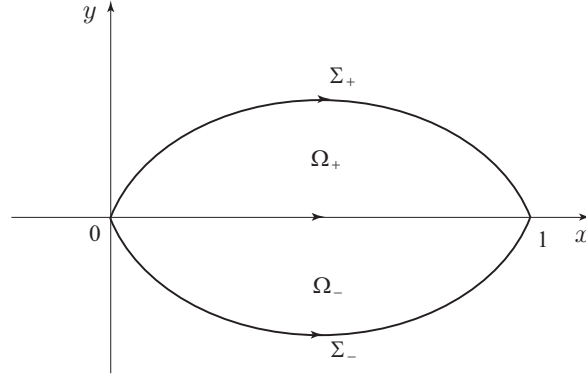


Figure 1 The domains  $\Omega_{\pm}$  and the contour  $\Sigma$

(R<sub>a</sub>)  $R(z)$  is analytic in  $\mathbb{C} \setminus \Sigma$ ;

(R<sub>b</sub>) for  $x \in (0, 1)$ ,

$$R_+(x) = R_-(x) \begin{pmatrix} 1 & r_n(x) \\ 0 & 1 \end{pmatrix}, \quad (3.2)$$

where

$$r_n(x) = (-1)^{N-1} \cos(N\pi x) e^{-N\nu x} \frac{1}{\prod_{k=0}^{N-1} (x - x_{N,k})}; \quad (3.3)$$

for  $z \in \Sigma_+ \cup \Sigma_-$ ,

$$R_+(z) = R_-(z) \begin{pmatrix} 1 & \tilde{r}_{n,\pm}(z) \\ 0 & 1 \end{pmatrix}, \quad (3.4)$$

where

$$\tilde{r}_{n,\pm}(z) = -\frac{1}{2} e^{\mp iN\pi(1-z)} e^{-N\nu z} \frac{1}{\prod_{k=0}^{N-1} (z - x_{N,k})}; \quad (3.5)$$

(R<sub>c</sub>) as  $z \rightarrow \infty$ ,

$$R(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix};$$

(R<sub>d</sub>) as  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$R(z) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (3.6)$$

where the  $O$ -symbol on the right-hand side is used to mean that all entries in the matrix are  $O(1)$ .

By using l'Hospital's rule, it can be easily verified that the entry  $r_n(x)$  in the jump matrix (3.2) is continuous in  $(0, 1)$  and can be extended into the complex plane. More precisely, by using the Gamma function and its reflection formula  $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ , (3.3) can be rewritten as

$$r_n(z) = -\frac{N^N \pi e^{-N\nu z}}{\Gamma(Nz + \frac{1}{2}) \Gamma(N(1-z) + \frac{1}{2})}, \quad (3.7)$$

which is an entire function.

Because of the term  $e^{\mp iN\pi(1-z)}$  in the definition of  $\tilde{r}_{n,\pm}(z)$ , we find that it is exponentially small as  $N \rightarrow \infty$ , comparing with  $r_n(x)$  in the jump matrix on  $(0, 1)$ . This fact indicates that the jump matrix on  $(0, 1)$  dominates the jump matrices on  $\Sigma_+$  and  $\Sigma_-$ .

Before we set out to solve our problem, we need several auxiliary functions. First, we assume that the equilibrium measure  $\mu_n(x)$  related to the weight function  $r_n(x)$  is supported on the interval  $[\alpha_n, \beta_n] \subset [0, 1]$ , where the constants  $\alpha_n, \beta_n$  are to be determined later. Thus,  $\mu_n(x) \geq 0$  for  $\alpha_n \leq x \leq \beta_n$  and

$$\int_{\alpha_n}^{\beta_n} \mu_n(x) dx = 1. \quad (3.8)$$

Then, we introduce two auxiliary functions given below.

**Definition 3.1** *The so-called  $g$ -function is the complex logarithmic potential of the corresponding measure  $\mu_n$ , that is*

$$g_n(z) := \int_{\alpha_n}^{\beta_n} \log(z-s) \mu_n(s) ds, \quad z \in \mathbb{C} \setminus (-\infty, \beta_n]. \quad (3.9)$$

**Definition 3.2** *The  $\phi$ -function is defined by*

$$\phi_n(z) := \int_{\beta_n}^z v_n(s) ds, \quad z \in \mathbb{C} \setminus (-\infty, \beta_n] \cup [1, +\infty). \quad (3.10)$$

Here  $\mu_n(x)$  and  $v_n(z)$  are two measures to be determined later. Furthermore,  $v_n(z)$  is a complex measure extended from  $\mu_n(x)$  and satisfies the requirement

$$v_{n,\pm}(x) = \pm \pi i \mu_n(x) \quad \text{for } x \in (\alpha_n, \beta_n). \quad (3.11)$$

### 3.1 Determination of the auxiliary functions

Following the usual argument, we first find a probability measure  $\mu_n(x)$  such that the related  $g$ -function satisfies the property

$$\left(n + \frac{1}{2}\right)(g_{n,+}(x) + g_{n,-}(x)) + \log \left\{ -\frac{N^N \pi e^{-N\nu x}}{\Gamma(Nx + \frac{1}{2}) \Gamma(N(1-x) + \frac{1}{2})} \right\} = 0 \quad (3.12)$$

for  $x \in (\alpha_n, \beta_n)$ . (This is essentially what is needed in the normalization of the RHP for  $R$ .) Differentiating both sides yields

$$\left(n + \frac{1}{2}\right)(g'_{n,+}(x) + g'_{n,-}(x)) - N \left[ \nu + \psi \left( Nx + \frac{1}{2} \right) - \psi \left( \frac{1}{2} + N - Nx \right) \right] = 0,$$

where  $\psi(z)$  is the Psi function, which is the logarithmic derivative of the Gamma function given by

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Therefore, we get

$$g'_{n,+}(x) + g'_{n,-}(x) = \frac{1}{\tilde{c}_n} \left( \nu + \psi \left( Nx + \frac{1}{2} \right) - \psi \left( \frac{1}{2} + N - Nx \right) \right), \quad (3.13)$$

where  $\tilde{c}_n = \frac{n+1}{N}$  and tends to the limit  $c$  as  $n \rightarrow \infty$ . For simplicity, we introduce the notation

$$h_n(x) := \nu + \psi \left( Nx + \frac{1}{2} \right) - \psi \left( \frac{1}{2} + N - Nx \right). \quad (3.14)$$

We need one more function  $G_n(z)$  defined by

$$G_n(z) := \frac{1}{\pi i} \int_{\alpha_n}^{\beta_n} \frac{\mu_n(s)}{s-z} ds = \frac{i}{\pi} g'_n(z), \quad z \in \mathbb{C} \setminus [\alpha_n, \beta_n]. \quad (3.15)$$

The following result provides an important relation between the functions  $g_n(z)$  and  $\phi_n(z)$ .

**Proposition 3.1** *Let  $l_n$  be defined as*

$$l_n := 2g_n(\beta_n) + \frac{\log r_n(\beta_n)}{n + \frac{1}{2}}. \quad (3.16)$$

*Then, the following relation between the  $g$ -function and  $\phi$ -function holds*

$$g_n(z) + \phi_n(z) = -\frac{1}{2n+1} \log r_n(z) + \frac{l_n}{2} \quad (3.17)$$

*for  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ , where  $r_n(z)$  is given in (3.7).*

**Proof** Coupling (3.13) and (3.15) yields

$$G_{n,+}(x) + G_{n,-}(x) = \frac{i}{\pi \tilde{c}_n} h_n(x). \quad (3.18)$$

Since  $G_{n,\pm}(x) = \lim_{\varepsilon \rightarrow 0} G_n(x \pm i\varepsilon)$ , it follows that

$$G_{n,\pm}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi i} \int_{\alpha_n}^{\beta_n} \frac{\mu_n(s)(s - x \pm i\varepsilon)}{(s-x)^2 + \varepsilon^2} ds = \pm \mu_n(x) + \frac{1}{\pi i} \text{P.V.} \int_{\alpha_n}^{\beta_n} \frac{\mu_n(s)}{s-x} ds. \quad (3.19)$$



This means  $\operatorname{Re} G_{n,\pm}(x) = \pm\mu_n(x)$  and  $\operatorname{Im} G_n(x) \equiv \operatorname{Im} G_{n,\pm}(x)$  for  $x \in (\alpha_n, \beta_n)$ . Furthermore, from (3.18) we also have

$$\operatorname{Im} G_n(x) = \frac{1}{2\tilde{c}_n\pi} h_n(x). \quad (3.20)$$

Recalling the requirement (3.11) between  $\mu_n(x)$  and  $v_n(z)$ , we need

$$v_{n,+}(x) = \pi i \mu_n(x) = \pi i \operatorname{Re} G_{n,+}(x). \quad (3.21)$$

This evokes us to define  $v_n(z)$  to be

$$v_n(z) = \pi i (G_n(z) - i \operatorname{Im} G_n(z)).$$

From (3.20), we get

$$v_n(z) = \pi i G_n(z) + \frac{1}{2\tilde{c}_n} h_n(z). \quad (3.22)$$

Since  $g'_n(z) = -\pi i G_n(z)$ , one easily obtains (3.17) by integrating both sides of the formula (3.22) from  $\beta_n$  to  $z$ . Furthermore, since  $\phi_n(\beta_n) = 0$ , letting  $z = \beta_n$  in (3.17) immediately gives (3.16).

Now, let us find  $\alpha_n$ ,  $\beta_n$  and  $\mu_n(x)$ . Equation (3.18) is actually a scalar RHP, and can be solved explicitly as

$$G_n(z) = \frac{\sqrt{(z - \alpha_n)(z - \beta_n)}}{2\pi i} \int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\tilde{c}_n \pi \sqrt{(s - \alpha_n)(\beta_n - s)}} \frac{ds}{s - z} \quad (3.23)$$

for  $z \in \mathbb{C} \setminus [\alpha_n, \beta_n]$ . From the definition of  $g_n(z)$ , we know that

$$g'_n(z) = \int_{\alpha_n}^{\beta_n} \frac{1}{z - s} \mu_n(s) ds = \frac{1}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty.$$

Since  $G_n(z) = \frac{i}{\pi} g'_n(z)$ , from the above formula we can obtain  $G_n(z) \rightarrow 0$  and  $zG_n(z) \rightarrow \frac{i}{\pi}$  as  $z \rightarrow \infty$ . Therefore, one gets two integral equations for  $\alpha_n$  and  $\beta_n$ :

$$\int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\sqrt{(s - \alpha_n)(\beta_n - s)}} ds = 0 \quad (3.24)$$

and

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{s h_n(s)}{\sqrt{(s - \alpha_n)(\beta_n - s)}} ds = \tilde{c}_n, \quad (3.25)$$

where  $\tilde{c}_n = \frac{n+\frac{1}{2}}{N}$  as before. To solve them, we need more information about the function  $h_n(x)$ . Recall the asymptotic expansion for the  $\psi$ -function given by

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2r} z^{-2r} \quad \text{as } z \rightarrow \infty \text{ in } |\arg z| < \pi,$$

where  $B_{2r}$  are the Bernoulli numbers (see [1, (6.3.18)]). Therefore,  $h_n(x)$  has an asymptotic expansion of the form

$$h_n(x) \sim \nu + \ln x - \ln(1-x) + \sum_{k=2}^{\infty} \frac{h^{(k)}(x)}{n^k} \quad \text{as } n \rightarrow \infty, \quad (3.26)$$

where the coefficient functions  $h^{(k)}(x)$  can be given explicitly; for example, we have

$$h^{(2)}(x) = \frac{c^2}{24} \left( \frac{1}{x^2} - \frac{1}{(1-x)^2} \right).$$

Moreover, to derive the asymptotic expansions of  $\alpha_n$  and  $\beta_n$ , we assume  $c_n$  has an asymptotic expansion of the form

$$c_n \sim c + \sum_{k=1}^{\infty} \frac{c_k}{n^k}. \quad (3.27)$$

Substituting (3.26) and (3.27) into (3.24) and (3.25), we also obtain the asymptotic expansions

$$\alpha_n \sim \alpha + \sum_{k=1}^{\infty} \frac{\alpha^{(k)}}{n^k}, \quad \beta_n \sim \beta + \sum_{k=1}^{\infty} \frac{\beta^{(k)}}{n^k}, \quad (3.28)$$

where  $\alpha$  and  $\beta$  are given in (2.6). Since  $\mu_n(x) = \operatorname{Re} G_{n,+}(x)$ , we have from (3.23)

$$\mu_n(x) = \frac{\sqrt{(x - \alpha_n)(\beta_n - x)}}{2\pi} \operatorname{P.V.} \int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\tilde{c}_n \pi \sqrt{(s - \alpha_n)(\beta_n - s)} (s - x)} ds \quad (3.29)$$

for  $x \in (\alpha_n, \beta_n)$ , and it can be shown that

$$\mu_n(x) = \mu(x) + O\left(\frac{1}{n}\right) \quad (3.30)$$

uniformly for  $x \in [\alpha, \beta]$ , where  $\mu(x)$  is given in (2.7).

Once the measure  $\mu_n(x)$  is determined,  $v_n(z)$  is well defined. Furthermore, one can obtain the important mapping properties of the function  $\phi_n(z)$  as shown in Figure 2, where we have used the same letters to indicate corresponding points on the boundary. From the definition of  $\phi_n(z)$  in (3.10), it is easy to see that  $\phi_n(\beta_n) = 0$  and  $\phi_{n,+}(\alpha_n) = -\pi i$ .

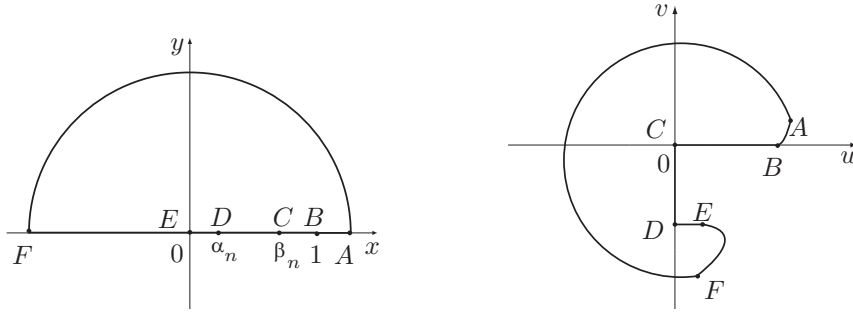


Figure 2 The upper half plane under the transformation of  $\phi_n(z)$

### 3.2 Construction of the parametrix

As usual, to simplify the jump conditions in  $(R_b)$ , we introduce the transformation  $R \rightarrow V$  defined by

$$V(z) := e^{-\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} R(z) r_n(z)^{\frac{1}{2}\sigma_3}, \quad (3.31)$$

where  $r_n(z)$  is given in (3.7) and  $\sigma_3$  is the Pauli matrix. It is readily seen that  $V$  is a solution of the following RHP:

$$(V_a) \quad V(z) \text{ is analytic for } z \in \mathbb{C} \setminus \Sigma;$$

$$(V_b) \quad \text{for } x \in (0, 1),$$

$$V_+(x) = V_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad (3.32)$$

for  $z \in \Sigma_{\pm}$ ,

$$V_+(z) = V_-(z) \begin{pmatrix} 1 & \frac{1}{1+e^{\mp 2N\pi iz}} \\ 0 & 1 \end{pmatrix}; \quad (3.33)$$

$$(V_c) \quad \text{as } z \rightarrow \infty,$$

$$V(z) = \left( I + O\left(\frac{1}{z}\right) \right) z^{-\frac{1}{2}\sigma_3} e^{-(n+\frac{1}{2})\phi_n(z)\sigma_3},$$

$$(V_d) \quad \text{as } z \rightarrow 0 \text{ and } z \rightarrow 1,$$

$$V(z) = O\left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right).$$

To construct a parametrix for the above RHP, we recall the properties of  $\phi_n(z)$  illustrated in Figure 2, which are very similar to those of the function

$$f(\xi) := \xi\sqrt{\xi^2 - 1} - \log(\xi + \sqrt{\xi^2 - 1}), \quad (3.34)$$

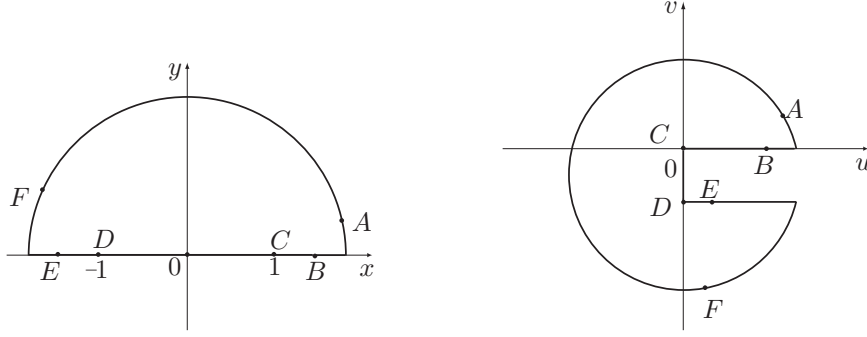
where  $\xi \in \mathbb{C} \setminus (-\infty, 1]$  (see Figure 3). Simple calculation gives  $f(1) = 0$  and  $f_+(-1) = -\pi i$ . The function  $f(\xi)$  plays an important role in describing the asymptotic behavior of the parabolic cylinder function  $U(-\tau, 2\sqrt{\tau}\xi)$  as  $\tau \rightarrow \infty$  (see [1, 18]). This fact invokes us to construct our approximate solution by using the parabolic cylinder function, and to introduce the mapping between  $\xi \leftrightarrow z$  defined by

$$f(\xi(z)) = \phi_n(z), \quad (3.35)$$

or equivalently

$$\xi(z) = (f^{-1} \circ \phi_n)(z). \quad (3.36)$$

As  $z$  traverses the boundary of the semi-circular region in the upper half-plane once, the image point  $\phi_n(z)$  also traverses once the corresponding boundary in Figure 2. Similarly, as  $\xi$  describes the semi-circular region once,  $f(\xi)$  goes once along the corresponding curve in Figure 3. This

Figure 3 The upper half plane under the transformation of  $f(\xi)$ 

establishes the one-to-oneness of the mapping  $z \leftrightarrow \xi$  on the boundary of the semi-circular region. By Theorem 4.5 in [17, Vol. 2, p. 118], this mapping is also one-to-one in the interior of the region. By Schwarz's reflection principle, the transformation  $z \leftrightarrow \xi$  defined in (3.35) is, in fact, one-to-one and analytic in the whole  $z$ -plane with two cuts  $(-\infty, 0]$  and  $[1, \infty)$ . From condition  $(V_c)$  and the asymptotic property of  $U(-\tau, 2\sqrt{\tau}\xi)$ , it is readily seen that we should take the parameter  $\tau$  to be  $n + \frac{1}{2}$ . The advantage of adopting the parabolic cylinder function, over the Airy function as done in [5], is that the region of validity of the asymptotic expansion we get here can include both the critical values  $\alpha_n$  and  $\beta_n$ , whereas the region of validity of the Airy-type expansion given in [5] includes only one critical value.

We now begin to construct the parametrix. From formula (19.4.7) in [1], we have

$$\sqrt{2\pi} U(a, \pm x) = \Gamma\left(\frac{1}{2} - a\right) \{e^{-i\pi(\frac{1}{2}a + \frac{1}{4})} U(-a, \pm ix) + e^{i\pi(\frac{1}{2}a + \frac{1}{4})} U(-a, \mp ix)\}, \quad (3.37)$$

which provides the matrix equation

$$\begin{aligned} & \begin{pmatrix} U(-\tau, 2\sqrt{\tau}\xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi\frac{n}{2}} U(\tau, 2i\sqrt{\tau}\xi) \\ U'(-\tau, 2\sqrt{\tau}\xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi\frac{n+1}{2}} U'(\tau, 2i\sqrt{\tau}\xi) \end{pmatrix} \\ &= \begin{pmatrix} U(-\tau, 2\sqrt{\tau}\xi) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi\frac{n}{2}} U(\tau, -2i\sqrt{\tau}\xi) \\ U'(-\tau, 2\sqrt{\tau}\xi) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi\frac{n+1}{2}} U'(\tau, -2i\sqrt{\tau}\xi) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (3.38)$$

where

$$\tau = \tau_n := n + \frac{1}{2}. \quad (3.39)$$

For convenience, we sometimes suppress the dependence of  $\tau$  on the large variable  $n$ . We also recall the asymptotic expansions

$$\begin{aligned} U(-\tau, 2\sqrt{\tau}\xi) &\sim 2^{-\frac{1}{2}} e^{-\frac{1}{2}\tau} \tau^{\frac{n}{2}} \frac{e^{-\tau f(\xi)}}{(\xi^2 - 1)^{\frac{1}{4}}}, \\ U'(-\tau, 2\sqrt{\tau}\xi) &\sim -2^{-\frac{1}{2}} e^{-\frac{1}{2}\tau} \tau^{\frac{1}{2}(\tau + \frac{1}{2})} (\xi^2 - 1)^{\frac{1}{4}} e^{-\tau f(\xi)} \end{aligned}$$

as  $n \rightarrow \infty$ , uniformly for  $\xi \in \mathbb{C} \setminus (-\infty, 1]$  (see [18, p. 140]). With the above results, simple calculation yields

$$\begin{aligned} & \begin{pmatrix} U(-\tau, 2\sqrt{\tau} \xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n}{2}} U(\tau, 2i\sqrt{\tau} \xi) \\ U'(-\tau, 2\sqrt{\tau} \xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n+1}{2}} U'(\tau, 2i\sqrt{\tau} \xi) \end{pmatrix} \\ & \sim \frac{1}{\sqrt{2}} (\xi^2 - 1)^{-\frac{1}{4}\sigma_3} \begin{pmatrix} m_{11} e^{-\tau f(\xi)} & m_{12} e^{\tau f(\xi)} \\ m_{21} e^{-\tau f(\xi)} & m_{22} e^{\tau f(\xi)} \end{pmatrix}, \end{aligned} \quad (3.40)$$

where the constants  $m_{ij}$  are given by

$$\begin{aligned} m_{11} &:= e^{-\frac{1}{2}(n+\frac{1}{2})(n+\frac{1}{2})\frac{n}{2}}, & m_{12} &:= -i \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{\frac{1}{2}(n+\frac{1}{2})(n+\frac{1}{2})-\frac{n+1}{2}}, \\ m_{21} &:= -e^{-\frac{1}{2}(n+\frac{1}{2})(n+\frac{1}{2})\frac{n+1}{2}}, & m_{22} &:= -i \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{\frac{1}{2}(n+\frac{1}{2})(n+\frac{1}{2})-\frac{n}{2}}. \end{aligned} \quad (3.41)$$

Note that there is a relation between these constants, namely,  $-\frac{m_{11}}{m_{21}} = \frac{m_{12}}{m_{22}}$ .

For  $z \in \mathbb{C}_+$ , we define

$$\begin{aligned} Q(z) &:= \sqrt{2} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2z-\alpha_n-\beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} \left( \frac{(\xi^2 - 1)^{\frac{1}{4}}}{b_n(z)} \right)^{\sigma_3} \\ &\quad \times \begin{pmatrix} U(-\tau, 2\sqrt{\tau} \xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n}{2}} U(\tau, 2i\sqrt{\tau} \xi) \\ U'(-\tau, 2\sqrt{\tau} \xi) & \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n+1}{2}} U'(\tau, 2i\sqrt{\tau} \xi) \end{pmatrix}, \end{aligned} \quad (3.42)$$

where  $b_n(z)$  is the analytic function in  $\mathbb{C} \setminus (-\infty, \beta_n]$  given by

$$b_n(z) := (z - \beta_n)^{\frac{1}{4}} (z - \alpha_n)^{\frac{1}{4}}. \quad (3.43)$$

The reason why we divide  $(\xi^2 - 1)^{\frac{1}{4}}$  by  $b_n(z)$  is to make sure that  $(\xi^2 - 1)^{\frac{1}{4}}/b_n(z)$  is analytic in  $\mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ , with no jump on the interval  $(0, 1)$ . Furthermore, it can be easily verified that  $Q(z)$  satisfies the same large- $z$  behavior as  $V(z)$  given in  $(V_c)$ .

Similarly, we can construct the parametrix in the lower half plane. For  $z \in \mathbb{C}_-$ , we define

$$\begin{aligned} Q(z) &:= \sqrt{2} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2z-\alpha_n-\beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} \left( \frac{(\xi^2 - 1)^{\frac{1}{4}}}{b_n(z)} \right)^{\sigma_3} \\ &\quad \times \begin{pmatrix} U(-\tau, 2\sqrt{\tau} \xi) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi \frac{n}{2}} U(\tau, -2i\sqrt{\tau} \xi) \\ U'(-\tau, 2\sqrt{\tau} \xi) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi \frac{n+1}{2}} U'(\tau, -2i\sqrt{\tau} \xi) \end{pmatrix}. \end{aligned} \quad (3.44)$$

It is easy to see that  $Q(z)$  satisfies all four conditions in the RHP for  $V$ , except for a new jump on  $(-\infty, 0) \cup (1, \infty)$ . In view of the transformation from  $V(z)$  to  $R(z)$  introduced in (3.31), a reasonable parametrix to the RHP for  $R$  is given by

$$\tilde{R}(z) := e^{\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} Q(z) r_n(z)^{-\frac{1}{2}\sigma_3}. \quad (3.45)$$

Since  $r_n(z)$  in (3.7) is an entire function and  $\xi(z)$  in (3.36) is analytic on  $(0, 1)$ , it is readily verified that

$$\tilde{R}_+(x) = \tilde{R}_-(x) \begin{pmatrix} 1 & r_n(x) \\ 0 & 1 \end{pmatrix} \quad \text{for } x \in (0, 1).$$

But there is an additional jump matrix on the part  $(-\infty, 0] \cup [1, \infty)$  of the real axis. In the following subsection, we will show that this jump matrix tends to the identity matrix.

### 3.3 Uniform asymptotic expansions

Define the matrix

$$D(z) := e^{-\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} R(z) \tilde{R}^{-1}(z) e^{\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3}. \quad (3.46)$$

Since  $\tilde{R}(z)$  has the same jump as  $R(z)$  on the interval  $(0, 1)$ , the matrix  $D(z)$  satisfies the relation

$$D_+(x) = D_-(x), \quad x \in (0, 1).$$

Furthermore, since  $R(z)$  is analytic on  $(-\infty, 0) \cup (1, \infty)$ , it is easy to verify that  $D(z)$  is a solution of the following RHP:

(D<sub>a</sub>)  $D(z)$  is analytic in  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ ;

(D<sub>b</sub>) for  $x \in (-\infty, 0) \cup (1, \infty)$ ,

$$D_+(x) = D_-(x) J_D(x), \quad (3.47)$$

where

$$J_D(x) := e^{-\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} \tilde{R}_-(x) (\tilde{R}_+(x))^{-1} e^{\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3}; \quad (3.48)$$

(D<sub>c</sub>) for  $z \in \mathbb{C} \setminus \mathbb{R}$ ,

$$D(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty;$$

(D<sub>d</sub>) as  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$D(z) = O\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right).$$

To solve this problem, we need to derive the explicit asymptotic expansion of  $J_D(x)$ . Note that for  $x \in (1, \infty)$ , we have

$$U(-\tau, 2\sqrt{\tau} \xi_+(x)) \sim \rho(\sqrt{2\tau}) \frac{e^{-\tau f(\xi_+(x))}}{(\xi_+^2(x) - 1)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{A_s(\xi_+(x))}{(2\tau)^s}, \quad (3.49a)$$

$$U(\tau, 2i\sqrt{\tau} \xi_+(x)) \sim \rho(i\sqrt{2\tau}) \frac{e^{\tau f(\xi_+(x))}}{(\xi_+^2(x) - 1)^{\frac{1}{4}}} \sum_{s=0}^{\infty} \frac{A_s(\xi_+(x))}{(-2\tau)^s}, \quad (3.49b)$$

$$U'(-\tau, 2\sqrt{\tau} \xi_+(x)) \sim -\sqrt{\tau} \rho(\sqrt{2\tau}) (\xi_+^2(x) - 1)^{\frac{1}{4}} e^{-\tau f(\xi_+(x))} \sum_{s=0}^{\infty} \frac{B_s(\xi_+(x))}{(2\tau)^s}, \quad (3.49c)$$

$$U'(\tau, 2i\sqrt{\tau} \xi_+(x)) \sim -i\sqrt{\tau} \rho(i\sqrt{2\tau}) (\xi_+^2(x) - 1)^{\frac{1}{4}} e^{\tau f(\xi_+(x))} \sum_{s=0}^{\infty} \frac{B_s(\xi_+(x))}{(-2\tau)^s}, \quad (3.49d)$$

where

$$\rho(\sqrt{2\tau}) \sim 2^{-\frac{1}{2}} e^{-\frac{1}{2}\tau} \tau^{\frac{\tau}{2}-\frac{1}{4}} \left[ 1 + \frac{1}{2} \sum_{s=1}^{\infty} \frac{\gamma_s}{\tau^s} \right], \quad (3.50)$$

$$A_s(\xi) = \frac{u_s(\xi)}{(\xi^2 - 1)^{\frac{3}{2}s}}, \quad B_s(\xi) = \frac{v_s(\xi)}{(\xi^2 - 1)^{\frac{3}{2}s}} \quad (3.51)$$

(see [18, p. 140]). Here  $u_0(\xi) = v_0(\xi) = 1$ , and when  $s > 0$ , both  $u_s(\xi)$  and  $v_s(\xi)$  are polynomials of degree  $3s$  if  $s$  is odd, and  $3s - 2$  if  $s$  is even. Moreover, the polynomials  $u_s(\xi)$  and  $v_s(\xi)$  can be successively determined through the following recurrence relations:

$$(\xi^2 - 1)u'_s(\xi) - 3s\xi u_s(\xi) = r_{s-1}(\xi), \quad v_s(\xi) = u_s(\xi) + \frac{1}{2}\xi u_{s-1}(\xi) - r_{s-2}(\xi)$$

with  $r_{-1}(\xi) = 0$  and

$$8r_s(\xi) = (3\xi^2 + 2)u_s(\xi) - 12(s+1)\xi r_{s-1}(\xi) + 4(\xi^2 - 1)r'_{s-1}(\xi).$$

Therefore, it is easily seen from (3.51) that  $A_0(\xi) = B_0(\xi) \equiv 1$ , and for  $s = 1, 2, \dots$ ,

$$A_{2s}(\xi) = O\left(\frac{1}{\xi^2}\right), \quad B_{2s}(\xi) = O\left(\frac{1}{\xi^2}\right), \quad A_{2s-1} = O(1), \quad B_{2s-1} = O(1) \quad \text{as } \xi \rightarrow \infty. \quad (3.52)$$

The coefficients  $\gamma_s$  in the expansion (3.50) are given by the following formula

$$\Gamma\left(z + \frac{1}{2}\right) \sim (2\pi)^{\frac{1}{2}} e^{-z} z^z \sum_{s=0}^{\infty} \frac{\gamma_s}{z^s}, \quad |\arg z| < \pi,$$

and the first three terms are

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{24}, \quad \gamma_2 = \frac{1}{1152}.$$

Now let us construct the asymptotic expansion of  $J_D(x)$ . For  $x \in (1, \infty)$ , we get from (3.45) and (3.48)

$$\begin{aligned} J_D(x) &= e^{-\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} \tilde{R}_-(x) (\tilde{R}_+(x))^{-1} e^{\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} \\ &= Q_-(x) r_{n,-}(x)^{-\frac{1}{2}\sigma_3} r_{n,+}(x)^{\frac{1}{2}\sigma_3} Q_+(x)^{-1}. \end{aligned} \quad (3.53)$$

From the Wronskian

$$i U(-\tau, 2\sqrt{\tau} \xi) U'(\tau, 2i\sqrt{\tau} \xi) - U'(-\tau, 2\sqrt{\tau} \xi) U(\tau, 2i\sqrt{\tau} \xi) = -i e^{-\frac{n\pi i}{2}},$$

we have

$$\begin{aligned} J_D(x) &= \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} \left( \frac{(\xi_-^2(x) - 1)^{\frac{1}{4}}}{b_n(x)} \right)^{\sigma_3} \\ &\quad \times \begin{pmatrix} U(-\tau, 2\sqrt{\tau} \xi_-(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi \frac{n}{2}} U(\tau, -2i\sqrt{\tau} \xi_-(x)) \\ U'(-\tau, 2\sqrt{\tau} \xi_-(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-i\pi \frac{n+1}{2}} U'(\tau, -2i\sqrt{\tau} \xi_-(x)) \end{pmatrix} \\ &\quad \times r_{n,-}(x)^{-\frac{1}{2}\sigma_3} r_{n,+}(x)^{\frac{1}{2}\sigma_3} \frac{i\sqrt{2\pi}}{\Gamma(n+1)} \end{aligned}$$

$$\begin{aligned} & \times \begin{pmatrix} \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n+1}{2}} U'(\tau, 2i\sqrt{\tau} \xi_+(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{i\pi \frac{n}{2}} U(\tau, 2i\sqrt{\tau} \xi_+(x)) \\ -U'(-\tau, 2\sqrt{\tau} \xi_+(x)) & U(-\tau, 2\sqrt{\tau} \xi_+(x)) \end{pmatrix} \\ & \times \left( \frac{(\xi_-^2(x) - 1)^{\frac{1}{4}}}{b_n(x)} \right)^{-\sigma_3} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix}^{-1}. \end{aligned}$$

Inserting the expansions in (3.49) into the last formula yields

$$\begin{aligned} J_D(x) & \sim \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} b_n(x)^{-\sigma_3} \sum_{s=0}^{\infty} \frac{M_s(x)}{(2\tau)^s} \begin{pmatrix} \rho(\sqrt{2\tau}) & 0 \\ 0 & \rho(-i\sqrt{2\tau}) \end{pmatrix} \\ & \times e^{-(n+\frac{1}{2})f(\xi_-(x))\sigma_3} r_{n,-}(x)^{-\frac{1}{2}\sigma_3} r_{n,+}(x)^{\frac{1}{2}\sigma_3} e^{(n+\frac{1}{2})f(\xi_+(x))\sigma_3} \frac{i\sqrt{2\pi}}{\Gamma(n+1)} \\ & \times \begin{pmatrix} \rho(i\sqrt{2\tau}) & 0 \\ 0 & \rho(\sqrt{2\tau}) \end{pmatrix} \sum_{s=0}^{\infty} \frac{N_s(x)}{(2\tau)^s} b_n(x)^{\sigma_3} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix}^{-1}, \quad (3.54) \end{aligned}$$

where

$$\begin{aligned} M_s(x) & := \begin{pmatrix} A_s(\xi_-(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-\pi i \frac{n}{2}} (-1)^s A_s(\xi_-(x)) \\ -\sqrt{\tau} B_s(\xi_-(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{-\pi i \frac{n}{2}} \sqrt{\tau} (-1)^s B_s(\xi_-(x)) \end{pmatrix}, \\ N_s(x) & := \begin{pmatrix} \frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{\pi i \frac{n}{2}} \sqrt{\tau} (-1)^s B_s(\xi_+(x)) & -\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{\pi i \frac{n}{2}} (-1)^s A_s(\xi_+(x)) \\ \sqrt{\tau} B_s(\xi_+(x)) & A_s(\xi_+(x)) \end{pmatrix}. \end{aligned}$$

From (3.9), it is readily seen that  $g_{n,+}(x) = g_{n,-}(x)$  for  $x \in (1, \infty)$ . Hence, by using (3.17), it can be shown that

$$e^{-(n+\frac{1}{2})f(\xi_-(x))\sigma_3} r_{n,-}(x)^{-\frac{1}{2}\sigma_3} r_{n,+}(x)^{\frac{1}{2}\sigma_3} e^{(n+\frac{1}{2})f(\xi_+(x))\sigma_3} = I.$$

Furthermore, since  $\rho(i\sqrt{2\tau}) = (-1)^{\tau+\frac{1}{2}}\rho(-i\sqrt{2\tau})$  (see (3.50)), equation (3.54) can be written in the form

$$\begin{aligned} J_D(x) & \sim i e^{\pi i \frac{n}{2}} \rho(\sqrt{2\tau}) \rho(i\sqrt{2\tau}) \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} b_n(x)^{-\sigma_3} \\ & \times \left\{ \begin{pmatrix} 2\sqrt{\tau} & 0 \\ 0 & 2\sqrt{\tau} \end{pmatrix} + \sum_{t=1}^{\infty} \sum_{j+k=t} \frac{M_{j,k}(x)}{(2\tau)^t} \right\} b_n(x)^{\sigma_3} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix}^{-1} \quad (3.55) \end{aligned}$$

for  $x \in (1, \infty)$ , where  $M_{j,k}(x)$  is the matrix

$$\begin{pmatrix} \sqrt{\tau} [(-1)^k + (-1)^j] A_j(\xi_-) B_k(\xi_+) & [(-1)^{k+1} + (-1)^j] A_j(\xi_-) A_k(\xi_+) \\ \tau [(-1)^{k+1} + (-1)^j] B_j(\xi_-) B_k(\xi_+) & \sqrt{\tau} [(-1)^k + (-1)^j] B_j(\xi_-) A_k(\xi_+) \end{pmatrix}. \quad (3.56)$$



Also from (3.50), we have

$$\rho(\sqrt{2\tau}) \rho(i\sqrt{2\tau}) \sim \frac{1}{2} e^{-(\frac{\tau}{2} + \frac{1}{4})\pi i} \tau^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{\eta_s}{\tau^s},$$

where  $\eta_0 = 1$  and

$$\eta_s = \frac{1}{2} [(-1)^s + 1] \gamma_s + \frac{1}{4} \sum_{\substack{j+k=s \\ j,k \neq 0}} (-1)^k \gamma_j \gamma_k \quad \text{for } s = 1, 2, \dots.$$

Therefore, we get

$$J_D(x) \sim I + \sum_{m=1}^{\infty} \frac{J_D^{(m)}(x)}{(2n+1)^m}, \quad (3.57)$$

where  $J_D^{(m)}(x)$  is equal to

$$\sum_{s+j+k=m} \frac{2^{s-1} \eta_s}{\sqrt{n + \frac{1}{2}}} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix} b_n(x)^{-\sigma_3} M_{j,k}(x) b_n(x)^{\sigma_3} \begin{pmatrix} \frac{1}{m_{11}} & 0 \\ \frac{2x - \alpha_n - \beta_n}{4m_{12}} & \frac{1}{2m_{22}} \end{pmatrix}^{-1}.$$

Using (3.52) and (3.56), it can be shown that

$$J_D^{(m)}(x) = O\left(\frac{1}{x^2}\right), \quad m = 1, 2, \dots, \quad \text{as } x \rightarrow \infty.$$

Thus, the expansion in (3.57) is uniformly valid for all  $x \in [1, \infty)$ .

For convenience, we put  $J_D^*(x) := J_D(x) - I$  so that

$$J_D^*(x) \sim \sum_{m=1}^{\infty} \frac{J_D^{(m)}(x)}{(2n+1)^m}.$$

From (D<sub>b</sub>) and (3.57), we get

$$D_+(x) = D_-(x)[I + J_D^*(x)] = D_-(x) \left[ I + O\left(\frac{1}{n}\right) \right] \quad \text{for } x \in (1, \infty). \quad (3.58)$$

In a similar way, it can be shown that (3.58) also holds for  $x \in (-\infty, 0)$ . Therefore, we have established the equation

$$(D_+(x) - I) - (D_-(x) - I) = D_-(x) J_D^*(x) \quad \text{for } x \in (-\infty, 0) \cup (1, \infty).$$

As in [5], we first derive formally the expansion

$$D(z) \sim I + \sum_{k=1}^{\infty} \frac{D_k(z)}{(2n+1)^k} \quad \text{as } n \rightarrow \infty. \quad (3.59)$$

Let  $D_0(z) = I$ , and define  $D_k(z)$  recursively by

$$D_k(z) = \frac{1}{2\pi i} \left( \int_{-\infty}^0 + \int_1^{\infty} \right) \left[ \sum_{j=1}^k (D_{k-j})_-(x) J_D^{(j)}(x) \right] \frac{dx}{x-z} \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty).$$

Using induction, it can be verified that for  $k = 1, 2, \dots$ ,

$$D_k(z) = O\left(\frac{1}{|z|}\right) \quad \text{as } z \rightarrow \infty.$$

Furthermore, by successive approximation (see [5]), it can be easily demonstrated that the expansion (3.59) holds uniformly for all  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ .

**Theorem 3.1** *Let  $r_n(z)$ ,  $l_n$  and  $\xi(z)$  be defined in (3.7), (3.16) and (3.35), respectively. Then the asymptotic expansion of the monic polynomial  $\pi_{N,n}(z)$  in (1.7) is given by*

$$\begin{aligned} \pi_{N,n}(z) = & \sqrt{2} r_n(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n+\frac{1}{2})l_n} \left[ U\left(-n - \frac{1}{2}, 2\sqrt{n + \frac{1}{2}} \xi(z)\right) A(z, n) \right. \\ & \left. + U'\left(-n - \frac{1}{2}, 2\sqrt{n + \frac{1}{2}} \xi(z)\right) B(z, n) \right], \end{aligned} \quad (3.60)$$

where  $A(z, n)$  and  $B(z, n)$  are analytic functions of  $z$  in  $\mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ . Furthermore, the asymptotic expansions

$$A(z, n) \sim \frac{(\xi^2 - 1)^{\frac{1}{4}}}{m_{11}(z - \beta_n)^{\frac{1}{4}}(z - \alpha_n)^{\frac{1}{4}}} \left[ 1 + \sum_{k=1}^{\infty} \frac{A_k(z)}{(2n+1)^k} \right], \quad (3.61)$$

$$B(z, n) \sim \frac{(z - \beta_n)^{\frac{1}{4}}(z - \alpha_n)^{\frac{1}{4}}}{m_{11}(\xi^2 - 1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{B_k(z)}{(2n+1)^{k+\frac{1}{2}}} \quad (3.62)$$

hold uniformly for  $z$  bounded away from  $(-\infty, 0] \cup [1, \infty)$ , where the constant  $m_{11}$  is given in (3.41).

**Proof** Let  $R_{ij}(z)$  and  $\tilde{R}_{ij}(z)$  denote the elements in  $R(z)$  and  $\tilde{R}(z)$ , respectively. Since  $R(z) = e^{\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} D(z) e^{-\frac{1}{2}(n+\frac{1}{2})l_n\sigma_3} \tilde{R}(z)$ , we have

$$\pi_{N,n}(z) = y_{11}(z) = R_{11}(z) = D_{11}(z)\tilde{R}_{11}(z) + D_{12}(z)e^{(n+\frac{1}{2})l_n}\tilde{R}_{21}(z). \quad (3.63)$$

By (3.59), we have

$$D_{11}(z) \sim 1 + \sum_{k=1}^{\infty} \frac{D_{11}^{(k)}(z)}{(2n+1)^k}, \quad (3.64)$$

$$D_{12}(z) \sim \sum_{k=1}^{\infty} \frac{D_{12}^{(k)}(z)}{(2n+1)^k}. \quad (3.65)$$

From the definition of  $\tilde{R}(z)$  in (3.45), we also know that for  $z$  in either  $\mathbb{C}_+$  or  $\mathbb{C}_-$ , the entries in the first column of this matrix are the same. More precisely, we have

$$\tilde{R}(z) = \sqrt{2} \begin{pmatrix} \tilde{R}_{11}(z) & * \\ \tilde{R}_{21}(z) & * \end{pmatrix},$$

where

$$\begin{aligned} \tilde{R}_{11}(z) &= \frac{1}{m_{11}} \frac{(\xi^2 - 1)^{\frac{1}{4}}}{b_n(z)} U\left(-n - \frac{1}{2}, 2\sqrt{n + \frac{1}{2}} \xi\right) r_n(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n+\frac{1}{2})l_n}, \\ \tilde{R}_{21}(z) &= e^{-\frac{1}{2}(n+\frac{1}{2})l_n} r_n(z)^{-\frac{1}{2}} \left( \frac{(\xi^2 - 1)^{\frac{1}{4}}}{b_n(z)} U\left(-n - \frac{1}{2}, 2\sqrt{n + \frac{1}{2}} \xi\right) \frac{2z - \alpha_n - \beta_n}{4m_{12}} \right) \end{aligned}$$

$$+ \frac{b_n(z)}{(\xi^2 - 1)^{\frac{1}{4}}} U' \left( -n - \frac{1}{2}, 2\sqrt{n + \frac{1}{2}} \xi \right) \frac{1}{2m_{22}} \Big).$$

By (3.63), one easily obtains (3.60), with

$$A(z, n) = \frac{(\xi^2 - 1)^{\frac{1}{4}}}{b_n(z)} \left[ D_{11}(z) \frac{1}{m_{11}} + D_{12}(z) \frac{2z - \alpha_n - \beta_n}{4m_{12}} \right],$$

$$B(z, n) = \frac{b_n(z)}{2m_{22}(\xi^2 - 1)^{\frac{1}{4}}} D_{12}(z).$$

Note that  $A(z, n)$  and  $B(z, n)$  are analytic for  $z$  in  $\mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$ . Hence, by (3.64) and (3.65), we have the asymptotic expansions in (3.61) and (3.62). Here, one additional thing that needs attention is  $\frac{1}{m_{22}} \sim \frac{i}{\sqrt{n + \frac{1}{2}}} \frac{1}{m_{11}}$  as  $n \rightarrow \infty$ . This completes the proof of the theorem.

## 4 Case II $p < c < \frac{1}{2}$

Since the parameter  $c$  is defined in terms of the degree of the polynomial, the number of zeros of the polynomial increases as  $c$  increases. Moreover, the zeros near the origin become as dense as the nodes (see [3, Theorem 2.12] and [19, Theorem 2]). Also, the density function reaches its upper constraint (see (2.4) and (2.8)). As we mentioned before, this fact is a crucial difference between the discrete orthogonal polynomials and the continuous ones, see the paragraph following (2.5).

Furthermore, since  $\mu(x)$  is not differentiable at the point  $\alpha$  in this case, the functions  $v_+(x)$  and  $\phi_+(x)$  are not analytic in the neighborhood of  $x = \alpha$ , and we can not expect to obtain a globally uniform asymptotic expansion (by using parabolic cylinder functions) in a region which includes both of the critical values  $\alpha$  and  $\beta$ . However, as we shall see, each of these values lies in a region in which there is a globally uniform asymptotic expansion in terms of the Airy function; the two regions overlap, and together cover the whole plane with two cuts along  $(-\infty, 0]$  and  $[1, \infty)$ .

### 4.1 Preliminary work

Now we need to modify our method to handle this case. First, we want to remove the saturated region. Define

$$\sigma_n := \frac{1}{2}(x_{N, k_0-1} + x_{N, k_0}) = \frac{k_0}{N}, \quad (4.1)$$

and choose  $k_0$  so that  $\sigma_n$  tends to a limit  $\sigma \in (0, 1)$ . Here,  $k_0$  is somewhat arbitrary, as long as  $x_{N, k_0-1}$  is in the band, not tending to its boundary and not asymptotically equal to  $n$ . Under this assumption, we shall see later that  $\alpha < \sigma < \beta$  and  $\sigma \neq c$ . Our first transformation is

$$H(z) := Y(z) \begin{pmatrix} \prod_{j=0}^{k_0-1} (z - x_{N,j})^{-1} & 0 \\ 0 & \prod_{j=0}^{k_0-1} (z - x_{N,j}) \end{pmatrix}. \quad (4.2)$$

It is easy to see that  $H$  is a solution of the following RHP:

(H<sub>a</sub>)  $H(z)$  is analytic for  $z \in \mathbb{C} \setminus X_N$ ;

(H<sub>b</sub>) the residue at the simple pole  $x_{N,k}$  is given by

$$\operatorname{Res}_{z=x_{N,k}} H(z; N, n) = \lim_{z \rightarrow x_{N,k}} H(z; N, n) \begin{pmatrix} 0 & \frac{w_{n,k}}{2i} \prod_{j=0}^{k_0-1} (x_{N,k} - x_{N,j})^2 \\ 0 & 0 \end{pmatrix} \quad (4.3)$$

for  $k = k_0, \dots, N-1$ , and

$$\operatorname{Res}_{z=x_{N,k}} H(z; N, n) = \lim_{z \rightarrow x_{N,k}} H(z; N, n) \begin{pmatrix} 0 & 0 \\ \frac{2i}{w_{n,k}} \prod_{\substack{j=0 \\ j \neq k}}^{k_0-1} (x_{N,k} - x_{N,j})^{-2} & 0 \end{pmatrix} \quad (4.4)$$

for  $k = 0, \dots, k_0-1$ ;

(H<sub>c</sub>) as  $z \rightarrow \infty$ ,

$$H(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^{n-k_0} & 0 \\ 0 & z^{-n+k_0} \end{pmatrix}.$$

To introduce the second transformation, which removes the poles and transforms the discrete RHP into a continuous one, we define

$$R^*(z) := H(z) \begin{pmatrix} 1 & 0 \\ \mp \frac{1}{2} e^{\mp i N \pi (1-z)} e^{N \nu z} \frac{\prod_{j=k_0}^{N-1} (z - x_{N,j})}{\prod_{j=0}^{k_0-1} (z - x_{N,j})} & 1 \end{pmatrix} \quad \text{for } z \in \Omega_{\pm}^{\Delta}, \quad (4.5)$$

$$R^*(z) := H(z) \begin{pmatrix} 1 & \mp \frac{1}{2} e^{\mp i N \pi (1-z)} e^{-N \nu z} \frac{\prod_{j=0}^{k_0-1} (z - x_{N,j})}{\prod_{j=k_0}^{N-1} (z - x_{N,j})} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Omega_{\pm}^{\nabla}, \quad (4.6)$$

$$R^*(z) := H(z) \quad \text{for all other } z \in \mathbb{C} \setminus \Sigma^*, \quad (4.7)$$

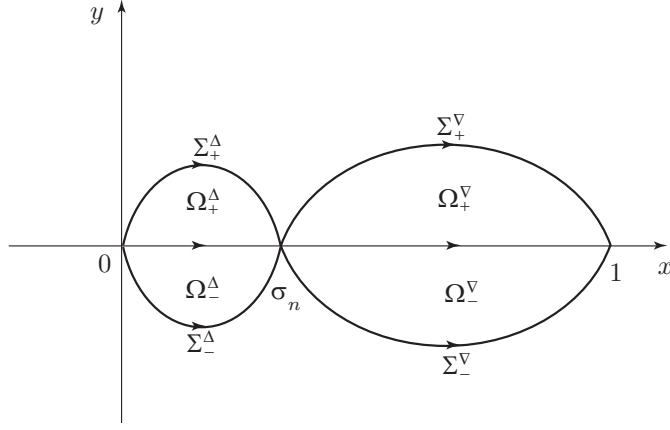
where  $\Sigma^* = (0, 1) \cup \Sigma_{\pm}^{\Delta} \cup \Sigma_{\pm}^{\nabla}$ . For the description of the domains  $\Omega_{\pm}^{\Delta}$ ,  $\Omega_{\pm}^{\nabla}$  and the contour  $\Sigma^*$ , see Figure 4. In this section, we use the superscript  $*$  to indicate that we are considering Case II.

The matrix  $R^*(z)$  is a solution of the following continuous RHP:

(R<sub>a</sub><sup>\*</sup>)  $R^*(z)$  is analytic for  $z \in \mathbb{C} \setminus \Sigma^*$ ;

(R<sub>b</sub><sup>\*</sup>) the jump conditions on the curve  $\Sigma^*$ : for  $x \in (0, \sigma_n)$ ,

$$R_+^*(x) = R_-^*(x) \begin{pmatrix} 1 & 0 \\ r_{1,n}(x) & 1 \end{pmatrix}, \quad (4.8)$$

Figure 4 The domains  $\Omega_{\pm}^{\Delta}$ ,  $\Omega_{\pm}^{\nabla}$  and the contour  $\Sigma^*$ 

where

$$r_{1,n}(x) = (-1)^{N-1} \cos(N\pi x) e^{N\nu x} \frac{\prod_{j=k_0}^{N-1} (x - x_{N,j})}{\prod_{j=0}^{k_0-1} (x - x_{N,j})}; \quad (4.9)$$

for  $x \in (\sigma_n, 1)$ ,

$$R_+^*(x) = R_-^*(x) \begin{pmatrix} 1 & r_{2,n}(x) \\ 0 & 1 \end{pmatrix}, \quad (4.10)$$

where

$$r_{2,n}(x) = (-1)^{N-1} \cos(N\pi x) e^{-N\nu x} \frac{\prod_{j=0}^{k_0-1} (x - x_{N,j})}{\prod_{j=k_0}^{N-1} (x - x_{N,j})}; \quad (4.11)$$

for  $z \in \Sigma_{\pm}^{\Delta}$ ,

$$R_+^*(z) = R_-^*(z) \begin{pmatrix} 1 & 0 \\ r_{\pm}^{\Delta}(z) & 1 \end{pmatrix}, \quad (4.12)$$

where

$$r_{\pm}^{\Delta}(z) = \frac{1}{2} e^{\mp iN\pi(1-z)} e^{N\nu z} \frac{\prod_{j=k_0}^{N-1} (z - x_{N,j})}{\prod_{j=0}^{k_0-1} (z - x_{N,j})}; \quad (4.13)$$

for  $z \in \Sigma_{\pm}^{\nabla}$ ,

$$R_+^*(z) = R_-^*(z) \begin{pmatrix} 1 & r_{\pm}^{\nabla}(z) \\ 0 & 1 \end{pmatrix}, \quad (4.14)$$

where

$$r_{\pm}^{\nabla}(z) = \frac{1}{2} e^{\mp i N \pi (1-z)} e^{-N \nu z} \frac{\prod_{j=0}^{k_0-1} (z - x_{N,j})}{\prod_{j=k_0}^{N-1} (z - x_{N,j})}; \quad (4.15)$$

$(R_c^*)$  as  $z \rightarrow \infty$ ,

$$R^*(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^{n-k_0} & 0 \\ 0 & z^{-n+k_0} \end{pmatrix};$$

$(R_d^*)$  as  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$R^*(z) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

As in Case I, due to the term  $e^{\mp i N \pi (1-z)}$  in the definitions of  $r_{\pm}^{\Delta}(z)$  and  $r_{\pm}^{\nabla}(z)$ , we shall concentrate on the jump conditions on the real line. Again by using the Gamma function,  $r_{1,n}(z)$  and  $r_{2,n}(z)$  can be written as

$$r_{1,n}(z) = - \frac{N^{2k_0-N} \pi e^{N \nu z} \Gamma(N(1-z) + \frac{1}{2})}{\Gamma(Nz + \frac{1}{2}) \Gamma^2(k_0 - Nz + \frac{1}{2})}, \quad (4.16)$$

$$r_{2,n}(z) = - \frac{N^{N-2k_0} \pi e^{-N \nu z} \Gamma(Nz + \frac{1}{2})}{\Gamma^2(Nz - k_0 + \frac{1}{2}) \Gamma(N(1-z) + \frac{1}{2})}. \quad (4.17)$$

As usual, we now define the auxiliary functions  $g^*$  and  $\phi^*$ .

**Definition 4.1** *The  $g$ -functions are defined by*

$$\tilde{g}_n^*(z) := \int_{\alpha_n^*}^{\beta_n^*} \log(z-s) \mu_n^*(s) ds, \quad z \in \mathbb{C} \setminus [\alpha_n^*, \infty), \quad (4.18)$$

$$g_n^*(z) := \int_{\alpha_n^*}^{\beta_n^*} \log(z-s) \mu_n^*(s) ds, \quad z \in \mathbb{C} \setminus (-\infty, \beta_n^*], \quad (4.19)$$

where the measure  $\mu_n^*(x)$  and the Mhaskar-Rakhmanov-Saff numbers  $\alpha_n^*$  and  $\beta_n^*$  will be determined later.

**Definition 4.2** *The  $\phi$ -functions are defined by*

$$\tilde{\phi}_n^*(z) := - \int_{\alpha_n^*}^z \tilde{v}_n^*(s) ds \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0] \cup [\alpha_n^*, \infty), \quad (4.20)$$

$$\phi_n^*(z) := \int_{\beta_n^*}^z v_n^*(s) ds \quad \text{for } z \in \mathbb{C} \setminus (-\infty, \beta_n^*] \cup [1, \infty). \quad (4.21)$$

Here the measures  $\tilde{v}_n^*(z)$  and  $v_n^*(z)$  are defined in the complex plane and satisfy

$$\frac{n}{n-k_0} \tilde{v}_{n,\pm}^*(x) = \pm \pi i \mu_n^*(x) \quad \text{for } x \in (\alpha_n^*, \sigma_n), \quad (4.22)$$

$$\frac{n}{n-k_0} v_{n,\pm}^*(x) = \pm \pi i \mu_n^*(x) \quad \text{for } x \in (\sigma_n, \beta_n^*). \quad (4.23)$$

To find the measures in the above definitions, we need the following equations:

$$-(n - k_0)(\tilde{g}_{n,+}^*(x) + \tilde{g}_{n,-}^*(x)) + \log r_{1,n}(x) = 0 \quad \text{for } x \in (\alpha_n^*, \sigma_n), \quad (4.24)$$

$$(n - k_0)(g_{n,+}^*(x) + g_{n,-}^*(x)) + \log r_{2,n}(x) = 0 \quad \text{for } x \in (\sigma_n, \beta_n^*). \quad (4.25)$$

These formulas correspond to (3.12) in Case I; they are what is required in the normalization of the RHP for  $R^*$ .

**Proposition 4.1** *With the constants defined by*

$$l_n^* := 2g_n^*(\beta_n^*) + \frac{\log r_{2,n}(\beta_n^*)}{n - k_0}, \quad (4.26)$$

$$\tilde{l}_n^* := 2\tilde{g}_n^*(\alpha_n^*) - \frac{\log r_{1,n}(\alpha_n^*)}{n - k_0}, \quad (4.27)$$

*the following connection formulas between the  $g$ -function ( $\tilde{g}$ -function) and the  $\phi$ -function ( $\tilde{\phi}$ -function) hold*

$$g_n^*(z) + \frac{n}{n - k_0} \phi_n^*(z) = -\frac{1}{2(n - k_0)} \log r_{2,n}(z) + \frac{l_n^*}{2} \quad \text{for } z \in \mathbb{C} \setminus (-\infty, \sigma_n] \cup [1, \infty), \quad (4.28)$$

$$\tilde{g}_n^*(z) - \frac{n}{n - k_0} \tilde{\phi}_n^*(z) = \frac{1}{2(n - k_0)} \log r_{1,n}(z) + \frac{\tilde{l}_n^*}{2} \quad \text{for } z \in \mathbb{C} \setminus (-\infty, 0] \cup [\sigma_n, \infty). \quad (4.29)$$

Furthermore, we have

$$\tilde{g}_n^*(z) = \begin{cases} g_n^*(z), & z \in \mathbb{C}_+, \\ g_n^*(z) + 2\pi i, & z \in \mathbb{C}_-, \end{cases} \quad (4.30)$$

$$\tilde{\phi}_n^*(z) = \begin{cases} -\phi_n^*(z) - \frac{n - k_0}{n} \pi i - \frac{1}{2n} \log r_{1,n}(z) r_{2,n}(z), & z \in \mathbb{C}_+, \\ -\phi_n^*(z) + \frac{n - k_0}{n} \pi i - \frac{1}{2n} \log r_{1,n}(z) r_{2,n}(z), & z \in \mathbb{C}_-, \end{cases} \quad (4.31)$$

$$\tilde{l}_n^* = l_n + 2\pi i. \quad (4.32)$$

**Proof** The proof is similar to that of Proposition 3.1, and we only give a sketch of it. Corresponding to (3.22), here we have

$$\begin{aligned} \frac{n}{n - k_0} \tilde{v}_n^*(z) &= -\tilde{g}_n^*(z) + \frac{1}{2(c_n - \sigma_n)} \left[ \nu + 2\psi \left( N(\sigma_n - z) + \frac{1}{2} \right) \right. \\ &\quad \left. - \psi \left( Nz + \frac{1}{2} \right) - \psi \left( N(1 - z) + \frac{1}{2} \right) \right] \end{aligned} \quad (4.33)$$

for  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [\alpha_n^*, \infty)$ , and

$$\begin{aligned} \frac{n}{n - k_0} v_n^*(z) &= -g_n^*(z) + \frac{1}{2(c_n - \sigma_n)} \left[ \nu + 2\psi \left( N(z - \sigma_n) + \frac{1}{2} \right) \right. \\ &\quad \left. - \psi \left( Nz + \frac{1}{2} \right) - \psi \left( N(1 - z) + \frac{1}{2} \right) \right] \end{aligned} \quad (4.34)$$

for  $z \in \mathbb{C} \setminus (-\infty, \beta_n^*] \cup [1, \infty)$ . Integrating  $v_n^*(z)$  and  $\tilde{v}_n^*(z)$  from  $\beta_n^*$  and  $\alpha_n^*$  to  $z$ , respectively, we obtain (4.28) and (4.29). Moreover, (4.26) and (4.27) follow immediately.

Equation (4.30) is obtained by observing the branch cut of  $\log(z - s)$  in (4.18) and (4.19). To get (4.31), we note that by using (4.33) and (4.34),

$$\tilde{v}_n^*(z) = v_n^*(z) + \frac{1}{2n} \frac{d}{dz} \log r_{1,n}(z) r_{2,n}(z). \quad (4.35)$$

From (4.20) and (4.21), it follows that

$$\begin{aligned} \tilde{\phi}_n^*(z) &= - \int_{\alpha_n^*}^z \tilde{v}_n^*(s) ds = - \int_{\alpha_n^*}^{\beta_n^*} \tilde{v}_{n,\pm}^*(s) ds - \int_{\beta_n^*}^z \tilde{v}_n^*(s) ds \\ &= \mp \frac{n - k_0}{n} \pi i - \phi_n^*(z) - \frac{1}{2n} \log r_{1,n}(z) r_{2,n}(z) \quad \text{for } z \in \mathbb{C}_{\pm}. \end{aligned} \quad (4.36)$$

In reaching the last equality, use has been made of (4.22), (4.23) and (4.35). This establishes (4.31). Subtracting (4.29) from (4.28), we obtain (4.32) from (4.30) and (4.31).

From (4.24) and (4.25), one can derive the asymptotic expansions of  $\alpha_n^*$  and  $\beta_n^*$ ; they have the same form as given in (3.28). It turns out that these expansions are independent of the choice of  $k_0$  and the corresponding number  $\sigma_n$  in (4.1). Furthermore,  $\mu_n^*(x)$  can be shown to satisfy the relation

$$\mu_n^*(x) = \mu^*(x) + O\left(\frac{1}{n}\right) \quad (4.37)$$

uniformly for  $x \in [\alpha, \beta]$  (see the corresponding result (3.30) in Case I). Note that due to the transformation in (4.2), here  $\mu^*(x)$  is discontinuous at the point  $\sigma$ . Moreover, we have

$$\mu^*(x) = \begin{cases} \frac{c}{c - \sigma} \left( \mu(x) - \frac{1}{c} \right), & x \in (0, \sigma), \\ \frac{c}{c - \sigma} \mu(x), & x \in (\sigma, \beta), \end{cases}$$

where  $\mu(x)$  is defined in (2.8).

## 4.2 Construction of the parametrix

We now define the final transformation

$$V^*(z) := \begin{cases} e^{-\frac{1}{2}(n-k_0)l_n^* \sigma_3} R^*(z) r_{2,n}(z)^{\frac{1}{2}\sigma_3} & \text{for } \operatorname{Re} z > \sigma_n, \\ e^{-\frac{1}{2}(n-k_0)\tilde{l}_n^* \sigma_3} R^*(z) r_{1,n}(z)^{-\frac{1}{2}\sigma_3} & \text{for } \operatorname{Re} z < \sigma_n. \end{cases} \quad (4.38)$$

Let  $\Gamma$  be the line  $\operatorname{Re} z = \sigma_n$ ; see Figure 5. Using the relations (4.28) and (4.29), it can be verified that  $V^*(z)$  satisfies the RHP:

(V<sub>a</sub><sup>\*</sup>)  $V^*(z)$  is analytic in  $\mathbb{C} \setminus \Sigma^* \cup \Gamma$ ;

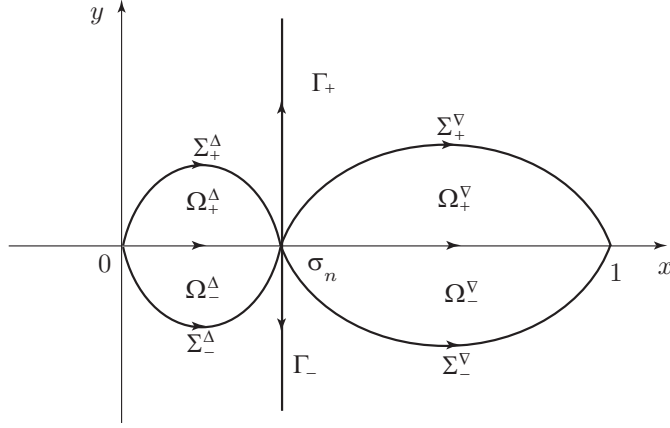
(V<sub>b</sub><sup>\*</sup>) the jump conditions on the contour  $\Sigma^* \cup \Gamma$ : for  $x \in (\sigma_n, 1)$ ,

$$V_+^*(x) = V_-^*(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad (4.39)$$

for  $x \in (0, \sigma_n)$ ,

$$V_+^*(x) = V_-^*(x) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad (4.40)$$



Figure 5 The domains  $\Omega_{\pm}^{\Delta}$ ,  $\Omega_{\pm}^{\nabla}$  and the contours  $\Sigma^*$  and  $\Gamma$ 

for  $z \in \Sigma_{\pm}^{\nabla}$ ,

$$V_+^*(z) = V_-^*(z) \begin{pmatrix} 1 & -\frac{1}{1+e^{\mp 2N\pi iz}} \\ 0 & 1 \end{pmatrix}; \quad (4.41)$$

for  $z \in \Sigma_{\pm}^{\Delta}$ ,

$$V_+^*(z) = V_-^*(z) \begin{pmatrix} 1 & 0 \\ -\frac{1}{1+e^{\mp 2N\pi iz}} & 1 \end{pmatrix}; \quad (4.42)$$

for  $z \in \Gamma_{\pm}$ ,

$$V_+^*(z) = (-1)^{n-k_0} V_-^*(z) \begin{pmatrix} \frac{i}{e^{N\pi iz} + e^{-N\pi iz}} & 0 \\ 0 & -i(e^{N\pi iz} + e^{-N\pi iz}) \end{pmatrix}^{\pm 1}; \quad (4.43)$$

$(V_c^*)$  as  $z \rightarrow \infty$ ,

$$V^*(z) = \left( I + O\left(\frac{1}{z}\right) \right) e^{-n\phi_n^*(z)\sigma_3} \quad (4.44)$$

for  $\operatorname{Re} z > \sigma_n$ , and

$$V^*(z) = \left( I + O\left(\frac{1}{z}\right) \right) e^{n\tilde{\phi}_n^*(z)\sigma_3} \quad (4.45)$$

for  $\operatorname{Re} z < \sigma_n$ ;

$(V_d^*)$  as  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$V^*(z) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

From the well-known formula

$$\text{Ai}(z) + \omega \text{Ai}(\omega z) + \omega^2 \text{Ai}(\omega^2 z) = 0,$$

one can get the following matrix equations

$$\begin{pmatrix} \text{Ai}(f^*(z)) & -\omega^2 \text{Ai}(\omega^2 f^*(z)) \\ \text{Ai}'(f^*(z)) & -\omega \text{Ai}'(\omega^2 f^*(z)) \end{pmatrix} = \begin{pmatrix} \text{Ai}(f^*(z)) & \omega \text{Ai}(\omega f^*(z)) \\ \text{Ai}'(f^*(z)) & \omega^2 \text{Ai}'(\omega f^*(z)) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (4.46)$$

$$\begin{pmatrix} -\omega^2 \text{Ai}(\omega^2 f^*(z)) & \text{Ai}(f^*(z)) \\ -\omega \text{Ai}'(\omega^2 f^*(z)) & \text{Ai}'(f^*(z)) \end{pmatrix} = \begin{pmatrix} \omega \text{Ai}(\omega f^*(z)) & \text{Ai}(f^*(z)) \\ \omega^2 \text{Ai}'(\omega f^*(z)) & \text{Ai}'(f^*(z)) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (4.47)$$

We divide the complex plane into four regions by using  $\Gamma$  and the real axis, see Figure 6.

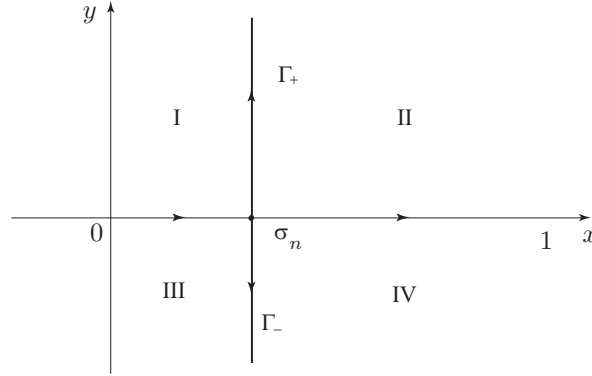


Figure 6 The domains I, II, III and IV

With a similar technique as given in [5], we construct the parametrix of the RHP for  $V^*$  by using the Airy functions in these four regions. Define

$$Q^*(z) := 2\sqrt{\pi} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} \left( \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} \right)^{\sigma_3} \begin{pmatrix} \text{Ai}(f^*(z)) & -\omega^2 \text{Ai}(\omega^2 f^*(z)) \\ \text{Ai}'(f^*(z)) & -\omega \text{Ai}'(\omega^2 f^*(z)) \end{pmatrix} \quad (4.48)$$

for  $z \in \text{II}$ ,

$$Q^*(z) := 2\sqrt{\pi} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} \left( \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} \right)^{\sigma_3} \begin{pmatrix} \text{Ai}(f^*(z)) & \omega \text{Ai}(\omega f^*(z)) \\ \text{Ai}'(f^*(z)) & \omega^2 \text{Ai}'(\omega f^*(z)) \end{pmatrix} \quad (4.49)$$

for  $z \in \text{IV}$ ,

$$Q^*(z) := 2\sqrt{\pi} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}^{-1} (\tilde{f}^*(z)^{\frac{1}{4}} a_n(z))^{\sigma_3} \begin{pmatrix} -\omega^2 \text{Ai}(\omega^2 \tilde{f}^*(z)) & \text{Ai}(\tilde{f}^*(z)) \\ -\omega \text{Ai}'(\omega^2 \tilde{f}^*(z)) & \text{Ai}'(\tilde{f}^*(z)) \end{pmatrix} \quad (4.50)$$

for  $z \in \text{I}$  and

$$Q^*(z) := 2\sqrt{\pi} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}^{-1} (\tilde{f}^*(z)^{\frac{1}{4}} a_n(z))^{\sigma_3} \begin{pmatrix} \omega \text{Ai}(\omega \tilde{f}^*(z)) & \text{Ai}(\tilde{f}^*(z)) \\ \omega^2 \text{Ai}'(\omega \tilde{f}^*(z)) & \text{Ai}'(\tilde{f}^*(z)) \end{pmatrix} \quad (4.51)$$

for  $z \in \text{III}$ , where  $f^*(z)$  and  $\tilde{f}^*(z)$  are defined by

$$f^*(z) := \left[ \frac{3}{2} n \phi_n^*(z) \right]^{\frac{2}{3}}, \quad \tilde{f}^*(z) := \left[ \frac{3}{2} n \tilde{\phi}_n^*(z) \right]^{\frac{2}{3}}, \quad (4.52)$$

and  $a_n(z)$  is the analytic function in  $\mathbb{C} \setminus [\alpha_n^*, \beta_n^*]$  given by

$$a_n(z) := \frac{(z - \beta_n^*)^{\frac{1}{4}}}{(z - \alpha_n^*)^{\frac{1}{4}}}. \quad (4.53)$$

In view of the relations between  $V^*(z)$  and  $R^*(z)$  given in (4.38), a reasonable parametrix of the RHP for  $R^*$  is

$$\tilde{R}^*(z) := \begin{cases} e^{\frac{1}{2}(n-k_0)l_n^* \sigma_3} Q^*(z) r_{2,n}(z)^{-\frac{1}{2} \sigma_3}, & z \in \text{II} \cup \text{IV}, \\ e^{\frac{1}{2}(n-k_0)\tilde{l}_n^* \sigma_3} Q^*(z) r_{1,n}(z)^{\frac{1}{2} \sigma_3}, & z \in \text{I} \cup \text{III}. \end{cases} \quad (4.54)$$

From these definitions, it is easy to verify that

$$\begin{aligned} \tilde{R}_+^*(x) &= \tilde{R}_-^*(x) \begin{pmatrix} 1 & 0 \\ r_{1,n}(x) & 1 \end{pmatrix}, \quad x \in (0, \sigma_n), \\ \tilde{R}_+^*(x) &= \tilde{R}_-^*(x) \begin{pmatrix} 1 & r_{2,n}(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (\sigma_n, 1), \end{aligned}$$

and that  $\tilde{R}^*(z)$  satisfies the large- $z$  behavior given in  $(R_c^*)$ . The only difference between  $R^*(z)$  and  $\tilde{R}^*(z)$  is that there are new jump matrices on  $(-\infty, 0) \cup (1, \infty)$  and the vertical line  $\Gamma$ . In the subsequent analysis, we will show that all these jump matrices tend to the identity matrix as  $n \rightarrow \infty$ .

### 4.3 Uniform asymptotic expansions

Define the matrix

$$S(z) := e^{-\frac{1}{2}(n-k_0)l_n^* \sigma_3} R^*(z) (\tilde{R}^*(z))^{-1} e^{\frac{1}{2}(n-k_0)l_n^* \sigma_3}. \quad (4.55)$$

From the construction of  $\tilde{R}^*(z)$ , it is easy to see that

$$S_+(x) = S_-(x), \quad x \in (0, 1).$$

Furthermore, it can be verified that  $S(z)$  is a solution to the RHP:

(S<sub>a</sub>)  $S(z)$  is analytic for  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty) \cup \Gamma$ ;

(S<sub>b</sub>) for  $z \in (-\infty, 0] \cup [1, \infty) \cup \Gamma$ ,

$$S_+(z) = S_-(z) J_S(z), \quad (4.56)$$

where

$$J_S(z) := e^{-\frac{1}{2}(n-k_0)l_n^* \sigma_3} \tilde{R}_-^*(z) (\tilde{R}_+^*(z))^{-1} e^{\frac{1}{2}(n-k_0)l_n^* \sigma_3}; \quad (4.57)$$

(S<sub>c</sub>) for  $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty) \cup \Gamma$ ,

$$S(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty;$$

(S<sub>d</sub>) as  $z \rightarrow 0$  and  $z \rightarrow 1$ ,

$$S(z) = O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We consider only the case on the line  $\Gamma$ , since the discussions for the other cases on the cuts  $(-\infty, 0)$  and  $(1, \infty)$  are very similar. Let  $\Gamma_{\pm} \equiv \Gamma \cap \mathbb{C}_{\pm}$ . For  $z \in \Gamma_+$ , we recall the asymptotic expansions of the Airy functions in [1, p. 448]

$$\begin{aligned} \text{Ai}(f^*(z)) &\sim \frac{1}{2\sqrt{\pi}} (f^*(z))^{-\frac{1}{4}} e^{-n\phi_n^*(z)} \sum_{k=0}^{\infty} (-1)^k c_k (n\phi_n^*(z))^{-k}, \\ \text{Ai}'(f^*(z)) &\sim -\frac{1}{2\sqrt{\pi}} (f^*(z))^{\frac{1}{4}} e^{-n\phi_n^*(z)} \sum_{k=0}^{\infty} (-1)^k d_k (n\phi_n^*(z))^{-k}, \\ \text{Ai}(\omega^2 f^*(z)) &\sim \frac{e^{\frac{i\pi}{6}}}{2\sqrt{\pi}} (f^*(z))^{-\frac{1}{4}} e^{n\phi_n^*(z)} \sum_{k=0}^{\infty} c_k (n\phi_n^*(z))^{-k}, \\ \text{Ai}'(\omega^2 f^*(z)) &\sim -\frac{e^{-\frac{i\pi}{6}}}{2\sqrt{\pi}} (f^*(z))^{\frac{1}{4}} e^{n\phi_n^*(z)} \sum_{k=0}^{\infty} d_k ((n-k_0)\phi_n^*(z))^{-k}, \end{aligned} \tag{4.58}$$

where  $c_0 = d_0 = 1$  and for  $k = 1, 2, \dots$ ,

$$c_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})}, \quad d_k = -\frac{6k+1}{6k-1} c_k.$$

Corresponding results can be given for  $\text{Ai}(\tilde{f}^*(z))$ ,  $\text{Ai}'(\tilde{f}^*(z))$ ,  $\text{Ai}(\omega^2 \tilde{f}^*(z))$  and  $\text{Ai}'(\omega^2 \tilde{f}^*(z))$ .

Now we set out to derive the asymptotic expansion of  $J_S(z)$  for  $z \in \Gamma_+$ . From (4.54) and (4.57), we have

$$\begin{aligned} J_S(z) &= \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}^{-1} \left( \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} \right)^{\sigma_3} \begin{pmatrix} \text{Ai}(f^*(z)) & -\omega^2 \text{Ai}(\omega^2 f^*(z)) \\ \text{Ai}'(f^*(z)) & -\omega \text{Ai}'(\omega^2 f^*(z)) \end{pmatrix} r_{2,n}(z)^{-\frac{1}{2}\sigma_3} \\ &\quad \times r_{1,n}(z)^{-\frac{1}{2}\sigma_3} \begin{pmatrix} -\omega^2 \text{Ai}(\omega^2 \tilde{f}^*(z)) & \text{Ai}(\tilde{f}^*(z)) \\ -\omega \text{Ai}'(\omega^2 \tilde{f}^*(z)) & \text{Ai}'(\tilde{f}^*(z)) \end{pmatrix}^{-1} \left( \frac{1}{\tilde{f}^*(z)^{\frac{1}{4}} a_n(z)} \right)^{\sigma_3} \\ &\quad \times \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} e^{-\frac{1}{2}(n-k_0)\tilde{l}_n^* \sigma_3} e^{\frac{1}{2}(n-k_0)l_n^* \sigma_3}. \end{aligned}$$

Coupling (4.32) and the well-known formula (10.4.12) in [1, p. 446]

$$\omega \text{Ai}(\tilde{f}^*(z)) \text{Ai}'(\omega^2 \tilde{f}^*(z)) - \omega^2 \text{Ai}'(\tilde{f}^*(z)) \text{Ai}(\omega^2 \tilde{f}^*(z)) = \frac{1}{2\pi i},$$

we get

$$\begin{aligned}
J_S(z) &= (-1)^{n-k_0} \pi i \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \left( \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} \right)^{\sigma_3} \begin{pmatrix} \text{Ai}(f^*(z)) & -\omega^2 \text{Ai}(\omega^2 f^*(z)) \\ \text{Ai}'(f^*(z)) & -\omega \text{Ai}'(\omega^2 f^*(z)) \end{pmatrix} \\
&\quad \times r_{2,n}(z)^{-\frac{1}{2}\sigma_3} r_{1,n}(z)^{-\frac{1}{2}\sigma_3} \begin{pmatrix} \text{Ai}'(\tilde{f}^*(z)) & -\text{Ai}(\tilde{f}^*(z)) \\ \omega \text{Ai}'(\omega^2 \tilde{f}^*(z)) & -\omega^2 \text{Ai}(\omega^2 \tilde{f}^*(z)) \end{pmatrix} \\
&\quad \times \left( \frac{1}{\tilde{f}^*(z)^{\frac{1}{4}} a_n(z)} \right)^{\sigma_3} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}.
\end{aligned}$$

Substituting the expansions in (4.58) into the last equation and taking into account Proposition 4.1, we readily see that

$$\begin{aligned}
J_S(z) &\sim \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a_n(z)^{-\sigma_3} \left\{ \sum_{j=0}^{\infty} \begin{pmatrix} (-1)^j c_j & i c_j \\ (-1)^{j+1} d_j & i d_j \end{pmatrix} \left( \frac{1}{n \phi_n^*(z)} \right)^j \right\} \\
&\quad \times \left\{ \sum_{k=0}^{\infty} \begin{pmatrix} (-1)^{k+1} i d_k & (-1)^{k+1} i c_k \\ d_k & -c_k \end{pmatrix} \left( \frac{1}{n \tilde{\phi}_n^*(z)} \right)^k \right\} a_n(z)^{-\sigma_3} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}.
\end{aligned}$$

Straightforward calculation yields

$$\begin{aligned}
J_S(z) &\sim \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a_n(z)^{-\sigma_3} \left\{ \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} + \sum_{m=1}^{\infty} \sum_{j+k=m} \frac{M_{j,k}^*(z)}{n^m} \right\} \\
&\quad \times a_n(z)^{-\sigma_3} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \quad \text{for } z \in \Gamma_+,
\end{aligned} \tag{4.59}$$

where

$$M_{j,k}^*(z) = i \begin{pmatrix} [(-1)^{m+1} + 1] c_j d_k & [(-1)^{m+1} - 1] c_j c_k \\ [(-1)^m + 1] d_j d_k & [(-1)^m - 1] d_j c_k \end{pmatrix} \left( \frac{1}{\phi_n^*(z)} \right)^j \left( \frac{1}{\tilde{\phi}_n^*(z)} \right)^k. \tag{4.60}$$

Thus, we obtain

$$J_S(z) \sim I + \sum_{m=1}^{\infty} \frac{J_m(z)}{n^m}, \tag{4.61}$$

where

$$J_m(z) = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} a_n(z)^{-\sigma_3} \sum_{j+k=m} M_{j,k}^*(z) a_n(z)^{-\sigma_3} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}. \tag{4.62}$$

From (4.60) and (4.62), it can be easily shown that

$$J_m(z) = O\left(\frac{1}{(\phi_n^*(z))^m}\right) = O\left(\frac{1}{z^m}\right), \quad m = 1, 2, \dots, \quad \text{as } z \rightarrow \infty. \tag{4.63}$$

This, in particular, infers that the expansion in (4.61) is uniformly valid for all  $z \in \Gamma_+$ . Similar results hold for  $z \in \Gamma_- \cup (-\infty, 0) \cup (1, \infty)$ . Therefore, formally we have

$$S(z) \sim I + \sum_{k=1}^{\infty} \frac{S_k(z)}{n^k} \quad \text{as } n \rightarrow \infty, \quad (4.64)$$

where

$$S_k(z) = \frac{1}{2\pi i} \left( \int_{\Gamma} + \int_{-\infty}^0 + \int_1^{\infty} \right) \left[ \sum_{j=1}^k (S_{k-j})_-(t) J_j(t) \right] \frac{dt}{t-z} \quad (4.65)$$

for  $z \in \mathbb{C} \setminus \Gamma \cup (-\infty, 0] \cup [1, \infty)$ , with  $S_0(z) = I$ . By induction, it can be verified that for  $k = 1, 2, \dots$ ,

$$S_k(z) = O\left(\frac{1}{|z|}\right) \quad \text{as } z \rightarrow \infty.$$

Using the usual method of successive approximation, we can show that the formal expansion in (4.64) is actually uniformly valid for  $z \in \mathbb{C} \setminus \Gamma \cup (-\infty, 0] \cup [1, \infty)$ . Thus, we arrive at our second main result stated below.

**Theorem 4.1** *Let  $r_n(z)$ ,  $l_n^*$  and  $f^*(z)$  be defined in (3.7), (4.26) and (4.52), respectively. The asymptotic expansion of the monic polynomial  $\pi_{N,n}(z)$  is given by*

$$\pi_{N,n}(z) = \sqrt{\pi} r_n(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)l_n^*} [\text{Ai}(f^*(z))A^*(z, n) + \text{Ai}'(f^*(z))B^*(z, n)], \quad (4.66)$$

where  $A^*(z, n)$  and  $B^*(z, n)$  are analytic functions of  $z$  in  $\mathbb{C} \setminus [1, \infty)$  and  $\text{Re } z > \alpha$ .

Similarly, with  $\tilde{l}_n^*$  and  $\tilde{f}^*(z)$  defined in (4.27) and (4.52),

$$\begin{aligned} \pi_{N,n}(z) &= \frac{(-1)^{N-1}}{2} \sqrt{\pi} r_n(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} \\ &\quad \times \{ [\cos(N\pi z) \text{Bi}(\tilde{f}^*(z)) - \sin(N\pi z) \text{Ai}(\tilde{f}^*(z))] \tilde{A}(z, n) \\ &\quad + [\cos(N\pi z) \text{Bi}'(\tilde{f}^*(z)) - \sin(N\pi z) \text{Ai}'(\tilde{f}^*(z))] \tilde{B}(z, n) \}, \end{aligned} \quad (4.67)$$

where  $\tilde{A}(z, n)$  and  $\tilde{B}(z, n)$  are analytic functions of  $z$  in  $\mathbb{C} \setminus (-\infty, 0]$  and  $\text{Re } z < \beta$ .

Furthermore,  $A^*(z, n)$  and  $B^*(z, n)$  have the asymptotic expansions

$$A^*(z, n) \sim \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} \left[ 1 + \sum_{k=1}^{\infty} \frac{A_k^*(z)}{n^k} \right], \quad B^*(z, n) \sim \frac{a_n(z)}{f^*(z)^{\frac{1}{4}}} \left[ -1 + \sum_{k=1}^{\infty} \frac{B_k^*(z)}{n^k} \right], \quad (4.68)$$

and  $\tilde{A}(z, n)$ ,  $\tilde{B}(z, n)$  have the asymptotic expansions

$$\tilde{A}(z, n) \sim \tilde{f}^*(z)^{\frac{1}{4}} a_n(z) \left[ 1 + \sum_{k=1}^{\infty} \frac{\tilde{A}_k(z)}{n^k} \right], \quad \tilde{B}(z, n) \sim \frac{1}{\tilde{f}^*(z)^{\frac{1}{4}} a_n(z)} \left[ 1 + \sum_{k=1}^{\infty} \frac{\tilde{B}_k(z)}{n^k} \right]. \quad (4.69)$$

All four expansions hold uniformly for  $z$  in their respective regions of analyticity.

**Proof** From the definition of  $S(z)$  in (4.55), we have

$$R^*(z) = e^{\frac{1}{2}(n-k_0)l_n^* \sigma_3} S(z) e^{-\frac{1}{2}(n-k_0)l_n^* \sigma_3} \tilde{R}^*(z).$$

For any matrix  $X$ , we shall denote by  $X_{ij}$  the  $(i, j)$  element in  $X$ . The above formula then gives

$$R_{11}^*(z) = S_{11}(z)\tilde{R}_{11}^*(z) + S_{12}(z)\tilde{R}_{21}^*(z)e^{(n-k_0)l_n^*}$$

and

$$R_{12}^*(z) = S_{11}(z)\tilde{R}_{12}^*(z) + S_{12}(z)\tilde{R}_{22}^*(z)e^{(n-k_0)l_n^*}.$$

First, let us consider  $z$  in the half plane on the right side of  $\Gamma$ . Recalling the definition of  $\tilde{R}^*(z)$  in (4.54), one obtains

$$R_{11}^*(z) = \sqrt{\pi} r_{2,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)l_n^*} [\text{Ai}(f^*(z))A^*(z, n) + \text{Ai}'(f^*(z))B^*(z, n)], \quad (4.70)$$

where

$$A^*(z, n) = \frac{f^*(z)^{\frac{1}{4}}}{a_n(z)} (S_{11}(z) - iS_{12}(z)) \quad \text{and} \quad B^*(z, n) = \frac{a_n(z)}{f^*(z)^{\frac{1}{4}}} (-S_{11}(z) - iS_{12}(z)).$$

From (4.2) and (4.6), it also follows that

$$\pi_{N,n}(z) = Y_{11}(z) = R_{11}^*(z) \prod_{j=0}^{k_0-1} (z - x_{N,j}).$$

This, together with the fact that

$$r_n(z) = r_{2,n}(z) \prod_{j=0}^{k_0-1} (z - x_{N,j})^{-2} \quad (4.71)$$

(see (3.3) and (4.11)), gives us the main result (4.66). From the asymptotic expansion of  $S(z)$  in (4.64), we immediately obtain (4.68).

Next, let us consider  $z$  in the half plane on the left of  $\Gamma$ , and restrict it to the region I indicated in Figure 6. From (4.54), we have

$$R_{11}^*(z) = \sqrt{\pi} r_{1,n}(z)^{\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} [e^{-\frac{1}{6}\pi i} \text{Ai}(\omega^2 \tilde{f}^*(z))\tilde{A}(z, n) - e^{\frac{1}{6}\pi i} \text{Ai}'(\omega^2 \tilde{f}^*(z))\tilde{B}(z, n)]$$

and

$$R_{12}^*(z) = \sqrt{\pi} r_{1,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} [-i \text{Ai}(\tilde{f}^*(z))\tilde{A}(z, n) - i \text{Ai}'(\tilde{f}^*(z))\tilde{B}(z, n)],$$

where

$$\tilde{A}(z, n) = \tilde{f}^*(z)^{\frac{1}{4}} a_n(z) (S_{11}(z) + iS_{12}(z))$$

and

$$\tilde{B}(z, n) = \frac{1}{\tilde{f}^*(z)^{\frac{1}{4}} a_n(z)} (S_{11}(z) - iS_{12}(z)).$$

Similarly, for  $z \in \text{III}$ ,

$$R_{11}^*(z) = \sqrt{\pi} r_{1,n}(z)^{\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} [e^{\frac{1}{6}\pi i} \text{Ai}(\omega \tilde{f}^*(z))\tilde{A}(z, n) - e^{-\frac{1}{6}\pi i} \text{Ai}'(\omega \tilde{f}^*(z))\tilde{B}(z, n)]$$

and

$$R_{12}^*(z) = \sqrt{\pi} r_{1,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} [-i \operatorname{Ai}(\tilde{f}^*(z))\tilde{A}(z, n) - i \operatorname{Ai}'(\tilde{f}^*(z))\tilde{B}(z, n)].$$

From (4.5), we know that  $H_{11}(z)$  has different expressions in different parts of regions I and III. Let us first consider regions  $\Omega_+^\Delta$  and  $\Omega_-^\Delta$  shown in Figure 4. For  $z \in \Omega_+^\Delta$ , we have from (4.5)

$$\begin{aligned} H_{11}(z) &= R_{11}^*(z) + \frac{1}{2} e^{-iN\pi(1-z)} e^{N\nu z} \frac{\prod_{j=k_0}^{N-1} (z - x_{N,j})}{\prod_{j=0}^{k_0-1} (z - x_{N,j})} R_{12}^*(z) \\ &= -\frac{1}{2} \sqrt{\pi} r_{2,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} \{ (e^{-iN\pi(1-z)} + e^{iN\pi(1-z)}) \\ &\quad \times [e^{-\frac{1}{6}\pi i} \operatorname{Ai}(\omega^2 \tilde{f}^*(z))\tilde{A}(z, n) - e^{\frac{1}{6}\pi i} \operatorname{Ai}'(\omega^2 \tilde{f}^*(z))\tilde{B}(z, n)] \\ &\quad + i e^{-iN\pi(1-z)} [\operatorname{Ai}(\tilde{f}^*(z))\tilde{A}(z, n) + \operatorname{Ai}'(\tilde{f}^*(z))\tilde{B}(z, n)] \}. \end{aligned} \quad (4.72)$$

In view of the well-known formula of the Airy functions (see [1, p. 446])

$$\operatorname{Bi}(z) = \pm i [2e^{\mp \frac{1}{3}\pi i} \operatorname{Ai}(\omega^{\pm 1} z) - \operatorname{Ai}(z)], \quad (4.73)$$

equation (4.72) can be rewritten as

$$\begin{aligned} H_{11}(z) &= \frac{(-1)^{N-1}}{2} \sqrt{\pi} r_{2,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} \\ &\quad \times \{ [\cos(N\pi z) \operatorname{Bi}(\tilde{f}^*(z)) - \sin(N\pi z) \operatorname{Ai}(\tilde{f}^*(z))] \tilde{A}(z, n) \\ &\quad + [\cos(N\pi z) \operatorname{Bi}'(\tilde{f}^*(z)) - \sin(N\pi z) \operatorname{Ai}'(\tilde{f}^*(z))] \tilde{B}(z, n) \}. \end{aligned} \quad (4.74)$$

Similarly, for  $z \in \Omega_-^\Delta$ ,

$$\begin{aligned} H_{11}(z) &= R_{11}^*(z) - \frac{1}{2} e^{iN\pi(1-z)} e^{N\nu z} \frac{\prod_{j=k_0}^{N-1} (z - x_{N,j})}{\prod_{j=0}^{k_0-1} (z - x_{N,j})} R_{12}^*(z) \\ &= -\frac{1}{2} \sqrt{\pi} r_{2,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} \{ (e^{-iN\pi(1-z)} + e^{iN\pi(1-z)}) \\ &\quad \times [e^{\frac{1}{6}\pi i} \operatorname{Ai}(\omega \tilde{f}^*(z))\tilde{A}(z, n) - e^{-\frac{1}{6}\pi i} \operatorname{Ai}'(\omega \tilde{f}^*(z))\tilde{B}(z, n)] \\ &\quad - i e^{iN\pi(1-z)} [\operatorname{Ai}(\tilde{f}^*(z))\tilde{A}(z, n) + \operatorname{Ai}'(\tilde{f}^*(z))\tilde{B}(z, n)] \}. \end{aligned} \quad (4.75)$$

Again by (4.73), we get exactly the same formula given in (4.74). On account of (4.2) and (4.71), one now easily gets the main result (4.67) for  $z \in \Omega_\pm^\Delta \cup (0, \sigma_n)$ . Using the asymptotic formula of  $S(z)$  in (4.64) again, we obtain (4.69).

Now, we show that the asymptotic expansion of  $H_{11}(z)$  in (4.74) holds for  $z$  in I and III, rather than for  $z$  only in  $\Omega_\pm^\Delta$ . Recall that, in our analysis the choice of  $\Omega_\pm^\Delta$  is quite arbitrary and these regions may be large. Moreover, from the relation between  $R^*(z)$  and  $H(z)$  in (4.7),



we know that  $H_{11}(z) = R_{11}^*(z)$  for  $z \in \text{I} \cup \text{III} \setminus \Omega_{\pm}^{\Delta}$ . In contrast to the expansions in (4.72) and (4.75), for  $z$  outside  $\Omega_{\pm}^{\Delta}$  the terms

$$\mp \frac{i}{2} \sqrt{\pi} r_{2,n}(z)^{-\frac{1}{2}} e^{\frac{1}{2}(n-k_0)\tilde{l}_n^*} e^{\mp iN\pi(1-z)} [\text{Ai}(\tilde{f}^*(z))\tilde{A}(z, n) + \text{Ai}'(\tilde{f}^*(z))\tilde{B}(z, n)] \quad (4.76)$$

do not appear. Note that, due to the quantity  $e^{\mp iN\pi(1-z)}$ , these two terms are exponentially small as  $n$  goes to infinity in comparison with the other term in (4.72) and (4.75), respectively. This suggests that the region of validity of the expansion given in (4.74) can be extended to  $z \in \text{I} \cup \text{III}$ . As a consequence, (4.67) holds for  $z \in \mathbb{C} \setminus (-\infty, 0]$  and  $\text{Re } z < \sigma$ .

So far we have established the asymptotic expansions of  $\pi_{N,n}(z)$  in the form of (4.66) and (4.67) only for  $z > \sigma$  and  $z < \sigma$ , respectively. Recall that the choice of  $\sigma_n$  in (4.1) is also somewhat arbitrary, as long as  $\alpha < \sigma < \beta$  and  $\sigma \neq c$ . Hence, by choosing appropriate  $\sigma_n$ , we can make the regions of validity of (4.66) and (4.67) bigger. Indeed, the expansion (4.66) is valid as long as  $\text{Re } z > \alpha$ , and the expansion (4.67) is valid for  $\text{Re } z < \beta$ .

This completes the proof of Theorem 4.1.

**Acknowledgement** The authors are grateful to Dr. Z. Wang for his helpful discussions and kind suggestions.

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