Chin. Ann. Math.
28B(1), 2007, 35–54
DOI: 10.1007/s11401-005-0572-3

Chinese Annals of Mathematics, Series B

An Inverse Problem for Maxwell's Equations in Anisotropic Media**

Shumin LI* Masahiro YAMAMOTO*

Abstract The authors consider Maxwell's equations for an isomagnetic anisotropic and inhomogeneous medium in two dimensions, and discuss an inverse problem of determining the permittivity tensor $\begin{pmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_3 \end{pmatrix}$ and the permeability μ in the constitutive relations from a finite number of lateral boundary measurements. Applying a Carleman estimate, the authors prove an estimate of the Lipschitz type for stability, provided that ε_1, ε_2, ε_3, μ satisfy some a priori conditions.

Keywords Anisotropic media, Inverse problem, Maxwell's equations, Carleman estimate, Lipschitz stability
2000 MR Subject Classification 35R25, 35R30, 35Q60

1 Introduction and Main Results

We consider Maxwell's equations for an isomagnetic anisotropic and inhomogeneous medium (see, e.g., [16, 17]) in two dimensions:

$$\begin{cases} \partial_t D_1(x,t) - \partial_2 H_3(x,t) = 0, & (x,t) \in G \equiv \Omega \times (0,T), \\ \partial_t D_2(x,t) + \partial_1 H_3(x,t) = 0, & (x,t) \in G, \\ \partial_t B_3(x,t) + \partial_1 E_2(x,t) - \partial_2 E_1(x,t) = 0, & (x,t) \in G, \\ \partial_1 D_1(x,t) + \partial_2 D_2(x,t) = 0, & (x,t) \in G, \\ D_1(x,0) = d_1(x), \quad D_2(x,0) = d_2(x), \quad B_3(x,0) = b(x), & x \in \Omega, \\ \nu_1(x) E_2(x,t) - \nu_2(x) E_1(x,t) = 0, & (x,t) \in \Sigma \equiv \partial\Omega \times (0,T) \end{cases} \tag{1.1}$$

with the constitutive relations

$$\begin{cases} \begin{pmatrix} D_1(x,t) \\ D_2(x,t) \end{pmatrix} = \begin{pmatrix} \varepsilon_1(x) & \varepsilon_2(x) \\ \varepsilon_2(x) & \varepsilon_3(x) \end{pmatrix} \begin{pmatrix} E_1(x,t) \\ E_2(x,t) \end{pmatrix}, & (x,t) \in G, \\ B_3(x,t) = \mu(x) H_3(x,t), & (x,t) \in G, \end{cases} \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, Ω is a bounded domain in $\mathbb{R}^2$ with the C^2-boundary $\partial\Omega$, $\partial_k = \frac{\partial}{\partial x_k}$ for $k = 1, 2$, $\partial_t = \frac{\partial}{\partial t}$, and $(\nu_1(x), \nu_2(x))$ denotes the outward unit normal vector to $\partial\Omega$ at x.

Manuscript received December 27, 2005.
*Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba Meguro Tokyo 153, Japan. E-mail: lism@ms.u-tokyo.ac.jp myama@ms.u-tokyo.ac.jp
**Project supported by the Rotary Yoneyama Doctor Course Scholarship (Japan), the Fujyu-kai (Tokyo, Japan), the 21st Century Center of Excellence Program at Graduate School of Mathematical Sciences, the University of Tokyo, the Japan Society for the Promotion of Science (No. 15340027) and the Ministry of Education, Cultures, Sports and Technology (No. 17654019).

Here $\begin{pmatrix} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & \varepsilon_3 \end{pmatrix}$ is the permittivity tensor and μ is the permeability. The boundary condition of E means that Ω is bounded by a superconductive material. For mathematical treatments (see also [3]). Throughout this paper, we assume that $\varepsilon_j(\,\cdot\,)$, $j = 1, 2, 3$, $\mu(\,\cdot\,) \in C^2(\overline{\Omega})$ satisfy

$$\varepsilon_1(x), \quad \varepsilon_3(x), \quad \varepsilon_1(x)\varepsilon_3(x) - \varepsilon_2^2(x), \quad \mu(x) > 0, \quad x \in \overline{\Omega}.$$

These conditions guarantee the hyperbolicity of (1.1), that is, the well-posedness of the boundary-value/initial-value problem follows.

We assume that the initial data d_1, d_2, b in (1.1) are sufficiently smooth and satisfy sufficient compatibility conditions. Throughout this paper, we set

$$\epsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \Phi = (d_1, d_2, b), \quad x_3 = t, \quad \partial_3 = \partial_t, \quad \nabla_x = (\partial_1, \partial_2), \quad \nabla_{x,t} = (\partial_1, \partial_2, \partial_t).$$

By $D_k(\epsilon, \mu; \Phi)(x, t)$, $E_k(\epsilon, \mu; \Phi)(x, t)$, $B_3(\epsilon, \mu; \Phi)(x, t)$ and $H_3(\epsilon, \mu; \Phi)(x, t)$, $k = 1, 2$, we denote the sufficiently smooth solution to (1.1) and (1.2).

In this paper, we consider an inverse problem of determining $(\varepsilon_1(x), \varepsilon_2(x), \varepsilon_3(x), \mu(x))$ for $x \in \Omega$ from the observation data

$$D_k(\epsilon, \mu; \Phi)(x, t), \quad B_3(\epsilon, \mu; \Phi)(x, t), \quad (x, t) \in \partial\Omega \times (0, T),\ k = 1, 2.$$

For inverse problems for Maxwell's equations, we can refer to Romanov [18, 19], Romanov and Kabanikhin [20], Sun and Uhlmann [21], Yamamoto [22, 23]. However, to our knowledge, there are few results on inverse problems of determining the coefficients in the constitutive relations for Maxwell's equations in an anisotropic medium with a finite number of measurements. This is the motivation of our consideration. In this paper, we will establish the uniqueness and the Lipschitz stability for our inverse problem with a finite number of measurements provided that unknown coefficients satisfy some a priori conditions.

To state our main results, we define some notation. Denote

$$\lambda = \sqrt{\inf_{x\in\Omega} |x - x^0|^2} > 0, \quad \Lambda = \sqrt{\sup_{x\in\Omega} |x - x^0|^2 - \lambda^2} > 0 \tag{1.3}$$

with some fixed $x^0 = (x_1^0, x_2^0) \in \mathbb{R}^2 \setminus \overline{\Omega}$.

The following sets are concerned with unknown coefficients $(\varepsilon_{1k}, \varepsilon_{2k}, \varepsilon_{3k}, \mu_k)$, $k = 1, 2$:

$$\begin{aligned}
\mathcal{V} &= \mathcal{V}_{M_0, M_1, \delta, \theta_0, \theta_1} \\
&= \Big\{(a_1, a_2, a_3) \in (C^2(\overline{\Omega}))^3 : \|a_1\|_{C^2(\overline{\Omega})}, \|a_2\|_{C^2(\overline{\Omega})}, \|a_3\|_{C^2(\overline{\Omega})} < M_1,\ \|a_2\|_{C^1(\overline{\Omega})} < \delta, \\
&\quad \|\nabla_x a_1\|_{C(\overline{\Omega})}, \|\nabla_x a_3\|_{C(\overline{\Omega})} < M_0,\ a_1(x), a_3(x) > \theta_1,\ x \in \overline{\Omega}, \\
&\quad \min\Big\{a_1(x)\Big[2 + \Big[\partial_1\Big(\ln\frac{a_1(x)}{a_3(x)}\Big)\Big] \cdot (x_1 - x_1^0)\Big], a_3(x)\Big[2 - \Big[\partial_2\Big(\ln\frac{a_1(x)}{a_3(x)}\Big)\Big] \cdot (x_2 - x_2^0)\Big]\Big\} \\
&\quad - a_1(x)[\partial_1(\ln a_3(x))] \cdot (x_1 - x_1^0) - a_3(x)[\partial_2(\ln a_1(x))] \cdot (x_2 - x_2^0) > \theta_0,\ x \in \overline{\Omega}\Big\},
\end{aligned} \tag{1.4}$$

$$
\begin{aligned}
\mathcal{U} &= \mathcal{U}_{M_0,M_1,\delta,\theta_0,\theta_1,\gamma_0,\mu_0} \\
&= \Big\{(\varepsilon_1,\varepsilon_2,\varepsilon_3,\mu) \in (C^2(\overline{\Omega}))^4 : \frac{\varepsilon_2}{\varepsilon_1\varepsilon_3-\varepsilon_2^2} = \gamma_0,\ \mu = \mu_0 \text{ on } \partial\Omega; \\
&\quad \|\varepsilon_1\|_{C^2(\overline{\Omega})},\ \|\varepsilon_3\|_{C^2(\overline{\Omega})},\ \|\mu\|_{C^2(\overline{\Omega})} < M_1; \\
&\quad \varepsilon_1(x),\ \varepsilon_3(x),\ \varepsilon_1(x)\varepsilon_3(x)-\varepsilon_2^2(x),\ \mu(x) > \theta_1,\ x \in \overline{\Omega}; \\
&\quad \Big(\frac{\varepsilon_1}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)},\ \frac{\varepsilon_2}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)},\ \frac{\varepsilon_3}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)}\Big) \in \mathcal{V}_{M_0,M_1,\delta,\theta_0,\theta_1}; \\
&\quad \|D_k(\epsilon,\mu;\Psi(j))\|_{W^{3,\infty}(G)},\ \|B_3(\epsilon,\mu;\Psi(j))\|_{W^{3,\infty}(G)} < M_1 \text{ for } k=1,2,\ j=1,\cdots,5\Big\}, \quad (1.5)
\end{aligned}
$$

where $M_0 > 0$, M_1, θ_0, $\theta_1 > 0$, $0 < \delta < \min\left\{\theta_1, \frac{\theta_0}{M_3}\right\}$ and smooth functions γ_0, μ_0 on $\partial\Omega$ are suitably given. Here

$$
M_3 = M_3(M_0,M_1,\theta_1,\Lambda,\lambda) = \frac{2\sqrt{\Lambda^2+\lambda^2}}{\theta_1}\Big(2\sqrt{\frac{M_1\theta_1}{\Lambda^2+\lambda^2}} + M_0\sqrt{\frac{M_1}{\theta_1}} + 3M_0 + 2\theta_1\Big). \qquad (1.6)
$$

The set $\mathcal{U}$ is an admissible set where unknown coefficients are considered, and we extra require the last inequality in (1.4) as well as the positivity. If $\|\nabla\varepsilon_j\|_{C(\overline{\Omega})}$, $j = 1, 3$, $\|\varepsilon_2\|_{C^1(\overline{\Omega})}$, and $\|\nabla\mu\|_{C(\overline{\Omega})}$ are sufficiently small, ε_1, ε_3, $\varepsilon_1\varepsilon_3-\varepsilon_2^2$, $\mu> 0$ on $\overline{\Omega}$ and ε_1, ε_2, ε_3, $\mu\in C^2(\overline{\Omega})$, then

$$
\Big(\frac{\varepsilon_1}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)},\ \frac{\varepsilon_2}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)},\ \frac{\varepsilon_3}{\mu(\varepsilon_1\varepsilon_3-\varepsilon_2^2)}\Big) \in \mathcal{V}_{M_0,M_1,\delta,\theta_0,\theta_1}.
$$

Therefore the set $\mathcal{U}$ is restrictive but can contain sufficiently many elements.

Furthermore we can take a constant β satisfying

$$
\begin{aligned}
0 < \beta < \min\Big\{&\frac{4\lambda^2(\theta_1-\delta)}{(\Lambda+\sqrt{\Lambda^2+4\lambda\sqrt{\theta_1-\delta}}\,)^2}, \\
&\frac{4(\theta_0-M_3\delta)^2\theta_1^2}{(5M_0\Lambda\sqrt{M_1}+\sqrt{25M_0^2\Lambda^2M_1+16(\theta_0-M_3\delta)\theta_1^2}\,)^2}\Big\}. \qquad (1.7)
\end{aligned}
$$

To guarantee the uniqueness and the stability of the inverse problem, we will use five sets of the initial data: $\Psi(j) = (d_1(j), d_2(j), b(j))$, $j = 1,\cdots,5$, and we state our main result.

Theorem 1.1 (Stability) *We assume that* $d_k(j), b(j) \in C^2(\overline{\Omega})$, $k = 1,2$, $j = 1,\cdots,5$, *satisfy*

$$
\begin{aligned}
&d_k(1)(x) = b(m)(x) = 0, && x \in \overline{\Omega},\ k=1,2,\ m=2,\cdots,5, && (1.8)\\
&[\partial_1 d_1(m)](x) + [\partial_2 d_2(m)](x) = 0, && x \in \overline{\Omega},\ m=2,\cdots,5, && (1.9)\\
&|b(1)(x)| \ge \theta_2 > 0, && x \in \overline{\Omega}, && (1.10)\\
&|\det(\mathbb{D}_k(x))| \ge \theta_2 > 0, && x \in \overline{\Omega},\ k=1,2 && (1.11)
\end{aligned}
$$

with a constant $\theta_2 > 0$. *Here we set*

$$
\mathbb{D}_1(x) = \begin{pmatrix}
[\partial_1 d_2(2)](x) & -[\partial_2 d_1(2)](x) & d_2(2)(x) & -d_1(2)(x)\\
[\partial_1 d_2(3)](x) & -[\partial_2 d_1(3)](x) & d_2(3)(x) & -d_1(3)(x)\\
[\partial_1 d_2(4)](x) & -[\partial_2 d_1(4)](x) & d_2(4)(x) & -d_1(4)(x)\\
[\partial_1 d_2(5)](x) & -[\partial_2 d_1(5)](x) & d_2(5)(x) & -d_1(5)(x)
\end{pmatrix}, \quad x \in \Omega,
$$

$$\mathbb{D}_2(x) = \begin{pmatrix} & & & & 0 & 2[\partial_1 d_1(2)](x) & 0 & 0 \\ & \mathbb{D}_1(x) & & & \vdots & \vdots & \vdots & \vdots \\ & & & & 0 & 2[\partial_1 d_1(5)](x) & 0 & 0 \\ 2[\partial_1 d_1(2)](x) & 0 & 0 & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & & \mathbb{D}_1(x) & & \\ 2[\partial_1 d_1(5)](x) & 0 & 0 & 0 & & & & \end{pmatrix}, \quad x \in \Omega.$$

Let the observation time $T > 0$ satisfy

$$T > \frac{\Lambda}{\sqrt{\beta}} \tag{1.12}$$

where β satisfies (1.7). *Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\sum_{l=1}^{3} \|\varepsilon_{l1} - \varepsilon_{l2}\|_{L^2(\Omega)} + \|\mu_1 - \mu_2\|_{L^2(\Omega)} \\ &\le C \sum_{j=1}^{5} \Big\{ \sum_{l=1}^{3} \Big[\sum_{k=1}^{2} \|\partial_l \partial_t [D_k(\epsilon_1, \mu_1; \Psi(j)) - D_k(\epsilon_2, \mu_2; \Psi(j))]\|_{L^2(\partial\Omega \times (0,T))} \\ &\quad + \|\partial_l \partial_t [B_3(\epsilon_1, \mu_1; \Psi(j)) - B_3(\epsilon_2, \mu_2; \Psi(j))]\|_{L^2(\partial\Omega \times (0,T))} \Big] \\ &\quad + \sum_{k=1}^{2} \|\partial_t [D_k(\epsilon_1, \mu_1; \Psi(j)) - D_k(\epsilon_2, \mu_2; \Psi(j))]\|_{L^2(\partial\Omega \times (0,T))} \\ &\quad + \|\partial_t [B_3(\epsilon_1, \mu_1; \Psi(j)) - B_3(\epsilon_2, \mu_2; \Psi(j))]\|_{L^2(\partial\Omega \times (0,T))} \Big\} \end{aligned} \tag{1.13}$$

for all $(\epsilon_1, \mu_1) = (\varepsilon_{11}, \varepsilon_{21}, \varepsilon_{31}, \mu_1)$, $(\epsilon_2, \mu_2) = (\varepsilon_{12}, \varepsilon_{22}, \varepsilon_{32}, \mu_2) \in \mathcal{U}$.

Here the constant $C > 0$ is independent of (ϵ_1, μ_1), $(\epsilon_2, \mu_2) \in \mathcal{U}$. For the Lipschitz stability in the inverse problem, we have to choose particular initial inputs $d_k(j)$, $b(j)$, $k = 1, 2$, $j = 1, \cdots, 5$ satisfying (1.8)–(1.11).

Similar kinds of positivity conditions are needed for inverse problems for a scalar hyperbolic equation (see, e.g., [7]) and it is extremely difficult to relax those conditions drastically. Moreover we need to change initial values five times and it may be possible to reduce the number, but here we do not further exploit. We can choose $d_k(m)$, $k = 1, 2$, $m = 2, \cdots, 5$, satisfying (1.9) and (1.11) as the following example shows.

Example We assume that

$$0 \notin \overline{\Omega} \quad \text{and} \quad \overline{\Omega} \cap \{(x_1, x_2); x_1^2 = x_2^2\} = \emptyset. \tag{1.14}$$

We choose

$$\begin{pmatrix} d_1(2)(x) \\ d_1(3)(x) \\ d_1(4)(x) \\ d_1(5)(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1^2 \\ 2x_1x_2 \end{pmatrix}, \quad \begin{pmatrix} d_2(2)(x) \\ d_2(3)(x) \\ d_2(4)(x) \\ d_2(5)(x) \end{pmatrix} = -\begin{pmatrix} x_2 \\ x_1 \\ 2x_1x_2 \\ x_2^2 \end{pmatrix}.$$

Then (1.9) is satisfied and

$$|\det(\mathbb{D}_1(x))| = |2x_1x_2(x_1^2 - x_2^2)|, \quad |\det(\mathbb{D}_2(x))| = 12x_1^2x_2^2(x_1^4 + x_1^2x_2^2 + x_2^4),$$

so that (1.14) implies (1.11).

In Section 2, we will show Carleman estimates as preliminaries for a hyperbolic equation and a first-order differential equation. We will prove Theorem 1.1 in Section 3.

Our proof is based on the methodology of Bukhgeim and Klibanov [2] or Klibanov [13]. Their methodology is by means of a Carleman estimate and there are succeeding publications [1, 4–12, 14, 15, 24] for example (see also the references therein). In this paper, we mainly use the argument of [7] which modifies [2] and further apply the ideas in Chapter 3.5 in Klibanov and Timonov [14] and Klibanov and Yamamoto [15] to prove the Lipschitz stability as well as the uniqueness.

2 Carleman Estimate for a Hyperbolic Equation

For β, λ and $x^0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, we define the functions $\psi = \psi(x,t)$ and $\varphi = \varphi(x,t)$ by

$$\psi(x,t) = |x - x^0|^2 - \beta t^2 - \lambda^2, \quad \varphi(x,t) = e^{\varrho\psi(x,t)} \tag{2.1}$$

with some large parameter $\varrho > 0$. We set $Q = \Omega \times (-T,T)$ for $T > 0$. Moreover we set, for $(x,t) \in Q$,

$$(Pv)(x,t) = (\partial_t^2 v)(x,t) - [a_1(x)(\partial_1^2 v)(x,t) + 2a_2(x)(\partial_1\partial_2 v)(x,t) + a_3(x)(\partial_2^2 v)(x,t)]. \tag{2.2}$$

We show a Carleman estimate for a general second-order hyperbolic equation in two dimensions which is derived from [10, Theorem 2.1] (or [9, Theorem 3.2.1]).

Proposition 2.1 *We assume that* $(a_1,a_2,a_3) \in \mathcal{V}$. *Let* β *satisfy* (1.7) *and let* $\varphi(x,t)$, P *be given by* (2.1), (2.2), *respectively. Then there exists a constant* $0 < \vartheta < 1$ *such that for* $T \in \left(0, \frac{\Lambda}{\sqrt{\beta}} + \vartheta\right)$, *for some* $\varrho > 0$, *there exists a constant* $K_1 = K_1(\vartheta, \varrho, M_0, M_1, \delta, \theta_0, \theta_1, \beta, \Omega, T, x^0) > 0$ *such that*

$$\begin{aligned} &\int_Q (s|\nabla_{x,t} y|^2 + s^3 y^2) e^{2s\varphi} dxdt \\ &\le K_1 \Big(\int_Q |Py|^2 e^{2s\varphi} dxdt + \int_{\partial Q} (s|\nabla_{x,t} y|^2 + s^3 y^2) e^{2s\varphi} d\sigma \Big) \quad \text{for all } s > K_1, \end{aligned} \tag{2.3}$$

provided that $y \in H^1(Q)$, $Py \in L^2(Q)$.

For the proof of Proposition 2.1, in terms of [10, Theorem 2.1], it is sufficient to verify that the weight function φ given by (2.1) satisfies some conditions called the pseudoconvexity for $(a_1,a_2,a_3) \in \mathcal{V}$. For completeness, we will give the verification of these conditions in the appendix.

Proposition 2.2 *Let* $\varphi(x,t)$ *be given by* (2.1). *Then there exists* $K_2 > 0$ *such that for* $s > K_2$ *we have*

$$\int_\Omega s|w|^2 e^{2s\varphi(x,0)} dx \le K_2 \int_\Omega |\nabla w|^2 e^{2s\varphi(x,0)} dx \quad \text{for all } w \in C_0^1(\overline{\Omega}).$$

For the proof of Proposition 2.2, we refer to [4, Lemma 3.6].

3 Proof of Theorem 1.1

We note that $0 < \vartheta < 1$ and $\varrho > 0$ are given in Proposition 2.1, $\beta > 0$ and φ are given by (1.7) and (2.1) respectively, and $(\epsilon_k, \mu_k) \in \mathcal{U},\ k = 1, 2$. For any $\vartheta_0 \in (0, \vartheta)$, we set

$$T = \frac{\Lambda}{\sqrt{\beta}} + \vartheta_0. \tag{3.1}$$

Then $T \in \left(0, \frac{\Lambda}{\sqrt{\beta}} + \vartheta\right)$ and under the assumption of Proposition 2.1, Carleman estimate (2.3) holds on $Q \equiv \Omega \times (-T, T)$.

In order to prove Theorem 1.1, it suffices to prove Theorem 1.1 for T which is given by (3.1).

We extend the functions $D_k(\epsilon_l, \mu_l; \Psi(j))$, $E_k(\epsilon_l, \mu_l; \Psi(j))$, $B_3(\epsilon_l, \mu_l; \Psi(j))$ and $H_3(\epsilon_l, \mu_l; \Psi(j))$, k, l= 1, 2 and j= 1, $\cdots$, 5, from $G = \Omega \times (0, T)$ by the following formulae:

$$\begin{aligned}
D_k(\epsilon_l, \mu_l; \Psi(m))(x, t) &= D_k(\epsilon_l, \mu_l; \Psi(m))(x, -t),\\
E_k(\epsilon_l, \mu_l; \Psi(m))(x, t) &= E_k(\epsilon_l, \mu_l; \Psi(m))(x, -t),\\
B_3(\epsilon_l, \mu_l; \Psi(m))(x, t) &= -B_3(\epsilon_l, \mu_l; \Psi(m))(x, -t),\\
H_3(\epsilon_l, \mu_l; \Psi(m))(x, t) &= -H_3(\epsilon_l, \mu_l; \Psi(m))(x, -t),\\
D_k(\epsilon_l, \mu_l; \Psi(1))(x, t) &= -D_k(\epsilon_l, \mu_l; \Psi(1))(x, -t),\\
E_k(\epsilon_l, \mu_l; \Psi(1))(x, t) &= -E_k(\epsilon_l, \mu_l; \Psi(1))(x, -t),\\
B_3(\epsilon_l, \mu_l; \Psi(1))(x, t) &= B_3(\epsilon_l, \mu_l; \Psi(1))(x, -t),\\
H_3(\epsilon_l, \mu_l; \Psi(1))(x, t) &= H_3(\epsilon_l, \mu_l; \Psi(1))(x, -t)
\end{aligned}$$

for all $(x, t) \in \Omega \times (-T, 0)$, $k, l = 1, 2$, and $m = 2, \cdots, 5$. For simplicity, we denote the extended functions by the same notation. By (1.8), we have $D_k(\epsilon_l, \mu_l; \Psi(1))(\,\cdot\,, 0) = B_3(\epsilon_l, \mu_l; \Psi(m))(\,\cdot\,, 0) = 0$ in Ω for $k, l = 1, 2$ and $m = 2, \cdots, 5$. Therefore, for $k, l = 1, 2$, $m = 2, \cdots, 5$, and $j = 1, \cdots, 5$, by (1.1), (1.2), and $D_k(\epsilon_l, \mu_l; \Psi(j))$, $B_3(\epsilon_l, \mu_l; \Psi(j)) \in W^{3,\infty}(G)$, we can verify that

$$\begin{aligned}
H_3(\epsilon_l, \mu_l; \Psi(m))(\,\cdot\,, 0) &= (\partial_t D_k(\epsilon_l, \mu_l; \Psi(m)))(\,\cdot\,, 0) = (\partial_t E_k(\epsilon_l, \mu_l; \Psi(m)))(\,\cdot\,, 0)\\
&= (\partial_t^2 B_3(\epsilon_l, \mu_l; \Psi(m)))(\,\cdot\,, 0) = (\partial_t^2 H_3(\epsilon_l, \mu_l; \Psi(m)))(\,\cdot\,, 0)\\
&= E_k(\epsilon_l, \mu_l; \Psi(1))(\,\cdot\,, 0) = (\partial_t B_3(\epsilon_l, \mu_l; \Psi(1)))(\,\cdot\,, 0)\\
&= (\partial_t H_3(\epsilon_l, \mu_l; \Psi(1)))(\,\cdot\,, 0) = (\partial_t^2 D_k(\epsilon_l, \mu_l; \Psi(1)))(\,\cdot\,, 0)\\
&= (\partial_t^2 E_k(\epsilon_l, \mu_l; \Psi(1)))(\,\cdot\,, 0) = 0 \quad \text{in } \Omega.
\end{aligned}$$

Therefore, $D_k(\epsilon_l, \mu_l; \Psi(j))$, $B_3(\epsilon_l, \mu_l; \Psi(j)) \in W^{3,\infty}(Q)$, and in both (1.1) and (1.2), G and Σ can be replaced with Q and $\partial\Omega \times (-T, T)$, respectively.

For all $(x, t) \in Q$, $k = 1, 2$, $j = 1, \cdots, 5$ and $m, l = 1, 2, 3$, we set

$$\begin{aligned}
y_k(x, t; j) &= [\partial_t D_k(\epsilon_1, \mu_1; \Psi(j))](x, t) - [\partial_t D_k(\epsilon_2, \mu_2; \Psi(j))](x, t),\\
y_3(x, t; j) &= [\partial_t B_3(\epsilon_1, \mu_1; \Psi(j))](x, t) - [\partial_t B_3(\epsilon_2, \mu_2; \Psi(j))](x, t),\\
R_k(x, t; j) &= [\partial_t D_k(\epsilon_2, \mu_2; \Psi(j))](x, t), \quad R_3(x, t; j) = [\partial_t B_3(\epsilon_2, \mu_2; \Psi(j))](x, t),
\end{aligned}$$

$$z_{ml}(x,t;j) = \partial_m y_l(x,t;j), \quad Y(x,t;j) = (y_1,y_2,y_3)(x,t;j), \quad |Y(x,t;j)|^2 = \sum_{l=1}^{3} |y_l(x,t;j)|^2,$$

$$Z(x,t;j) = (z_{ml}(x,t;j))_{1\le m,l\le 3}, \quad |Z(x,t;j)|^2 = \sum_{m,l=1}^{3} |z_{ml}(x,t;j)|^2,$$

$$\gamma_{lk}(x) = \frac{\varepsilon_{lk}(x)}{\varepsilon_{1k}(x)\varepsilon_{3k}(x) - \varepsilon_{2k}^2(x)}, \quad f_l(x) = \gamma_{l2}(x) - \gamma_{l1}(x),$$

$$f_4(x) = \frac{1}{\mu_2(x)} - \frac{1}{\mu_1(x)}, \quad f_5(x) = \partial_1 f_1(x) + \partial_2 f_2(x), \quad f_6(x) = \partial_1 f_2(x) + \partial_2 f_3(x).$$

Moreover, we set $(F_1,F_2,F_3,F_4)(x) = \partial_1(f_1,f_3,f_5,f_6)(x)$, $(F_5,F_6,F_7,F_8)(x) = \partial_2(f_1,f_3,f_5,f_6)(x)$, and

$$\mathcal{F}(x) = \sum_{k,l=1}^{2} |(\partial_k\partial_l f_4)(x)|^2 + \sum_{\substack{1\le k\le 2\\ 1\le l\le 6}} |\partial_k f_l(x)|^2 + \sum_{l=1}^{6} |f_l(x)|^2 \quad \text{for } x\in\overline{\Omega}. \tag{3.2}$$

Then we have, for $j = 1,\cdots,5$ and $m,l = 1,2,3$, $y_l(\,\cdot\,;j), R_l(\,\cdot\,;j) \in W^{2,\infty}(Q)$, $z_{ml}(\,\cdot\,;j) \in W^{1,\infty}(Q)$,

$$\partial_t y_1(\,\cdot\,;j) - \partial_2\Big[\frac{1}{\mu_1} y_3(\,\cdot\,;j)\Big] = -\partial_2[f_4 R_3(\,\cdot\,;j)] \quad \text{in } Q, \tag{3.3}$$

$$\partial_t y_2(\,\cdot\,;j) + \partial_1\Big[\frac{1}{\mu_1} y_3(\,\cdot\,;j)\Big] = \partial_1[f_4 R_3(\,\cdot\,;j)] \quad \text{in } Q, \tag{3.4}$$

$$\begin{aligned}&\partial_t y_3(\,\cdot\,;j) + \partial_1[-\gamma_{21}y_1(\,\cdot\,;j) + \gamma_{11}y_2(\,\cdot\,;j)] - \partial_2[\gamma_{31}y_1(\,\cdot\,;j) - \gamma_{21}y_2(\,\cdot\,;j)]\\ &= [\partial_1 R_2(\,\cdot\,;j)]f_1 - 2[\partial_1 R_1(\,\cdot\,;j)]f_2 - [\partial_2 R_1(\,\cdot\,;j)]f_3 + R_2(\,\cdot\,;j)f_5 - R_1(\,\cdot\,;j)f_6 \quad \text{in } Q,\end{aligned} \tag{3.5}$$

$$(\partial_1 y_1)(\,\cdot\,;j) + (\partial_2 y_2)(\,\cdot\,;j) = 0 \quad \text{in } Q, \tag{3.6}$$

$$y_1(\cdot,0;j) = -\partial_2[f_4 b(j)], \quad y_2(\,\cdot\,,0;j) = \partial_1[f_4 b(j)] \quad \text{in } \Omega, \tag{3.7}$$

$$y_3(\,\cdot\,,0;j) = [\partial_1 d_2(j)]f_1 - 2[\partial_1 d_1(j)]f_2 - [\partial_2 d_1(j)]f_3 + [d_2(j)]f_5 - [d_1(j)]f_6 \quad \text{in } \Omega. \tag{3.8}$$

In fact, by (1.2), we have

$$\begin{aligned}&H_3(\epsilon_1,\mu_1;\Psi(j)) - H_3(\epsilon_2,\mu_2;\Psi(j))\\ &= \frac{1}{\mu_1}B_3(\epsilon_1,\mu_1;\Psi(j)) - \frac{1}{\mu_2}B_3(\epsilon_2,\mu_2;\Psi(j))\\ &= \frac{1}{\mu_1}\left[B_3(\epsilon_1,\mu_1;\Psi(j)) - B_3(\epsilon_2,\mu_2;\Psi(j))\right] - f_4 B_3(\epsilon_2,\mu_2;\Psi(j)) \quad \text{in } Q.\end{aligned} \tag{3.9}$$

Using (1.2) and noting

$$\begin{pmatrix}\varepsilon_{1k} & \varepsilon_{2k}\\ \varepsilon_{2k} & \varepsilon_{3k}\end{pmatrix}\begin{pmatrix}\gamma_{3k} & -\gamma_{2k}\\ -\gamma_{2k} & \gamma_{1k}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{on } \overline{\Omega},\ k=1,2,$$

we have

$$\begin{aligned}&\begin{pmatrix}E_1(\epsilon_1,\mu_1;\Psi(j)) - E_1(\epsilon_2,\mu_2;\Psi(j))\\ E_2(\epsilon_1,\mu_1;\Psi(j)) - E_2(\epsilon_2,\mu_2;\Psi(j))\end{pmatrix}\\ &= \begin{pmatrix}\gamma_{31} & -\gamma_{21}\\ -\gamma_{21} & \gamma_{11}\end{pmatrix}\begin{pmatrix}D_1(\epsilon_1,\mu_1;\Psi(j))\\ D_2(\epsilon_1,\mu_1;\Psi(j))\end{pmatrix} - \begin{pmatrix}\gamma_{32} & -\gamma_{22}\\ -\gamma_{22} & \gamma_{12}\end{pmatrix}\begin{pmatrix}D_1(\epsilon_2,\mu_2;\Psi(j))\\ D_2(\epsilon_2,\mu_2;\Psi(j))\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \gamma_{31} & -\gamma_{21} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix} \begin{pmatrix} D_1(\epsilon_1,\mu_1;\Psi(j)) - D_1(\epsilon_2,\mu_2;\Psi(j)) \\ D_2(\epsilon_1,\mu_1;\Psi(j)) - D_2(\epsilon_2,\mu_2;\Psi(j)) \end{pmatrix}$$
$$- \begin{pmatrix} f_3 & -f_2 \\ -f_2 & f_1 \end{pmatrix} \begin{pmatrix} D_1(\epsilon_2,\mu_2;\Psi(j)) \\ D_2(\epsilon_2,\mu_2;\Psi(j)) \end{pmatrix} \quad \text{in } Q. \tag{3.10}$$

Differentiating (3.9) and (3.10) with respect to t and noting t-independence of μ_1, γ_{l1}, f_l $(l = 1,2,3)$, f_4, we have

$$[\partial_t H_3(\epsilon_1,\mu_1;\Psi(j))](\,\cdot\,) - [\partial_t H_3(\epsilon_2,\mu_2;\Psi(j))](\,\cdot\,) = \frac{1}{\mu_1} y_3(\,\cdot\,;j) - f_4 R_3(\,\cdot\,;j), \tag{3.11}$$

$$\begin{pmatrix} [\partial_t E_1(\epsilon_1,\mu_1;\Psi(j))](\,\cdot\,) - [\partial_t E_1(\epsilon_2,\mu_2;\Psi(j))](\,\cdot\,) \\ [\partial_t E_2(\epsilon_1,\mu_1;\Psi(j))](\,\cdot\,) - [\partial_t E_2(\epsilon_2,\mu_2;\Psi(j))](\,\cdot\,) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_{31} & -\gamma_{21} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix} \begin{pmatrix} y_1(\,\cdot\,;j) \\ y_2(\,\cdot\,;j) \end{pmatrix} - \begin{pmatrix} f_3 & -f_2 \\ -f_2 & f_1 \end{pmatrix} \begin{pmatrix} R_1(\,\cdot\,;j) \\ R_2(\,\cdot\,;j) \end{pmatrix} \quad \text{in } Q. \tag{3.12}$$

By (1.1), we have

$$y_1(\,\cdot\,;j) = \{\partial_2[H_3(\epsilon_1,\mu_1;\Psi(j)) - H_3(\epsilon_2,\mu_2;\Psi(j))]\}(\,\cdot\,), \tag{3.13}$$
$$y_2(\,\cdot\,;j) = -\{\partial_1[H_3(\epsilon_1,\mu_1;\Psi(j)) - H_3(\epsilon_2,\mu_2;\Psi(j))]\}(\,\cdot\,), \tag{3.14}$$
$$y_3(\,\cdot\,;j) = -\{\partial_1[E_2(\epsilon_1,\mu_1;\Psi(j)) - E_2(\epsilon_2,\mu_2;\Psi(j))]\}(\,\cdot\,) + \{\partial_2[E_1(\epsilon_1,\mu_1;\Psi(j)) - E_1(\epsilon_2,\mu_2;\Psi(j))]\}(\,\cdot\,) \quad \text{in } Q. \tag{3.15}$$

Differentiating (3.13) and (3.14) with respect to t and using (3.11), we can obtain (3.3) and (3.4). Differentiating (3.15) with respect to t and using (3.12), we have

$$\begin{aligned} &\partial_t y_3(\,\cdot\,;j) + \partial_1[-\gamma_{21} y_1(\,\cdot\,;j) + \gamma_{11} y_2(\,\cdot\,;j)] - \partial_2[\gamma_{31} y_1(\,\cdot\,;j) - \gamma_{21} y_2(\,\cdot\,;j)] \\ &= -\partial_1[f_2 R_1(\,\cdot\,;j) - f_1 R_2(\,\cdot\,;j)] + \partial_2[-f_3 R_1(\,\cdot\,;j) + f_2 R_2(\,\cdot\,;j)] \\ &= [\partial_1 R_2(\,\cdot\,;j)] f_1 + [\partial_2 R_2(\,\cdot\,;j) - \partial_1 R_1(\,\cdot\,;j)] f_2 - [\partial_2 R_1(\,\cdot\,;j)] f_3 \\ &\quad - (\partial_1 f_2 + \partial_2 f_3)\, R_1(\,\cdot\,;j) + (\partial_1 f_1 + \partial_2 f_2)\, R_2(\,\cdot\,;j) \quad \text{in } Q. \end{aligned}$$

Therefore, using $\partial_1 R_1(\,\cdot\,;j) + \partial_2 R_2(\,\cdot\,;j) = 0$ in Q and noting the definitions of f_5 and f_6, we have (3.5). By (1.1), we have (3.6). Moreover, by (1.1), (3.9), (3.13) and (3.14), we can obtain (3.7). By (1.1), (3.10) and (3.15), we have

$$\begin{aligned} y_3(\,\cdot\,,0;j) &= -\partial_1[f_2 d_1(j) - f_1 d_2(j)] + \partial_2[-f_3 d_1(j) + f_2 d_2(j)] \\ &= [\partial_1(d_2(j))] f_1 + [\partial_2 d_2(j) - \partial_1 d_1(j)] f_2 - [\partial_2(d_1(j))] f_3 \\ &\quad - (\partial_1 f_2 + \partial_2 f_3) d_1(j) + (\partial_1 f_1 + \partial_2 f_2) d_2(j) \quad \text{in } \Omega. \end{aligned}$$

Therefore, using (1.8) and (1.9) and noting the definitions of f_5 and f_6, we have (3.8).

Furthermore, for $j = 1, \cdots, 5$, by the extensions of $D_k(\epsilon_l,\mu_l;\Psi(j))$, $B_3(\epsilon_l,\mu_l;\Psi(j))$, $k,l = 1,2$, and the definitions of $Y(\,\cdot\,;j)$, $Z(\,\cdot\,;j)$, we see that

$$|Y(\,\cdot\,,t;j)| = |Y(\,\cdot\,,-t;j)|, \quad |Z(\,\cdot\,,t;j)| = |Z(\,\cdot\,,-t;j)|, \quad t > 0. \tag{3.16}$$

Differentiating (3.3) with respect to t, we have

$$\partial_t^2 y_1(\,\cdot\,;j) - \partial_2\Big[\frac{1}{\mu_1}(\partial_t y_3(\,\cdot\,;j))\Big] = -\partial_2[f_4(\partial_t R_3(\,\cdot\,;j))] \quad \text{in } Q \text{ for } j=1,\cdots,5. \tag{3.17}$$

We can replace $\partial_t y_3(\,\cdot\,;j)$ in the second term of (3.17) by using (3.5) and obtain that

$$\begin{aligned}&\partial_t^2 y_1(\,\cdot\,;j) - \frac{\gamma_{21}}{\mu_1}\partial_1\partial_2 y_1(\,\cdot\,;j) + \frac{\gamma_{11}}{\mu_1}\partial_1\partial_2 y_2(\,\cdot\,;j) - \frac{\gamma_{31}}{\mu_1}\partial_2^2 y_1(\,\cdot\,;j) + \frac{\gamma_{21}}{\mu_1}\partial_2^2 y_2(\,\cdot\,;j)\\ &= A_1(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j)) + \partial_2\Big\{\frac{1}{\mu_1}[(\partial_1 R_2(\,\cdot\,;j))f_1 - 2(\partial_1 R_1(\,\cdot\,;j))f_2\\ &\quad - (\partial_2 R_1(\,\cdot\,;j))f_3 + R_2(\,\cdot\,;j)f_5 - R_1(\,\cdot\,;j)f_6]\Big\} - \partial_2\{f_4[\partial_t R_3(\,\cdot\,;j)]\} \quad \text{in } Q\end{aligned} \tag{3.18}$$

for $j=1,\cdots,5$. Here and henceforth, $A_k(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j))$, $k=1,2,\cdots$, denote linear functions of elements of $Z(\,\cdot\,;j)$ and $Y(\,\cdot\,;j)$ with $C(\overline{\Omega})$-coefficients. Furthermore, we can replace $\partial_2 y_2(\,\cdot\,;j)$ in the left-hand side of (3.18) with $-\partial_1 y_1(\,\cdot\,;j)$ by using (3.6) and arrive at

$$\begin{aligned}&\partial_t^2 y_1(\,\cdot\,;j) - \Big[\frac{\gamma_{11}}{\mu_1}\partial_1^2 y_1(\,\cdot\,;j) + \frac{2\gamma_{21}}{\mu_1}\partial_1\partial_2 y_1(\,\cdot\,;j) + \frac{\gamma_{31}}{\mu_1}\partial_2^2 y_1(\,\cdot\,;j)\Big]\\ &= A_1(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j)) + \partial_2\Big\{\frac{1}{\mu_1}[(\partial_1 R_2(\,\cdot\,;j))f_1 - 2(\partial_1 R_1(\,\cdot\,;j))f_2\\ &\quad - (\partial_2 R_1(\,\cdot\,;j))f_3 + R_2(\,\cdot\,;j)f_5 - R_1(\,\cdot\,;j)f_6]\Big\} - \partial_2\{f_4[\partial_t R_3(\,\cdot\,;j)]\} \quad \text{in } Q \text{ for } j=1,\cdots,5.\end{aligned}$$

By the same argument, we can obtain that

$$\begin{aligned}&\partial_t^2 y_2(\,\cdot\,;j) - \Big[\frac{\gamma_{11}}{\mu_1}\partial_1^2 y_2(\,\cdot\,;j) + \frac{2\gamma_{21}}{\mu_1}\partial_1\partial_2 y_2(\,\cdot\,;j) + \frac{\gamma_{31}}{\mu_1}\partial_2^2 y_2(\,\cdot\,;j)\Big]\\ &= A_2(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j)) - \partial_1\Big\{\frac{1}{\mu_1}[(\partial_1 R_2(\,\cdot\,;j))f_1 - 2(\partial_1 R_1(\,\cdot\,;j))f_2\\ &\quad - (\partial_2 R_1(\,\cdot\,;j))f_3 + R_2(\,\cdot\,;j)f_5 - R_1(\,\cdot\,;j)f_6]\Big\} + \partial_1\{f_4[\partial_t R_3(\,\cdot\,;j)]\} \quad \text{in } Q,\\ &\partial_t^2 y_3(\,\cdot\,;j) - \Big[\frac{\gamma_{11}}{\mu_1}\partial_1^2 y_3(\,\cdot\,;j) + \frac{2\gamma_{21}}{\mu_1}\partial_1\partial_2 y_3(\,\cdot\,;j) + \frac{\gamma_{31}}{\mu_1}\partial_2^2 y_3(\,\cdot\,;j)\Big]\\ &= A_3(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j)) - \partial_1\{\gamma_{21}[\partial_2(f_4R_3(\,\cdot\,;j))] + \gamma_{11}[\partial_1(f_4R_3(\,\cdot\,;j))]\}\\ &\quad - \partial_2\{\gamma_{31}[\partial_2(f_4R_3(\,\cdot\,;j))] + \gamma_{21}[\partial_1(f_4R_3(\,\cdot\,;j))]t\} + [\partial_1\partial_t R_2(\,\cdot\,;j)]f_1 - 2[\partial_1\partial_t R_1(\,\cdot\,;j)]f_2\\ &\quad - [\partial_2\partial_t R_1(\,\cdot\,;j)]f_3 + [\partial_t R_2(\,\cdot\,;j)]f_5 - [\partial_t R_1(\,\cdot\,;j)]f_6 \quad \text{in } Q\end{aligned}$$

for $j=1,\cdots,5$. Therefore, by $(\epsilon_1,\mu_1)\in\mathcal{U}$, we have $\left(\frac{\gamma_{11}}{\mu_1},\frac{\gamma_{21}}{\mu_1},\frac{\gamma_{31}}{\mu_1}\right)\in\mathcal{V}$ and can apply Proposition 2.1 to $y_l(\,\cdot\,;j)$, $l=1,2,3$, so that by $\|R_l(\,\cdot\,;j)\|_{W^{2,\infty}(Q)}\le M_1$, $l=1,2,3$,

$$\begin{aligned}&\int_Q (s|Z(x,t;j)|^2 + s^3|Y(x,t;j)|^2)e^{2s\varphi(x,t)}dxdt\\ &\le C_1\Big\{\int_Q \mathcal{F}(x)e^{2s\varphi(x,t)}dxdt + \int_\Omega (s|Z(x,T;j)|^2 + s^3|Y(x,T;j)|^2)e^{2s\varphi(x,T)}dx\\ &\quad + s^3e^{2s\Phi}\Theta + \int_\Omega (s|Z(x,-T;j)|^2 + s^3|Y(x,-T;j)|^2)e^{2s\varphi(x,-T)}dx\Big\}\end{aligned} \tag{3.19}$$

for all sufficiently large $s>0$ and $j=1,\cdots,5$ where $\Phi\equiv e^{\varrho\Lambda^2}\ge 1$ and

$$\Theta \equiv \sum_{j=1}^{5}\int_{-T}^{T}\int_{\partial\Omega}(|Z(x,t;j)|^2 + |Y(x,t;j)|^2)d\sigma dt. \tag{3.20}$$

Here and henceforth, $C_k,\ k=1,2,\cdots,\ C_*,\ C_{**}$ denote positive constants which are dependent on Ω, T, x^0, $\Psi(1)$, $\Psi(2)$, $\Psi(3)$, $\Psi(4)$, $\Psi(5)$, M_0, M_1, δ, θ_0, θ_1, θ_2, γ_0, μ_0, ϱ, β, but independent of s and η. By (3.16), we can eliminate the last term in (3.19).

Noting the definition of $(z_{ml}(\,\cdot\,;j))_{1\le m,l\le 3}$ and differentiating (3.3)–(3.5) with respect to x_m, we can obtain that

$$\begin{aligned}&\partial_t z_{m1}(\,\cdot\,;j)-\partial_2\Big[\frac{1}{\mu_1}z_{m3}(\,\cdot\,;j)\Big]\\ =\ & A_{3m+1}(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j))-\partial_2\partial_m[f_4R_3(\,\cdot\,;j)]\quad\text{in } Q,\end{aligned}\tag{3.21}$$

$$\begin{aligned}&\partial_t z_{m2}(\,\cdot\,;j)+\partial_1\Big[\frac{1}{\mu_1}z_{m3}(\,\cdot\,;j)\Big]\\ =\ & A_{3m+2}(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j))+\partial_1\partial_m[f_4R_3(\,\cdot\,;j)]\quad\text{in } Q,\end{aligned}\tag{3.22}$$

$$\begin{aligned}&\partial_t z_{m3}(\,\cdot\,;j)+\partial_1[-\gamma_{21}z_{m1}(\,\cdot\,;j)+\gamma_{11}z_{m2}(\,\cdot\,;j)]-\partial_2[\gamma_{31}z_{m1}(\,\cdot\,;j)-\gamma_{21}z_{m2}(\,\cdot\,;j)]\\ =\ & A_{3m+3}(\,\cdot\,;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j))+\partial_m\{[\partial_1R_2(\,\cdot\,;j)]f_1-2[\partial_1R_1(\,\cdot\,;j)]f_2\\ &-[\partial_2R_1(\,\cdot\,;j)]f_3+R_2(\,\cdot\,;j)f_5-R_1(\,\cdot\,;j)f_6\}\quad\text{in } Q\end{aligned}\tag{3.23}$$

for $m=1,2,3$ and $j=1,\cdots,5$. For any $-T\le t_1<t_2\le T$, $s\ge 0$ and $\eta>0$, multiplying (3.21), (3.22) and (3.23) by $[\gamma_{31}z_{m1}(\,\cdot\,;j)-\gamma_{21}z_{m2}(\,\cdot\,;j)]\ e^{2s\varphi-\eta t}$, $[-\gamma_{21}z_{m1}(\,\cdot\,;j)+\gamma_{11}z_{m2}(\,\cdot\,;j)]\cdot e^{2s\varphi-\eta t}$, and $\frac{z_{m3}(\,\cdot\,;j)}{\mu_1}\ e^{2s\varphi-\eta t}$ respectively, adding them and integrating over $\Omega\times(t_1,t_2)$, we can obtain that

$$\begin{aligned}&\int_{t_1}^{t_2}\!\!\int_\Omega\Big\{\frac12\partial_t\mathcal{Z}_m(x,t;j)-\partial_2\Big[\frac{z_{m3}(x,t;j)}{\mu_1(x)}(\gamma_{31}(x)z_{m1}(x,t;j)-\gamma_{21}(x)z_{m2}(x,t;j))\Big]\\ &+\partial_1\Big[\frac{z_{m3}(x,t;j)}{\mu_1(x)}(-\gamma_{21}(x)z_{m1}(x,t;j)+\gamma_{11}(x)z_{m2}(x,t;j))\Big]\Big\}e^{2s\varphi(x,t)-\eta t}dxdt\\ =&\int_{t_1}^{t_2}\!\!\int_\Omega\Big\{\sum_{l=1}^{3}A_{9+3m+l}(x,t;j;Z(\,\cdot\,;j),Y(\,\cdot\,;j))z_{ml}(x,t;j)\\ &-[\partial_2\partial_m(f_4(x)R_3(x,t;j))][\gamma_{31}(x)z_{m1}(x,t;j)-\gamma_{21}(x)z_{m2}(x,t;j)]\\ &+[\partial_1\partial_m(f_4(x)R_3(x,t;j))][-\gamma_{21}(x)z_{m1}(x,t;j)+\gamma_{11}(x)z_{m2}(x,t;j)]\\ &+\{\partial_m[(\partial_1R_2(x,t;j))f_1(x)-2(\partial_1R_1(x,t;j))f_2(x)-(\partial_2R_1(x,t;j))f_3(x)\\ &+R_2(x,t;j)f_5(x)-R_1(x,t;j)f_6(x)]\}\frac{z_{m3}(x,t;j)}{\mu_1(x)}\Big\}e^{2s\varphi(x,t)-\eta t}dxdt,\end{aligned}\tag{3.24}$$

where $\mathcal{Z}_m(\,\cdot\,;j)\equiv\gamma_{31}z_{m1}^2(\,\cdot\,;j)-2\gamma_{21}z_{m1}(\,\cdot\,;j)z_{m2}(\,\cdot\,;j)+\gamma_{11}z_{m2}^2(\,\cdot\,;j)+\frac{z_{m3}^2(\,\cdot\,;j)}{\mu_1}$, $m=1,2,3$, and $j=1,\cdots,5$. We denote the left- and the right-hand sides of (3.24) by I_{1jm} and I_{2jm} respectively. Using integration by parts, we can obtain that

$$\begin{aligned}\mathrm{I}_{1jm}=&\frac12\int_\Omega\mathcal{Z}_m(x,t_2;j)e^{2s\varphi(x,t_2)-\eta t_2}dx-\frac12\int_\Omega\mathcal{Z}_m(x,t_1;j)e^{2s\varphi(x,t_1)-\eta t_1}dx\\ &+\int_{t_1}^{t_2}\!\!\int_{\partial\Omega}\Big\{\frac{z_{m3}(x,t;j)}{\mu_1(x)}[-\gamma_{21}(x)z_{m1}(x,t;j)+\gamma_{11}(x)z_{m2}(x,t;j)]\nu_1(x)\\ &-\frac{z_{m3}(x,t;j)}{\mu_1(x)}[\gamma_{31}(x)z_{m1}(x,t;j)-\gamma_{21}(x)z_{m2}(x,t;j)]\nu_2(x)\Big\}e^{2s\varphi(x,t)-\eta t}d\sigma dt\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \mathcal{Z}_m(x,t;j)[2s(\partial_t\varphi(x,t)) - \eta]e^{2s\varphi(x,t)-\eta t}dxdt\\
&- 2s\int_{t_1}^{t_2}\int_\Omega \Big\{\frac{z_{m3}(x,t;j)}{\mu_1(x)}[-\gamma_{21}(x)z_{m1}(x,t;j) + \gamma_{11}(x)z_{m2}(x,t;j)][\partial_1\varphi(x,t)]\\
&- \frac{z_{m3}(x,t;j)}{\mu_1(x)}[\gamma_{31}(x)z_{m1}(x,t;j) - \gamma_{21}(x)z_{m2}(x,t;j)][\partial_2\varphi(x,t)]\Big\}e^{2s\varphi(x,t)-\eta t}dxdt
\end{aligned}
$$

for $m = 1,2,3$, $j = 1,\cdots,5$, $\eta > 0$, and $s \geq 0$. Therefore, by the definition of φ, the inequality: $2|ab| \leq a^2 + b^2$ and $(\epsilon_1, \mu_1) \in \mathcal{U}$, we have

$$
\begin{aligned}
\mathrm{I}_{1jm} \geq\ & \frac{1}{2}\int_\Omega \mathcal{Z}_m(x,t_2;j)e^{2s\varphi(x,t_2)-\eta t_2}dx - \frac{1}{2}\int_\Omega \mathcal{Z}_m(x,t_1;j)e^{2s\varphi(x,t_1)-\eta t_1}dx\\
&+ \frac{\eta}{2}\int_{t_1}^{t_2}\int_\Omega \mathcal{Z}_m(x,t;j)e^{2s\varphi(x,t)-\eta t}dxdt - C_2\int_{t_1}^{t_2}\int_{\partial\Omega} e^{2s\varphi(x,t)-\eta t}\sum_{l=1}^{3}|z_{ml}(x,t;j)|^2 d\sigma dt\\
&- C_3 s\int_{t_1}^{t_2}\int_\Omega e^{2s\varphi(x,t)-\eta t}\sum_{l=1}^{3}|z_{ml}(x,t;j)|^2 dxdt
\end{aligned}
\tag{3.25}
$$

for $m = 1,2,3$, $j = 1,\cdots,5$, $\eta > 0$, and $s \geq 0$. Furthermore, by (3.2), the definition of $\mathcal{Z}_m(\,\cdot\,;j)$, the inequality: $2|ab| \leq a^2 + b^2$, $(\epsilon_1, \mu_1) \in \mathcal{U}$, and $\|R_l(\,\cdot\,;j)\|_{W^{2,\infty}(Q)} < M_1$, $l = 1,2,3$, we can obtain

$$
\begin{aligned}
\mathrm{I}_{2jm} \leq C_*\int_{t_1}^{t_2}\int_\Omega (|Z(x,t;j)|^2 + |Y(x,t;j)|^2)e^{2s\varphi(x,t)-\eta t}dxdt\\
+ C_4\int_{t_1}^{t_2}\int_\Omega \mathcal{F}(x)e^{2s\varphi(x,t)-\eta t}dxdt
\end{aligned}
\tag{3.26}
$$

for $m = 1,2,3$, $j = 1,\cdots,5$, $\eta > 0$, and $s \geq 0$. By $(\epsilon_1, \mu_1) \in \mathcal{U}$, we have $\mu_1(x)$, $\frac{\gamma_{11}(x)}{\mu_1(x)}$, $\frac{\gamma_{31}(x)}{\mu_1(x)} > \theta_1$ for $x \in \overline{\Omega}$, $\|\mu_1\|_{C^2(\overline{\Omega})} < M_1$, and $\|\frac{\gamma_{21}}{\mu_1}\|_{C^1(\overline{\Omega})} < \delta$. Therefore, by the inequality: $2|ab| \leq a^2 + b^2$, $\delta < \theta_1$ and the definition of $\mathcal{Z}_m(\,\cdot\,;j)$, we have

$$
\begin{aligned}
\frac{1}{2}\mathcal{Z}_m(x,t;j) &\geq \frac{\mu_1(x)}{2}\{\theta_1 z_{m1}^2 - \delta(z_{m1}^2 + z_{m2}^2) + \theta_1 z_{m2}^2\}(x,t;j) + \frac{1}{2\mu_1(x)}z_{m3}^2(x,t;j)\\
&\geq \frac{\theta_1}{2}(\theta_1 - \delta)(z_{m1}^2 + z_{m2}^2)(x,t;j) + \frac{1}{2M_1}z_{m3}^2(x,t;j) \geq h\sum_{l=1}^{3}z_{ml}^2(x,t;j),
\end{aligned}
\tag{3.27}
$$

where $(x,t) \in \overline{Q}$, $j = 1,\cdots,5$, $m = 1,2,3$, and $h = \frac{\min\{\theta_1(\theta_1-\delta), \frac{1}{M_1}\}}{2} > 0$. Summing (3.24) over $m = 1,2,3$ and using (3.25), (3.26), and (3.27), we can obtain that

$$
\begin{aligned}
& h\int_\Omega |Z(x,t_2;j)|^2 e^{2s\varphi(x,t_2)-\eta t_2}dx + \eta h\int_{t_1}^{t_2}\int_\Omega |Z(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}dxdt\\
\leq\ & C_5\Big\{\int_\Omega |Z(x,t_1;j)|^2 e^{2s\varphi(x,t_1)-\eta t_1}dx + \int_{t_1}^{t_2}\int_{\partial\Omega} |Z(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}d\sigma dt\\
& + s\int_{t_1}^{t_2}\int_\Omega |Z(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}dxdt + \int_{t_1}^{t_2}\int_\Omega \mathcal{F}(x)e^{2s\varphi(x,t)-\eta t}dxdt\Big\}\\
& + 3C_*\int_{t_1}^{t_2}\int_\Omega (|Z(x,t;j)|^2 + |Y(x,t;j)|^2)e^{2s\varphi(x,t)-\eta t}dxdt
\end{aligned}
\tag{3.28}
$$

for $j = 1, \cdots, 5$, $\eta > 0$, and $s \ge 0$. Moreover, using (3.3)–(3.5) and repeating the procedure of deriving (3.28) from (3.21)–(3.23), we can obtain

$$\begin{aligned} & h\int_\Omega |Y(x,t_2;j)|^2 e^{2s\varphi(x,t_2)-\eta t_2}dx + \eta h\int_{t_1}^{t_2}\int_\Omega |Y(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}dxdt \\ \le\ & C_6\Big\{\int_\Omega |Y(x,t_1;j)|^2 e^{2s\varphi(x,t_1)-\eta t_1}dx + \int_{t_1}^{t_2}\int_{\partial\Omega} |Y(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}d\sigma dt \\ & + s\int_{t_1}^{t_2}\int_\Omega |Y(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}dxdt + \int_{t_1}^{t_2}\int_\Omega \mathcal{F}(x) e^{2s\varphi(x,t)-\eta t}dxdt\Big\} \\ & + C_{**}\int_{t_1}^{t_2}\int_\Omega |Y(x,t;j)|^2 e^{2s\varphi(x,t)-\eta t}dxdt \end{aligned} \tag{3.29}$$

for $j = 1, \cdots, 5$, $\eta > 0$ and $s \ge 0$. Adding (3.28) and (3.29), taking $\eta > \frac{3C_* + C_{**}}{h}$, and noting $-T \le t_1 < t_2 \le T$ and (3.20), we can obtain that

$$\begin{aligned} & \int_\Omega (|Z(x,t_2;j)|^2 + |Y(x,t_2;j)|^2) e^{2s\varphi(x,t_2)}dx \\ \le\ & C_7\Big\{\int_\Omega (|Z(x,t_1;j)|^2 + |Y(x,t_1;j)|^2) e^{2s\varphi(x,t_1)}dx + e^{2s\Phi}\Theta \\ & + \int_Q \mathcal{F}(x) e^{2s\varphi(x,t)}dxdt + s\int_Q (|Z(x,t;j)|^2 + |Y(x,t;j)|^2) e^{2s\varphi(x,t)}dxdt\Big\} \end{aligned} \tag{3.30}$$

for $j = 1, \cdots, 5$ and $s \ge 0$.

Taking $t_2 = 0$, $t_1 = -T$, and $s > 0$ sufficiently large in (3.30), we can appply (3.19) to replace the last term in (3.30) and arrive at

$$\begin{aligned} & \int_\Omega (|Z(x,0;j)|^2 + |Y(x,0;j)|^2) e^{2s\varphi(x,0)}dx \\ \le\ & C_8\Big\{\int_\Omega (|Z(x,-T;j)|^2 + |Y(x,-T;j)|^2) e^{2s\varphi(x,-T)}dx + s^3 e^{2s\Phi}\Theta \\ & + \int_Q \mathcal{F}(x) e^{2s\varphi(x,t)}dxdt + \int_\Omega (s|Z(x,T;j)|^2 + s^3|Y(x,T;j)|^2) e^{2s\varphi(x,T)}dx\Big\} \\ \le\ & C_9\Big\{s^3 e^{2s\Upsilon}\int_\Omega (|Z(x,T;j)|^2 + |Y(x,T;j)|^2)dx + s^3 e^{2s\Phi}\Theta + \int_Q \mathcal{F}(x) e^{2s\varphi(x,t)}dxdt\Big\} \end{aligned} \tag{3.31}$$

for all sufficiently large $s > 0$ and $j = 1, \cdots, 5$. For the second inequality in (3.16), we have used (3.16) and $\varphi(x,T) = \varphi(x,-T) = e^{\varrho(|x-x^0|^2 - \beta T^2 - \lambda^2)} \le \Upsilon \equiv e^{\varrho(\Lambda^2 - \beta T^2)}$. By (1.12), we see that

$$0 < \Upsilon < 1. \tag{3.32}$$

Furthermore, taking $t_2 = T$, $t_1 = 0$ and $s = 0$ in (3.30), we see that

$$\begin{aligned} & \int_\Omega (|Z(x,T;j)|^2 + |Y(x,T;j)|^2)dx \\ \le\ & C_{10}\Big\{\int_\Omega (|Z(x,0;j)|^2 + |Y(x,0;j)|^2)dx + \Theta + \int_Q \mathcal{F}(x)dx\mathrm{d}t\Big\} \quad \text{for } j = 1, \cdots, 5. \end{aligned} \tag{3.33}$$

Substituting (3.33) into (3.31), we obtain

$$\int_\Omega (|Z(x,0;j)|^2 + |Y(x,0;j)|^2) e^{2s\varphi(x,0)} dx \le C_{11} \Big\{ s^3 e^{2s\Phi}\Theta + \int_Q \mathcal{F}(x)(e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon}) dxdt + \int_\Omega s^3 e^{2s\Upsilon}(|Z(x,0;j)|^2 + |Y(x,0;j)|^2) dx \Big\} \quad (3.34)$$

for all sufficiently large $s > 0$ and $j = 1, \cdots, 5$. Here we have used $\Upsilon \le \Phi$. By

$$\varphi(x,0) = e^{\varrho(|x-x^0|^2 - \lambda^2)} \ge 1, \quad x \in \overline{\Omega} \quad (3.35)$$

and (3.32), we have

$$0 \le s^3 e^{2s\Upsilon} \le s^3 e^{2s(\Upsilon-1)} e^{2s\varphi(x,0)} \ \text{ for } x \in \overline{\Omega} \ \text{ and } \lim_{s\to\infty} s^3 e^{2s(\Upsilon-1)} = 0. \quad (3.36)$$

Therefore, we can eliminate the last term in (3.34). Then, summing (3.34) over $j = 1, \cdots, 5$, we can obtain that

$$\sum_{j=1}^{5} \int_\Omega (|Z(x,0;j)|^2 + |Y(x,0;j)|^2) e^{2s\varphi(x,0)} dx \le C_{12} \Big\{ s^3 e^{2s\Phi}\Theta + \int_Q \mathcal{F}(x)(e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon}) dxdt \Big\} \quad (3.37)$$

for all sufficiently large $s > 0$.

On the other hand, by (1.10), (3.7), and the definitions of $Z(\,\cdot\,;j)$, $Y(\,\cdot\,;j)$, we have

$$\sum_{k=1}^{2} |\partial_k f_4|^2 \le C_{13}(|Y(\,\cdot\,,0;1)|^2 + |f_4|^2) \quad \text{in } \Omega, \quad (3.38)$$

$$\sum_{k,l=1}^{2} |\partial_l \partial_k f_4|^2 \le C_{14}\Big(|Z(\,\cdot\,,0;1)|^2 + \sum_{k=1}^{2} |\partial_k f_4|^2 + |f_4|^2\Big) \quad \text{in } \Omega. \quad (3.39)$$

Furthermore, by (3.8) and the definitions of $Z(\,\cdot\,;j)$, $Y(\,\cdot\,;j)$ and $\mathbb{D}_1$, we see that

$$\mathbb{D}_1 \begin{pmatrix} f_1 \\ f_3 \\ f_5 \\ f_6 \end{pmatrix} = \begin{pmatrix} y_3(\,\cdot\,,0;2) \\ y_3(\,\cdot\,,0;3) \\ y_3(\,\cdot\,,0;4) \\ y_3(\,\cdot\,,0;5) \end{pmatrix} + 2 \begin{pmatrix} [\partial_1 d_1(2)]\, f_2 \\ [\partial_1 d_1(3)]\, f_2 \\ [\partial_1 d_1(4)]\, f_2 \\ [\partial_1 d_1(5)]\, f_2 \end{pmatrix} \quad \text{in } \Omega, \quad (3.40)$$

$$\mathbb{D}_1 \begin{pmatrix} \partial_k f_1 \\ \partial_k f_3 \\ \partial_k f_5 \\ \partial_k f_6 \end{pmatrix} = \begin{pmatrix} z_{k3}(\,\cdot\,,0;2) \\ z_{k3}(\,\cdot\,,0;3) \\ z_{k3}(\,\cdot\,,0;4) \\ z_{k3}(\,\cdot\,,0;5) \end{pmatrix} + 2 \begin{pmatrix} \partial_k \{[\partial_1 d_1(2)]\, f_2\} \\ \partial_k \{[\partial_1 d_1(3)]\, f_2\} \\ \partial_k \{[\partial_1 d_1(4)]\, f_2\} \\ \partial_k \{[\partial_1 d_1(5)]\, f_2\} \end{pmatrix} - (\partial_k \mathbb{D}_1) \begin{pmatrix} f_1 \\ f_3 \\ f_5 \\ f_6 \end{pmatrix} \quad \text{in } \Omega, \quad (3.41)$$

where $k = 1, 2$. Noting the definitions of F_1, F_2, $\cdots$, F_8, f_5, f_6, we have

$$\partial_1 f_2 = f_6 - F_6, \quad \partial_2 f_2 = f_5 - F_1, \quad \text{in } \Omega. \quad (3.42)$$

Noting the definition of $\mathbb{D}_2$ and using (3.41) and (3.42), we can obtain that

$$\mathbb{D}_2 \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_8 \end{pmatrix} = \begin{pmatrix} z_{13}(\,\cdot\,,0;2) \\ \vdots \\ z_{13}(\,\cdot\,,0;5) \\ z_{23}(\,\cdot\,,0;2) \\ \vdots \\ z_{23}(\,\cdot\,,0;5) \end{pmatrix} + 2 \begin{pmatrix} [\partial_1 d_1(2)]\, f_6 \\ \vdots \\ [\partial_1 d_1(5)]\, f_6 \\ [\partial_1 d_1(2)]\, f_5 \\ \vdots \\ [\partial_1 d_1(5)]\, f_5 \end{pmatrix} + 2f_2 \begin{pmatrix} \partial_1^2 d_1(2) \\ \vdots \\ \partial_1^2 d_1(5) \\ \partial_1\partial_2 d_1(2) \\ \vdots \\ \partial_1\partial_2 d_1(5) \end{pmatrix} - \begin{pmatrix} \partial_1 \mathbb{D}_1 \\ \partial_2 \mathbb{D}_1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_3 \\ f_5 \\ f_6 \end{pmatrix} \tag{3.43}$$

in Ω. Therefore, by (1.11), (3.40) and (3.43), we have

$$\sum_{l=1}^{6} |f_l|^2 \le C_{15}\Big(\sum_{j=2}^{5} |Y(\,\cdot\,,0;j)|^2 + |f_2|^2 + |f_4|^2 \Big) \quad \text{in } \Omega, \tag{3.44}$$

$$\sum_{l=1}^{8} |F_l|^2 \le C_{16}\Big(\sum_{j=2}^{5} |Z(\,\cdot\,,0;j)|^2 + \sum_{l=1}^{6} |f_l|^2 \Big) \quad \text{in } \Omega. \tag{3.45}$$

Furthermore, by (3.42), we have

$$\sum_{k=1}^{2} |\partial_k f_2|^2 \le C_{17}(|F_1|^2 + |F_6|^2 + |f_5|^2 + |f_6|^2) \quad \text{in } \Omega. \tag{3.46}$$

Then it follows from (3.2), (3.38), (3.39), (3.44), (3.45) and (3.46) that

$$\mathcal{F} \le C_{18}\Big\{ \sum_{j=1}^{5} (|Z(\,\cdot\,,0;j)|^2 + |Y(\,\cdot\,,0;j)|^2) + |f_2|^2 + |f_4|^2 \Big\} \quad \text{in } \Omega. \tag{3.47}$$

Consequently (3.37) and (3.47) imply

$$\int_\Omega \mathcal{F}(x) e^{2s\varphi(x,0)} dx \le C_{19}\Big\{ \int_\Omega (|f_2(x)|^2 + |f_4(x)|^2) e^{2s\varphi(x,0)} dx + s^3 e^{2s\Phi}\Theta + \int_Q \mathcal{F}(x)(e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon}) dxdt \Big\} \tag{3.48}$$

for all sufficiently large $s > 0$.

Noting $(\epsilon_1, \mu_1), (\epsilon_2, \mu_2) \in \mathcal{U}$ and the definitions of f_2, f_4, we can apply Proposition 2.2 to get that

$$\int_\Omega (|f_2(x)|^2 + |f_4(x)|^2) e^{2s\varphi(x,0)} dx \le \frac{C_{20}}{s} \int_\Omega \Big(\sum_{k=1}^{2} (|\partial_k f_2(x)|^2 + |\partial_k f_4(x)|^2) \Big) e^{2s\varphi(x,0)} dx \tag{3.49}$$

for all sufficiently large $s > 0$. By (3.2), (3.48) and (3.49), we can obtain

$$\int_\Omega \mathcal{F}(x) e^{2s\varphi(x,0)} dx \le C_{21}\Big\{ s^3 e^{2s\Phi}\Theta + \int_Q \mathcal{F}(x)(e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon}) dxdt \Big\} \tag{3.50}$$

for all sufficiently large $s > 0$. By (3.35), we have

$$\varphi(x,t) - \varphi(x,0) = e^{\varrho(|x-x^0|^2 - \lambda^2)}(e^{-\varrho\beta t^2} - 1) \le e^{-\varrho\beta t^2} - 1$$

for $(x,t) \in \overline{Q}$ and consequently

$$\begin{aligned} e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon} &= (e^{2s(\varphi(x,t)-\varphi(x,0))} + s^3 e^{2s(\Upsilon-\varphi(x,0))})e^{2s\varphi(x,0)} \\ &\le (e^{2s(e^{-\varrho\beta t^2}-1)} + s^3 e^{2s(\Upsilon-1)})e^{2s\varphi(x,0)}, \quad (x,t) \in \overline{Q}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_Q \mathcal{F}(x)(e^{2s\varphi(x,t)} + s^3 e^{2s\Upsilon})dxdt \\ \le &\Big\{ \int_{-T}^{T} (e^{2s(e^{-\varrho\beta t^2}-1)} + s^3 e^{2s(\Upsilon-1)})dt \Big\} \Big(\int_\Omega \mathcal{F}(x) e^{2s\varphi(x,0)} dx \Big). \end{aligned} \tag{3.51}$$

Noting (3.32), we have

$$\lim_{s\to\infty} \int_{-T}^{T} (e^{2s(e^{-\varrho\beta t^2}-1)} + s^3 e^{2s(\Upsilon-1)})dt = 0. \tag{3.52}$$

It follows from (3.35), (3.50), (3.51), and (3.52) that

$$\int_\Omega \mathcal{F}(x)dx \le e^{-2s} \int_\Omega \mathcal{F}(x) e^{2s\varphi(x,0)} dx \le C_{22} s^3 e^{-2s+2s\Phi} \Theta$$

for all sufficiently large $s > 0$. Hence, taking $s > 0$ sufficiently large, and noting (3.2), we obtain that

$$\int_\Omega \Big(\sum_{l=1}^{4} |f_l(x)|^2 \Big) dx \le \int_\Omega \mathcal{F}(x)dx \le C_{23}\Theta. \tag{3.53}$$

Moreover, by direct calculations, we can verify that $\mu_1 - \mu_2 = \mu_1\mu_2 f_4$ and

$$\varepsilon_{l2} - \varepsilon_{l1} = \frac{f_l(\gamma_{11}\gamma_{31} - \gamma_{21}^2) + \gamma_{l1}[(\gamma_{22} + \gamma_{21})f_2 - \gamma_{12}f_3 - \gamma_{31}f_1]}{(\gamma_{12}\gamma_{32} - \gamma_{22}^2)(\gamma_{11}\gamma_{31} - \gamma_{21}^2)} \quad \text{in } \Omega,$$

where $l = 1,2,3$. Therefore we have

$$\sum_{l=1}^{3} \|\varepsilon_{l1} - \varepsilon_{l2}\|_{L^2(\Omega)} + \|\mu_1 - \mu_2\|_{L^2(\Omega)} \le C_{24} \sum_{l=1}^{4} \|f_l\|_{L^2(\Omega)}. \tag{3.54}$$

In terms of (3.16), (3.20), (3.53), (3.54) and the definitions of $Z(\,\cdot\,;j)$, $Y(\,\cdot\,;j)$, $j = 1, \cdots, 5$, the proof of Theorem 1.1 is complete.

Appendix Verification of the Conditions for Carleman Estimate (2.3)

We note that P and $\beta > 0$ are given by (2.2) and (1.7) respectively, $x^0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $T \in (0, \frac{\Lambda}{\sqrt{\beta}} + \vartheta)$ where $0 < \vartheta < 1$ is sufficiently small.

We set

$$P(x;\zeta) = -\zeta_3^2 + a_1(x)\zeta_1^2 + 2a_2(x)\zeta_1\zeta_2 + a_3(x)\zeta_2^2, \quad x \in \overline{\Omega},\ \zeta = (\zeta_1,\zeta_2,\zeta_3) \in \mathbb{C}^3. \tag{A.1}$$

Then, under the condition $(a_1, a_2, a_3) \in \mathcal{V}$, we have to verify that $\psi(x,t) = |x - x^0|^2 - \beta t^2 - \lambda^2$ satisfies (A.2), (A.3) and (A.5):

$$|(\nabla_{x,t}\psi)(x,t)| > 0, \quad (x,t) \in \overline{Q}. \tag{A.2}$$

$$\begin{aligned} I \equiv & \sum_{j,k=1}^{3} (\partial_j \partial_k \psi)(x,t) \frac{\partial P}{\partial \zeta_j} \overline{\frac{\partial P}{\partial \zeta_k}}(x;\xi) + \lim_{\tau \to 0} \tau^{-1} \Im \sum_{k=1}^{3} (\partial_k P) \overline{\frac{\partial P}{\partial \zeta_k}}(x; \xi + \sqrt{-1}\,\tau (\nabla_{x,t}\psi)(x,t)) \\ & \ge K|\xi|^2 \end{aligned} \tag{A.3}$$

for some positive constant K, for any $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and any point (x,t) of $\overline{Q}$ such that

$$P(x;\xi) = 0, \quad \sum_{j=1}^{3} \frac{\partial P}{\partial \zeta_j}(x;\xi)(\partial_j \psi)(x,t) = 0. \tag{A.4}$$

$$P(x; (\nabla_{x,t}\psi)(x,t)) \ne 0 \quad \text{for all } (x,t) \in \overline{Q}. \tag{A.5}$$

Here $\Im$ means the imaginary part. Then by [10, Theorem 2.1] we see Carleman estimate (2.3).

First, by direct calculations, we have

$$\begin{aligned} &(\nabla_{x,t}\psi)(x,t) = (2(x_1 - x_1^0), 2(x_2 - x_2^0), -2\beta t), \\ &(\partial_1\partial_2\psi)(x,t) = (\partial_1\partial_3\psi)(x,t) = (\partial_2\partial_3\psi)(x,t) = 0, \\ &(\partial_1^2\psi)(x,t) = (\partial_2^2\psi)(x,t) = 2, \quad (\partial_3^2\psi)(x,t) = -2\beta, \\ &(\partial_k P)(x;\zeta) = (\partial_k a_1)(x)\zeta_1^2 + 2(\partial_k a_2)\zeta_1\zeta_2 + (\partial_k a_3)(x)\zeta_2^2 \quad \text{for } k = 1,2, \ (\partial_3 P)(x;\zeta) = 0, \\ &\frac{\partial P}{\partial \zeta_1}(x;\zeta) = 2a_1(x)\zeta_1 + 2a_2(x)\zeta_2, \quad \frac{\partial P}{\partial \zeta_2}(x;\zeta) = 2a_2(x)\zeta_1 + 2a_3(x)\zeta_2, \quad \frac{\partial P}{\partial \zeta_3}(x;\zeta) = -2\zeta_3. \end{aligned}$$

Verification of (A.2)

By $x^0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, we have $|(\nabla_{x,t}\psi)(x,t)|^2 = 4|x - x^0|^2 + 4\beta^2 t^2 > 0$ for $(x,t) \in \overline{Q}$. Therefore (A.2) has been verified.

Verification of (A.3)

By direct calculations, we have, for $x \in \overline{\Omega}$,

$$\begin{aligned} \mathrm{I} = & 8\{[a_1(x)\xi_1 + a_2(x)\xi_2]^2 + [a_2(x)\xi_1 + a_3(x)\xi_2]^2 - \beta\xi_3^2\} \\ & + 4\sum_{k=1}^{2} \{2[a_k(x)\xi_1 + a_{k+1}(x)\xi_2][(\partial_k a_1(x))(x_1 - x_1^0)\xi_1 + (\partial_k a_3(x))(x_2 - x_2^0)\xi_2 \\ & + (\partial_k a_2(x))((x_2 - x_2^0)\xi_1 + (x_1 - x_1^0)\xi_2)] \\ & - [(x_1 - x_1^0)a_k(x) + (x_2 - x_2^0)a_{k+1}(x)][(\partial_k a_1(x))\xi_1^2 + 2(\partial_k a_2(x))\xi_1\xi_2 + (\partial_k a_3(x))\xi_2^2]\}. \end{aligned} \tag{A.6}$$

We divide the right-hand side of (A.6) into two terms: the first term I_1 is independent of $a_2(x)$ and the second term I_2 is dependent on $a_2(x)$. That is, $\mathrm{I} \equiv \mathrm{I}_1 + \mathrm{I}_2$ where

$$\begin{aligned} \mathrm{I}_1 = & 4\{2a_1^2(x)\xi_1^2 + 2a_3^2(x)\xi_2^2 + a_1(x)[\partial_1 a_1(x)](x_1 - x_1^0)\xi_1^2 + 2a_1(x)[\partial_1 a_3(x)](x_2 - x_2^0)\xi_1\xi_2 \\ & - a_1(x)[\partial_1 a_3(x)](x_1 - x_1^0)\xi_2^2 + 2a_3(x)[\partial_2 a_1(x)](x_1 - x_1^0)\xi_1\xi_2 \\ & + a_3(x)[\partial_2 a_3(x)](x_2 - x_2^0)\xi_2^2 - a_3(x)[\partial_2 a_1(x)](x_2 - x_2^0)\xi_1^2 - 2\beta\xi_3^2\}, \end{aligned} \tag{A.7}$$

$$
\begin{aligned}
\mathrm{I}_2 = {}& 8a_2^2(x)(\xi_1^2+\xi_2^2) + 4a_2(x)\{4a_1(x)\xi_1\xi_2 + 4a_3(x)\xi_1\xi_2 + 2[\partial_1 a_1(x)](x_1-x_1^0)\xi_1\xi_2 \\
& + [\partial_1 a_3(x)](x_2-x_2^0)\xi_2^2 - [\partial_1 a_1(x)](x_2-x_2^0)\xi_1^2 + [\partial_2 a_1(x)](x_1-x_1^0)\xi_1^2 \\
& + 2[\partial_2 a_3(x)](x_2-x_2^0)\xi_1\xi_2 - [\partial_2 a_3(x)](x_1-x_1^0)\xi_2^2 + 2[\partial_1 a_2(x)](x_1-x_1^0)\xi_2^2 \\
& + 2[\partial_2 a_2(x)](x_2-x_2^0)\xi_1^2\} + 8a_1(x)[\partial_1 a_2(x)](x_2-x_2^0)\xi_1^2 + 8a_3(x)[\partial_2 a_2(x)](x_1-x_1^0)\xi_2^2.
\end{aligned}
$$

On the other hand, condition (A.4) implies that, for $(x,t)\in\overline{Q}$,

$$a_1(x)\xi_1^2 + 2a_2(x)\xi_1\xi_2 + a_3(x)\xi_2^2 = \xi_3^2, \tag{A.8}$$

$$a_1(x)\xi_1(x_1-x_1^0) + a_3(x)\xi_2(x_2-x_2^0) = -\beta t\xi_3 - a_2(x)\xi_2(x_1-x_1^0) - a_2(x)\xi_1(x_2-x_2^0). \tag{A.9}$$

Moreover, by $\|a_2\|_{C^1(\overline{\Omega})} < \delta < \theta_1$, $a_1(x)$, $a_3(x) > \theta_1$, $x\in\overline{\Omega}$ and the inequality: $2|ab| \le a^2+b^2$, we see by (A.8) that for $x\in\overline{\Omega}$,

$$
\begin{aligned}
\xi_3^2 &= a_1(x)\xi_1^2 + 2a_2(x)\xi_1\xi_2 + a_3(x)\xi_2^2 \\
&\le a_1(x)\xi_1^2 + a_3(x)\xi_2^2 + \frac{|a_2(x)|}{\sqrt{a_1(x)a_3(x)}}|2\sqrt{a_1(x)}\,\xi_1\sqrt{a_3(x)}\,\xi_2| \\
&\le a_1(x)\xi_1^2 + a_3(x)\xi_2^2 + \frac{\delta}{\theta_1}[a_1(x)\xi_1^2 + a_3(x)\xi_2^2] \le 2[a_1(x)\xi_1^2 + a_3(x)\xi_2^2].
\end{aligned} \tag{A.10}
$$

Consequently, again by $a_1(x)$, $a_3(x) > \theta_1$ for $x\in\overline{\Omega}$, we have

$$4[a_1(x)\xi_1^2 + a_3(x)\xi_2^2] \ge 2[a_1(x)\xi_1^2 + a_3(x)\xi_2^2] + \xi_3^2 \ge \min\{2\theta_1, 1\}|\xi|^2, \quad x\in\overline{\Omega}. \tag{A.11}$$

Next we estimate I_1. By (A.9), we obtain

$$
\begin{aligned}
a_1(x)[\partial_1 a_3(x)](x_2-x_2^0)\xi_1\xi_2 &= a_1(x)[\partial_1(\ln a_3(x))]\xi_1[a_3(x)(x_2-x_2^0)\xi_2] \\
&= a_1(x)[\partial_1(\ln a_3(x))]\xi_1[-a_1(x)\xi_1(x_1-x_1^0) - \beta t\xi_3 \\
&\quad - a_2(x)\xi_2(x_1-x_1^0) - a_2(x)\xi_1(x_2-x_2^0)],
\end{aligned} \tag{A.12}
$$

$$
\begin{aligned}
a_3(x)[\partial_2 a_1(x)](x_1-x_1^0)\xi_1\xi_2 &= a_3(x)[\partial_2(\ln a_1(x))]\xi_2[a_1(x)(x_1-x_1^0)\xi_1] \\
&= a_3(x)[\partial_2(\ln a_1(x))]\xi_2[-a_3(x)\xi_2(x_2-x_2^0) - \beta t\xi_3 \\
&\quad - a_2(x)\xi_2(x_1-x_1^0) - a_2(x)\xi_1(x_2-x_2^0)].
\end{aligned} \tag{A.13}
$$

Substituting (A.12) and (A.13) into (A.7) and arranging it according to ξ_1^2, ξ_2^2, β and $a_2(x)$, we can obtain, for $(x,t)\in\overline{Q}$,

$$
\begin{aligned}
\mathrm{I}_1 = {}& 4a_1(x)\xi_1^2\{2a_1(x) + [\partial_1 a_1(x)](x_1-x_1^0) - a_3(x)[\partial_2(\ln a_1(x))](x_2-x_2^0) \\
& - 2a_1(x)[\partial_1(\ln a_3(x))](x_1-x_1^0)\} + 4a_3(x)\xi_2^2\{2a_3(x) + [\partial_2 a_3(x)](x_2-x_2^0) \\
& - a_1(x)[\partial_1(\ln a_3(x))](x_1-x_1^0) - 2a_3(x)[\partial_2(\ln a_1(x))](x_2-x_2^0)\} - 8\beta\xi_3^2 \\
& - 8\beta t\{a_1(x)[\partial_1(\ln a_3(x))]\xi_1\xi_3 + a_3(x)[\partial_2(\ln a_1(x))]\xi_2\xi_3\} \\
& - 8a_2(x)\{a_1(x)[\partial_1(\ln a_3(x))](x_1-x_1^0)\xi_1\xi_2 + a_1(x)[\partial_1(\ln a_3(x))](x_2-x_2^0)\xi_1^2 \\
& + a_3(x)[\partial_2(\ln a_1(x))](x_1-x_1^0)\xi_2^2 + a_3(x)[\partial_2(\ln a_1(x))](x_2-x_2^0)\xi_1\xi_2\}.
\end{aligned} \tag{A.14}
$$

Moreover, using $\|\nabla_x a_1\|_{C(\overline{\Omega})}$, $\|\nabla_x a_3\|_{C(\overline{\Omega})} < M_0$, $a_1(x)$, $a_3(x) > \theta_1$, we have

$$
\begin{aligned}
&2|\sqrt{a_1(x)}\,\xi_1\sqrt{a_3(x)}\,\xi_2| \le a_1(x)\xi_1^2 + a_3(x)\xi_2^2, \quad 2|\sqrt{a_1(x)}\,\xi_1\xi_3| \le a_1(x)\xi_1^2 + \xi_3^2,\\
&2|\sqrt{a_3(x)}\,\xi_2\xi_3| \le a_3(x)\xi_2^2 + \xi_3^2, \quad |\partial_j[\ln a_k(x)]| \le \frac{M_0}{\theta_1} \quad \text{for } j = 1,2 \text{ and } k = 1,3,\\
&\partial_1 a_1(x) = a_1(x)\partial_1[\ln a_1(x)], \quad \partial_2 a_3(x) = a_3(x)\partial_2[\ln a_3(x)], \quad x \in \overline{\Omega}.
\end{aligned}
$$

Therefore, using (1.3), (A.10), $\|a_1\|_{C^2(\overline{\Omega})}$, $\|a_3\|_{C^2(\overline{\Omega})} < M_1$, $\|a_2\|_{C^1(\overline{\Omega})} < \delta$, we have, for $x \in \overline{\Omega}$,

$$
\begin{aligned}
\mathrm{I}_1 \ge\ & 4a_1(x)\xi_1^2\Big\{2a_1(x) + a_1(x)\Big[\partial_1\Big(\ln\frac{a_1(x)}{a_3(x)}\Big)\Big](x_1 - x_1^0)\\
&- a_3(x)[\partial_2(\ln a_1(x))](x_2 - x_2^0) - a_1(x)[\partial_1(\ln a_3(x))](x_1 - x_1^0)\}\\
&+ 4a_3(x)\xi_2^2\Big\{2a_3(x) - a_3(x)\Big[\partial_2\Big(\ln\frac{a_1(x)}{a_3(x)}\Big)\Big](x_2 - x_2^0)\\
&- a_1(x)[\partial_1(\ln a_3(x))](x_1 - x_1^0) - a_3(x)[\partial_2(\ln a_1(x))](x_2 - x_2^0)\}\\
&- 16\beta[a_1(x)\xi_1^2 + a_3(x)\xi_2^2] - \frac{4M_0\sqrt{M_1}\beta T}{\theta_1}[a_1(x)\xi_1^2 + a_3(x)\xi_2^2 + 2\xi_3^2]\\
&- \frac{8\sqrt{M_1}M_0\sqrt{\Lambda^2+\lambda^2}\,\delta}{\sqrt{\theta_1}\,\theta_1}[a_1(x)\xi_1^2 + a_3(x)\xi_2^2] - \frac{8M_0\sqrt{\Lambda^2+\lambda^2}\,\delta}{\theta_1}[a_1(x)\xi_1^2 + a_3(x)\xi_2^2]\\
\ge\ & 4[a_1(x)\xi_1^2 + a_3(x)\xi_2^2]\Big[\theta_0 - 4\beta - \frac{5M_0\sqrt{M_1}\beta T}{\theta_1} - \frac{2M_0\sqrt{\Lambda^2+\lambda^2}\,\delta}{\theta_1}\Big(\sqrt{\frac{M_1}{\theta_1}} + 1\Big)\Big]. \qquad \text{(A.15)}
\end{aligned}
$$

For the second inequality, we have used $(a_1, a_2, a_3) \in \mathcal{V}$ and (A.10). Furthermore we can similarly estimate I_2 and have

$$
\mathrm{I}_2 \ge -8\delta\sqrt{\Lambda^2+\lambda^2}\Big[2\sqrt{\frac{M_1}{\theta_1(\Lambda^2+\lambda^2)}} + \frac{2M_0}{\theta_1} + \frac{\delta}{\theta_1} + 1\Big][a_1(x)\xi_1^2 + a_3(x)\xi_2^2], \quad x \in \overline{\Omega}. \qquad \text{(A.16)}
$$

Then (A.15) and (A.16) imply

$$
\begin{aligned}
\mathrm{I} &= \mathrm{I}_1 + \mathrm{I}_2\\
&\ge 4[a_1(x)\xi_1^2 + a_3(x)\xi_2^2]\Big[\theta_0 - 4\beta - \frac{5M_0\sqrt{M_1}\beta T}{\theta_1}\\
&\quad - \frac{2\sqrt{\Lambda^2+\lambda^2}\,\delta}{\theta_1}\Big(2\sqrt{\frac{M_1\theta_1}{\Lambda^2+\lambda^2}} + M_0\sqrt{\frac{M_1}{\theta_1}} + 3M_0 + \delta + \theta_1\Big)\Big]\\
&\ge \min\{2\theta_1, 1\}\Big[\theta_0 - 4\beta - \frac{5M_0\sqrt{M_1}\beta}{\theta_1}\Big(\frac{\Lambda}{\sqrt{\beta}} + \vartheta\Big)\\
&\quad - \frac{2\sqrt{\Lambda^2+\lambda^2}\,\delta}{\theta_1}\Big(2\sqrt{\frac{M_1\theta_1}{\Lambda^2+\lambda^2}} + M_0\sqrt{\frac{M_1}{\theta_1}} + 3M_0 + 2\theta_1\Big)\Big]|\xi|^2\\
&= \min\{2\theta_1, 1\}\Big[\theta_0 - M_3\delta - 4\beta - \frac{5M_0\sqrt{M_1}\beta}{\theta_1}\Big(\frac{\Lambda}{\sqrt{\beta}} + \vartheta\Big)\Big]|\xi|^2, \quad x \in \overline{\Omega}. \qquad \text{(A.17)}
\end{aligned}
$$

For the second inequality and the last equality, we have used $0 < T < \frac{\Lambda}{\sqrt{\beta}} + \vartheta$, $\delta < \theta_1$, (A.11) and (1.6), respectively. By (1.7), we have

$$
\theta_0 - M_3\delta - 4\beta - \Big(\frac{5M_0\sqrt{M_1}\,\Lambda}{\theta_1}\Big)\sqrt{\beta} > 0.
$$

Therefore, taking

$$0 < \vartheta < \frac{\theta_0 - M_3\delta - 4\beta - \left(\frac{5M_0\sqrt{M_1}\,\Lambda}{\theta_1}\right)\sqrt{\beta}}{\frac{5M_0\sqrt{M_1}\,\beta}{\theta_1}},$$

we obtain

$$\theta_0 - M_3\delta - 4\beta - \frac{5M_0\sqrt{M_1}\,\beta}{\theta_1}\left(\frac{\Lambda}{\sqrt{\beta}} + \vartheta\right) > 0.$$

Hence we have completed the verification of (A.3).

Verification of (A.5)

By $0 < \vartheta < 1$, $\delta < \theta_1$ and (1.7), we have

$$\beta\vartheta + \Lambda\sqrt{\beta} - \lambda\sqrt{\theta_1 - \delta} < \beta + \Lambda\sqrt{\beta} - \lambda\sqrt{\theta_1 - \delta} < 0.$$

Consequently, noting

$$0 < T < \frac{\Lambda}{\sqrt{\beta}} + \vartheta,$$

we can see

$$\lambda\sqrt{\theta_1 - \delta} > \beta\vartheta + \Lambda\sqrt{\beta} > \beta T.$$

Therefore, noting

$$\|a_2\|_{C^1(\overline{\Omega})} < \delta, \quad a_1(x),\ a_3(x) > \theta_1 \quad \text{for } x \in \overline{\Omega}$$

and the inequality

$$2|ab| \le a^2 + b^2,$$

we have

$$\begin{aligned}
& P(x; (\nabla_{x,t}\psi)(x,t)) \\
={}& 4[-\beta^2 t^2 + a_1(x)(x_1 - x_1^0)^2 + 2a_2(x)(x_1 - x_1^0)(x_2 - x_2^0) + a_3(x)(x_2 - x_2^0)^2] \\
\ge{}& 4[-\beta^2 T^2 + \theta_1(x_1 - x_1^0)^2 - \delta|x - x^0|^2 + \theta_1(x_2 - x_2^0)^2] \\
={}& 4[-\beta^2 T^2 + (\theta_1 - \delta)|x - x^0|^2] \ge 4[-\beta^2 T^2 + (\theta_1 - \delta)\lambda^2] \\
={}& 4(\lambda\sqrt{\theta_1 - \delta} + \beta T)(\lambda\sqrt{\theta_1 - \delta} - \beta T) > 0, \quad (x,t) \in \overline{Q}.
\end{aligned}$$

We have verified (A.5).

Thus we have completed the verification of the conditions (A.2), (A.3) and (A.5). We can apply [10, Theorem 2.1] to obtain Carleman estimate (2.3) in Q if $(a_1, a_2, a_3) \in \mathcal{V}$.

References

[1] Bukhgeim, A. L., Introduction to the Theory of Inverse Problems, VSP, Utrecht, 2000.

[2] Bukhgeim, A. L. and Klibanov, M. V., Global uniqueness of a class of multidimensional inverse problems, *Soviet Math. Dokl.*, **24**, 1981, 244–247.

[3] Duvaut, G. and Lions, J. L., Inequalities in Mechanics and Physics, Springer-Verlag, 1976.

[4] Imanuvilov, O. Y., Isakov, V. and Yamamoto, M., An inverse problem for the dynamical Lamé system with two sets of boundary data, *Comm. Pure Appl. Math.*, **56**, 2003, 1366–1382.

[5] Imanuvilov, O. Y. and Yamamoto, M., Global Lipschitz stability in an inverse hyperbolic problem by interior observations, *Inverse Problems*, **17**, 2001, 717–728.

[6] Imanuvilov, O. Y. and Yamamoto, M., Carleman estimate for a parabolic equation in a Sobolev space of negative order and its applications, Control of Nonlinear Distributed Parameter Systems, Lecture Notes in Pure and Appl. Math., Vol. 218, Marcel-Dekker, New York, 2001, 113–137.

[7] Imanuvilov, O. Y. and Yamamoto, M., Determination of a coefficient in an acoustic equation with a single measurement, *Inverse Problems*, **19**, 2003, 157–171.

[8] Isakov, V., Uniqueness of the continuation across a time-like hyperplane and related inverse problems for hyperbolic equations, *Comm. Partial Differential Equations*, **14**, 1989, 465–478.

[9] Isakov, V., Inverse Problems for Partial Differential Equations, Springer-Verlag, Berlin, 1998.

[10] Isakov, V., Carleman type estimates and their applications, New Analytic and Geometric Methods in Inverse Problems, Springer-Verlag, Berlin, 2004, 93–125.

[11] Khaĭdarov, A., Carleman estimates and inverse problems for second order hyperbolic equations, *Math. USSR Sbornik*, **58**, 1987, 267–277.

[12] Khaĭdarov, A., On stability estimates in multidimensional inverse problems for differential equations (English translation), *Soviet Math. Dokl.*, 1989, 614–617.

[13] Klibanov, M. V., Inverse problems and Carleman estimates, *Inverse Problems*, **8**, 1992, 575–596.

[14] Klibanov, M. V. and Timonov, A. A., Carleman Estimates for Coefficient Inverse Problems and Numerical Applications, VSP, Utrect, 2004.

[15] Klibanov, M. V. and Yamamoto, M., Lipschitz stability of an inverse problem for an accoustic equation, *Appl. Anal.*, **85**, 2006, 515–538.

[16] Kong, J. A., Electromagnetic Wave Theory, John-Wiley, New York, 1990.

[17] Landau, L. D. and Lifshitz, E. M., Electrodynamics of Continuous Media, Addison-Wesley, Reading, 1960.

[18] Romanov, V. G., Inverse Problem of Mathematical Physics, VNU Science Press, Utrecht, 1987.

[19] Romanov, V. G., An inverse problem for Maxwell's equation, *Phys. Lett.* A, **138**, 1989, 459–462.

[20] Romanov, V. G. and Kabanikhin, S. I., Inverse problems for Maxwell's equations, VSP, Utrecht, 1994.

[21] Sun, Z. and Uhlmann, G., An inverse boundary problem for Maxwell's equations, *Arch. Ration. Mech. Anal.*, **119**, 1992, 71–93.

[22] Yamamoto, M., A mathematical aspect of inverse problems for non-stationary Maxwell's equations, *Int. J. of Appl. Electromag. and Mech.*, **8**, 1997, 77–98.

[23] Yamamoto, M., On an inverse problem of determining source terms in Maxwell's equations with a single measurement, Inverse Problems, Tomography, and Image Processing, Plenum, New York, **15**, 1998, 241–256.

[24] Yamamoto, M., Uniqueness and stability in multidimensional hyperbolic inverse problems, *J. Math. Pure Appl.*, **78**, 1999, 65–98.