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Hyperbolic Mapping Classes and Their Lifts on the Bers Fiber Space

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Abstract Let S be a Riemann surface with genus p and n punctures. Assume that $3p-3+n>0$ and $n\geq 1$. Let a be a puncture of S and let $\widetilde{S}=S\cup\{a\}$. Then all mapping classes in the mapping class group Mod_S that fixes the puncture a can be projected to mapping classes of $\mathrm{Mod}_{\widetilde{S}}$ under the forgetful map. In this paper the author studies the mapping classes in Mod_S that can be projected to a given hyperbolic mapping class in $\mathrm{Mod}_{\widetilde{S}}$.

Keywords Riemann surfaces, Absolutely extremal Teichmüller mapping, Mapping classes, Teichmüller spaces, Bers fiber spaces
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1 Introduction

Let S be a Riemann surface with genus p and $n\geq 1$ punctures. We assume that $3p-3+n>0$. Let a be a puncture of S, and Mod_S^a be the subgroup of the mapping class group Mod_S that consists of elements fixing the puncture a. This means that every element θ in Mod_S^a can be projected to an element χ in the mapping class group $\mathrm{Mod}_{\widetilde{S}}$, where $\widetilde{S}=S\cup\{a\}$, under the forgetful map defined by filling in the puncture a. Let

$$i:\mathrm{Mod}_S^a\to\mathrm{Mod}_{\widetilde{S}}$$

denote the projection carrying θ to χ.

Throughout the paper we consider the situation that χ is induced by an irreducible self-map f of $\widetilde{S}$. Here by the term "irreducible" we mean that it does not leave invariant any system of non-contractible, homotopically independent and simple disjoint loops on $\widetilde{S}$. Following Bers [3], every mapping class can be considered an action on the corresponding Teichmüller space $T(\widetilde{S})$ and thus it can be classified as an elliptic, parabolic, hyperbolic or pseudo-hyperbolic modular transformation. In [3], Bers proved that χ is induced by an irreducible map if and only if it is hyperbolic, in which case, we can find a point in $T(\widetilde{S})$ so that χ can be induced by a so called absolutely extremal Teichmüller mapping on a surface that represents the point.

In [7], Kra investigated the problem of characterizing all possible mapping classes in Mod_S^a that project to a mapping class in $\mathrm{Mod}_{\widetilde{S}}$ with a given type. He showed that if $\chi\in\mathrm{Mod}_{\widetilde{S}}$ is hyperbolic or pseudo-hyperbolic, then $i^{-1}(\chi)$ consists of only hyperbolic or pseudo-hyperbolic elements. He also showed that $\theta\in i^{-1}(\chi)$ is hyperbolic whenever χ is hyperbolic and $\widetilde{S}$ can be deformed so that χ and θ are induced by the same absolutely extremal mapping that fixes a. All these results were obtained by using Bers' classification for elements of $\mathrm{Mod}_{\widetilde{S}}$ as well as

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the remarkable Royden's theorem (see [12, 5]) which states that the Kobayashi metric and the Teichmüller metric on $T(\widetilde{S})$ are the same.

In this paper, we continue to study this problem. We show that if $\widetilde{S}$ is a compact Riemann surface without boundary, or if $\widetilde{S}$ is non-compact and f fixes no punctures of $\widetilde{S}$, then $i^{-1}(\chi)$ consists of hyperbolic elements whenever χ is hyperbolic. Furthermore, we give a necessary and sufficient condition for an element $\theta \in i^{-1}(\chi)$ to be hyperbolic in the case that f fixes some punctures of $\widetilde{S}$. Among other things, by utilizing a careful analysis given in Marden and Strebel [8] on lifts of an absolutely extremal Teichmüller mapping to the universal cover, we show that if f fixes some punctures of $\widetilde{S}$, then there are infinitely many hyperbolic and infinitely many pseudo-hyperbolic mapping classes in the union $\bigcup_{n\geq 1}\{i^{-1}(\chi^n)\}$.

Main results are stated in Section 3.

2 Preliminaries

In this section we review some well-known facts and basic properties of Teichmüller Theory. More details can be found in [2, 3, 7, 8].

Let R be a Riemann surface of finite type (p, n), $2p-2+n>0$. Let q be a meromorphic quadratic differential on R that may have simple poles at some punctures of R. A horizontal (resp. vertical) trajectory is an smooth arc along which $q(z)dz^2>0$ (resp. $q(z)dz^2<0$) except at zeros of q. Following Bers [3], let r be the order of q at a point P ($r=-1$ if P is a puncture of R and q has a simple pole at P). There is a natural local parameter z at P, $z(P)=0$, such that

$$q(z)=\left(\frac{r+2}{2}\right)^2 z^r dz^2 \quad \text{near } P.$$

Teichmüller's theorem asserts that if R_1 and R_2 are two diffeomorphic Riemann surfaces, and $f: R_1 \to R_2$ is a quasiconformal map, then in the homotopy class of f there is a unique quasiconformal map $h: R_1 \to R_2$, called Teichmüller mapping if it is not conformal, whose complex dilation satisfies

$$\mu_h(z)=k\frac{\overline{q(z)}}{|q(z)|}, \quad k<1,$$

for a meromorphic quadratic differential on R_1 which may have simple poles at punctures of R_1. q is called the initial differential of h. If $h: R_1 \to R_2$ is a Teichmüller mapping with initial differential q and dilatation

$$K=\frac{1+k}{1-k},$$

then there is a unique meromorphic quadratic differential q' on R_2, called terminal differential of h, such that

$$\iint_{R_1}|q|=\iint_{R_2}|q'|$$

and the order r of q at a point P is the order of q' at $h(P)$. The mapping h^{-1} is also a Teichmüller mapping with initial and terminal quadratic differentials $-q'$ and $-q$, respectively.

In the case that $R_1=R_2=R$, a Teichmüller self-mapping h of R is called absolutely extremal (also the term stable is used) if its terminal quadratic differential coincides with its initial differential up to a positive constant multiple. For every irreducible self-map of R, Bers [3] showed that there is a conformal structure $\mu(R)$ such that the map can be realized as an absolutely extremal Teichmüller mapping on $\mu(R)$.

The Teichmüller space is defined as a set of equivalence classes of all conformal structures $\mu(R)$ equipped with quasiconformal homeomorphism $w : R \to \mu(R)$, where

$$(w : R \to \mu(R)) \sim (w' : R \to \mu'(R)),$$

if there is a conformal map (isometry) $w_0 : \mu(R) \to \mu'(R)$ so that $w_0 \circ w$ is isotopic to w'. Denote $[\mu]$ the equivalence class of $\mu(R)$. Ahlfors identifies a natural complex structure on $T(R)$, so $T(R)$ becomes a complex manifold with dimension $3p - 3 + n$.

The Teichmüller distance between two points $[\mu]$ and $[\mu']$ in $T(R)$ is defined by

$$\langle [\mu],\ [\mu'] \rangle = \frac{1}{2} \log K(h),$$

where $h : \mu(R) \to \mu'(R)$ is the (unique) extremal quasiconformal map in the isotopy class of $w' \circ w^{-1}$, and $K(h) \geq 1$ is the dilatation of h.

Let f be a self-map of R. f naturally defines a mapping class χ that can be regarded as an action on the corresponding $T(R)$. χ acts on $T(R)$ effectively when R is not in the list: $(2, 0)$, $(1, 2)$, $(1, 1)$, $(0, 4)$ and $(0, 3)$. Denote

$$a(\chi) = \inf_{[\mu] \in T(R)} \langle [\mu],\ \chi([\mu]) \rangle.$$

χ is called hyperbolic if $a(\chi) > 0$ and there is a point $x_0 \in T(R)$ such that $a(\chi) = \langle x_0, \chi(x_0) \rangle$; pseudo-hyperbolic if $a(\chi) > 0$ but no such x_0 can be found.

Let χ be hyperbolic, and let R be a representative of x_0. Then χ is induced by an absolutely extremal map h on R (see [3]). Assume that q is the associated quadratic differential on R. From Marden and Strebel [8], q has no critical trajectories connecting any two critical points. In particular, every trajectory of q on R is dense. Also, for any two points x and $y \in R$ fixed by h, we let α be an arc connecting x and y such that $\alpha \backslash \{x, y\} \subset R$ and α is not contractible. Then $h(\alpha)$ is not homotopic to α on R.

Let $\mathbb{H}^2$ denote the hyperbolic plane $\{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$. Each conformal structure $\mu(R)$ on R determines a global quasiconformal automorphism w^μ of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that fixes $0, 1, \infty$ and is conformal on $\overline{\mathbb{H}^2} = \{z \in \mathbb{C}; \operatorname{Im} z < 0\}$. $w^\mu(\mathbb{H}^2)$ is a Jordan domain (quasidisk) that only depends on the equivalence class $[\mu]$. The Bers fiber space $F(R)$ is defined as a collection of pairs endowed with a product structure:

$$F(R) = \{([\mu], z);\ [\mu] \in T(R),\ z \in w^\mu(\mathbb{H}^2)\}.$$

There is a natural projection $\pi : F(R) \to T(R)$ defined by sending point $([\mu], z)$ to $[\mu]$. π is holomorphic. From [2, Lemma 6.3], each point x in $F(R)$ is represented as $([\mu], w^\mu(a))$ for a fixed but arbitrarily chosen point $a \in R$. Let $R' = R \backslash \{a\}$. There is a map

$$\varphi : F(R) \to T(R')$$

defined by sending x to the equivalence class of the conformal structure on R' determined by $([\mu], w^\mu(a))$. [2, Theorem 9] asserts that φ is a biholomorphic map (isomorphism) that respects π and the forgetful map $\iota : T(R') \to T(R)$. That is, $\pi = \varphi \circ \iota$. See [2] for more details.

Every self-map f on R can be lifted to an automorphism $\hat{f} : \mathbb{H}^2 \to \mathbb{H}^2$ under the universal covering

$$\varrho : \mathbb{H}^2 \to R. \tag{2.1}$$

Let G denote the covering group of (2.1). Bers [2] introduced the group mod R that consists of equivalence classes $[\hat{f}]$ of $\hat{f}$ where two lifts $\hat{f}$ and $\hat{f}' : \mathbb{H}^2 \to \mathbb{H}^2$ are equivalent if they induce the same automorphism (by conjugation) on the covering group G. Since every lift of f is quasiconformal and it naturally extends to $\partial\mathbb{H}^2$, the above statement is equivalent to that $\hat{f}|_{\partial\mathbb{H}^2} = \hat{f}'|_{\partial\mathbb{H}^2}$. In this manner, $G \cong \pi_1(R, a)$ is regarded as a normal subgroup of mod R so that

$$\operatorname{mod} R \,/\, G$$

is isomorphic to Mod_R, the mapping class group of R. Every element $[\hat{f}\,]$ of mod R acts on $F(R)$ via:

$$[\hat{f}\,]\ ([\mu], z) = ([\nu], w^\nu \circ \hat{f} \circ (w^\mu)^{-1}(z)), \tag{2.2}$$

where ν is the Beltrami coefficient of $w^\mu \circ \hat{f}^{-1}$. In particular, for every $g \in G$,

$$g\ ([\mu], z) = ([\mu], g^\mu(z)), \quad \text{where} \;\; g^\mu = w^\mu \circ g \circ (w^\mu)^{-1}. \tag{2.3}$$

Every element in mod R acts effectively and fiber-preservingly on $F(R)$, while element in G maps each fiber of $F(R)$ to itself. The group mod R is isomorphic to $\mathrm{Mod}^a_{R'}$ under the isomorphism $\varphi^* : \operatorname{mod} R \to \mathrm{Mod}^a_{R'}$ defined as

$$\operatorname{mod} R \ni [\hat{f}] \overset{\varphi^*}{\longmapsto} \varphi \circ [\hat{f}] \circ \varphi^{-1} \in \mathrm{Mod}^a_{R'}.$$

We return to our original case, that is, $R = \widetilde{S}$ and $R' = S$, where S and $\widetilde{S}$ are defined in the introduction. Let χ be a hyperbolic mapping class in $\mathrm{Mod}_{\widetilde{S}}$ and be induced by a self-map f of $\widetilde{S}$. Without loss of generality we may assume that $\widetilde{S}$ is the surface on which f is an absolutely extremal Teichmüller mapping. Lift f to a self-map f^* of $\mathbb{H}^2$ under (2.1) for $R = \widetilde{S}$. Let $\theta = \varphi^*([f^*])$. $\theta \in \mathrm{Mod}^a_S$. Note that every lift of f is of the form:

$$g_1 \circ f^* \circ g_2, \quad g_1, g_2 \in G.$$

But $f^* \circ g_2 = \Phi(g_2) \circ f^*$, where $\Phi : G \to G$ is the isomorphism induced by f^* by conjugation. Thus every lift of f is of the form:

$$g \circ f^*, \quad g \in G. \tag{2.4}$$

So every element in $i^{-1}(\chi)$ is of the form $\varphi^*(g) \circ \theta$.

Write $g = g_1^{\alpha_1} \circ g_2^{\alpha_2} \circ \cdots \circ g_r^{\alpha_r}$, where α_i are integers and $\Gamma = \{g_1, \cdots g_r\}$ is a set of generators (may not be distinct) of G. Each element of Γ is either parabolic or simple hyperbolic.

From [7, 11, Theorem 2], each mapping class $\varphi^*(g_i)$ is induced by either a spin about a, or a Dehn twist along a loop bounding a punctured disk enclosing a and another puncture depending on whether or not g_i is simple hyperbolic or parabolic. More precisely, when g_i is simple hyperbolic and corresponds to a simple closed geodesic α on $\widetilde{S}$, $\varphi^*(g_i)$ is induced by a spin $t^{-1}_{\alpha''} \circ t_{\alpha'}$, where t_c is the Dehn twist along a simple loop c and α' and α'' are two parallel simple loops on S so that both loops are homotopic to α on $\widetilde{S}$. When g_i is parabolic, $\varphi^*(g_i)$ is induced by a simple Dehn twist along a loop on S that bounds a 2-punctured disk on S containing a and the other puncture of $\widetilde{S}$ determined by the conjugacy class of g_i under (2.1).

3 Main Results

We see from the previous section that every element in $i^{-1}(\chi) \subset \mathrm{Mod}^a_S$ is obtained from a "canonical" mapping class θ followed by a chain of spins or twists. We now further assume that

$f : \widetilde{S} \to \widetilde{S}$ fixes at least one puncture ζ of $\widetilde{S}$. Let $f^* : \mathbb{H}^2 \to \mathbb{H}^2$ be the lift of f with $f^*(\zeta^*) = \zeta^*$, where $\zeta^* \in \partial\mathbb{H}^2$ satisfies $\varrho(\zeta^*) = \zeta$. That is, ζ^* is the fixed point of a parabolic element T of G. Let $\{T\}$ denote the conjugacy class of T in G. $[f^*]$ is determined by the boundary value $f^*|_{\partial\mathbb{H}^2}$ and defines a modular transformation on $F(\widetilde{S})$. It determines an element $\varphi^*([f^*])$ in $i^{-1}(\chi)$. Recall that $\varphi^*([f^*])$ is denoted by θ.

The main results of this paper are the following:

Theorem 3.1 *Let $\chi \in \mathrm{Mod}_{\widetilde{S}}$ be a hyperbolic mapping class, and let $f : \widetilde{S} \to \widetilde{S}$ be a representative of χ.*

(1) *If either $\widetilde{S}$ is a compact Riemann surface without boundary, or $\widetilde{S}$ is non-compact and f fixes no punctures of $\widetilde{S}$, then every element in $i^{-1}(\chi) \subset \mathrm{Mod}_S^a$ is hyperbolic.*

(2) *If $\widetilde{S}$ is of finite type and f fixes a puncture of $\widetilde{S}$, then an element $\hat{\theta} = \varphi^*([\hat{f}]) \in \mathrm{Mod}_S^a$ is pseudo-hyperbolic if and only if as an element of $\mathrm{mod}\,\widetilde{S}$, $[\hat{f}]$ commutes with a parabolic element of G.*

By (2.4) we notice that $\hat{\theta}$ can always be written as $\varphi^*(g) \circ \theta$. Let $E_\zeta \subset \partial\mathbb{H}^2$ denote the set

$$\{g(\zeta^*) \in \partial\mathbb{H}^2;\ \text{for all } g \in G\}. \tag{3.1}$$

E_ζ is a dense subset of $\partial\mathbb{H}^2$ and consists of fixed points of elements in the conjugacy class $\{T\}$ of T.

As a consequence of Theorem 3.1, we obtain:

Corollary 3.1 *Assume that $\chi \in \mathrm{Mod}_{\widetilde{S}}$ is a hyperbolic mapping class induced by a map f that fixes a puncture of $\widetilde{S}$. Let $\hat{f}$ be a lift of f. Then $\hat{f}|_{\partial\mathbb{H}^2}$ has no fixed points in E_ζ for any puncture ζ of $\widetilde{S}$ if and only if $\varphi^*([\hat{f}]) \in i^{-1}(\chi)$ is a hyperbolic mapping class.*

Proof Suppose that there is a parabolic element $T \in G$ such that $T \circ [\hat{f}] = [\hat{f}] \circ T$. Here both elements are considered elements of mod $\widetilde{S}$. Let ζ^* denote its fixed point. Choose a sequence $([0], z_n) \in \mathbb{H}^2 \subset F(\widetilde{S})$ such that $z_n \to \zeta^*$. We have

$$T \circ [\hat{f}]([0], z_n) = [\hat{f}] \circ T([0], z_n).$$

From (2.2) and (2.3),

$$T \circ [\hat{f}]([0], z_n) = ([\nu], T^\nu(w^\nu \circ \hat{f}(z_n))) \quad \text{and} \quad [\hat{f}] \circ T([0], z_n) = ([\nu], w^\nu \circ \hat{f}(T(z_n))),$$

where ν is the Beltrami coefficient of f^{-1} and $T^\nu = w^\nu \circ T \circ (w^\nu)^{-1}$. We thus obtain

$$T^\nu(w^\nu \circ \hat{f}(z_n)) = w^\nu \circ \hat{f}(T(z_n)) \quad \text{for all } n. \tag{3.2}$$

Let $n \to \infty$, from (3.2), we get $T \circ \hat{f}(\zeta^*) = \hat{f}(\zeta^*)$. This implies that $\hat{f}(\zeta^*)$ is also fixed by T. But T is parabolic which has a unique fixed point in $\partial\mathbb{H}^2$. We conclude that $\hat{f}(\zeta^*) = \zeta^*$.

Conversely, if $\hat{f}$ and T share the same fixed point $\zeta^* \in \partial\mathbb{H}^2$, there is an integer n such that

$$[\hat{f}] \circ T \circ [\hat{f}]^{-1} = T^n. \tag{3.3}$$

From [7, 11, Theorem 2], $\varphi^*(T^n) = (\varphi^*(T))^n$ is a power of the Dehn twist along the boundary ∂D of a punctured disc D enclosing a and another puncture ζ. Thus $\varphi^*([\hat{f}]) \circ \varphi^*(T) \circ \varphi^*([\hat{f}])^{-1}$ keeps $\varphi^*([\hat{f}])(\partial D)$ invariant. From (3.3), $\varphi^*([\hat{f}])(\partial D) = \partial D$. Therefore, $\varphi^*([\hat{f}])$ is reduced by ∂D.

Remark 3.1 From the same argument as in Corollary 3.1, we see that the canonical mapping class $\theta = \varphi^*([f^*])$ is a pseudo-hyperbolic mapping class reduced by a simple loop c that is the boundary of a punctured disk D enclosing a and the puncture ζ of $\widetilde{S}$ determined by T. $S\backslash D = \widetilde{S}\backslash D$ is the single component on which $\varphi^*([f^*])$ is irreducible and is obtained from the map $f : \widetilde{S} \to \widetilde{S}$ by blowing up the puncture ζ.

Corollary 3.2 *Assume that f fixes a puncture of $\widetilde{S}$. Then there are infinitely many pseudo-hyperbolic mapping classes in $i^{-1}(\chi)$.*

Proof To see that there are infinitely many pseudo-hyperbolic mapping classes in $i^{-1}(\chi)$, we notice that for each fixed point x^* in E_ζ, there is an element $\gamma \in G$ such that $\gamma(\zeta^*) = x^*$. Consider $\tilde{f}_x = \gamma \circ f^* \circ \gamma^{-1}$. $\tilde{f}_x$ is also a lift of f with $\tilde{f}_x(x^*) = x^*$. By the same argument in Theorem 3.1, $\varphi^*([\tilde{f}_x])$ is pseudo-hyperbolic. Suppose that there are $x^*, y^* \in E_\zeta$ with $x^* \neq y^*$, such that $\tilde{f}_x = \tilde{f}_y$. This would imply that there is a lift $\tilde{f} = \tilde{f}_x$ of f that has two fixed points lying in E_ζ. [8, Lemma 5.4] claims that this is impossible.

It follows that there is an injection of E_ζ into a set F of lifts of f. Since E_ζ contains infinitely many elements, so does F. This proves the corollary.

The question as to whether there are always hyperbolic mapping classes in $i^{-1}(\chi^n)$ for each n has not been answered and will be pursued elsewhere. However, if $\widetilde{S}$ is compact, Theorem 3.1 asserts that $i^{-1}(\chi)$ consists of hyperbolic mapping classes. If $\widetilde{S}$ has some punctures, we can prove:

Theorem 3.2 *Assume that f fixes a puncture of $\widetilde{S}$. Then there is an integer N, depending only on the mapping class χ, such that $i^{-1}(\chi^n)$ contains infinitely many hyperbolic mapping classes whenever $n \geq N$. In particular, there are infinitely many hyperbolic mapping classes in the union $\bigcup_{n\geq 1} \{i^{-1}(\chi^n)\}$.*

Remark 3.2 Let us now consider a special case that χ is induced by a composition of Dehn twists. Let α and β be two simple loops on $\widetilde{S}$ such that $\alpha \cup \beta$ fills $\widetilde{S}$ in the sense that $\widetilde{S}\backslash\{\alpha \cup \beta\}$ is a union of disks and punctured disks. Let t_c denote the simple Dehn twist as well as its mapping class along a simple loop c. [6, Theorem III.3] asserts that for any non-negative integers m_1 and m_2, the composition $t_\beta^{-m_2} \circ t_\alpha^{m_1}$ is a hyperbolic mapping class. In this case, one can prove that the fiber $i^{-1}(t_\beta^{-m_2} \circ t_\alpha^{m_1})$ contains infinitely many hyperbolic mapping classes. The proof appears in [15].

Remark 3.3 It was shown in [7] that for any finite type Riemann surface with genus $p \geq 2$, there is an integer n so that $i^{-1}(\chi^n)$ contains a hyperbolic mapping class. Our approach is different than [7] and is valid for all cases including $p = 1$.

Theorem 3.2 follows from:

Theorem 3.3 *For every hyperbolic element $g \in G$, there is an integer N, depending on g and the mapping class of f, such that for $n \geq N$, $g \circ (f^*)^n$ has a unique fixed point z in $\partial\mathbb{H}^2$, but z does not belong to E_ζ for any puncture ζ of $\widetilde{S}$, where E_ζ is defined in* (3.1).

Proof of Theorem 3.2 By Theorem 3.3, for large n, $g \circ (f^*)^n$ does not commute with any parabolic elements of G. By Theorem 3.1, $\theta_0 = \varphi^*(g \circ [f^*]^n) = \varphi^*(g) \circ \theta^n$ is a hyperbolic mapping class in $i^{-1}(\chi^n)$. Once θ_0 is found, the set

$$\{\varphi^*(g') \circ \theta_0 \circ \varphi^*(g')^{-1};\ g' \in G\} \subset i^{-1}(\chi)$$

consists of infinitely many hyperbolic mapping classes. Theorem 3.2 is proved.

4 Parabolic Elements and 2-punctured Disks

In this section we assume that $\widetilde{S}$ has some punctures. That is, G contains some parabolic elements. Theorem 2 of [7, 11] asserts that for every parabolic element T_0 of G, $\varphi^*(T_0) \in \mathrm{Mod}_S^a$ is a Dehn twist along a boundary of a topological disk D on S that encloses a and another puncture determined by the conjugacy class of T_0 under (2.1). The converse is stated in the following lemma.

Lemma 4.1 *Let c be the boundary of an arbitrary topological disk $D \subset S$ that encloses a and another puncture ζ of S. Then there is a parabolic element T_0 of G such that $\varphi^*(T_0) \in Mod_S^a$ is induced by the Dehn twist along c. Moreover, T_0 corresponds to the puncture ζ under* (2.1) *if ζ is regarded as a puncture of $\widetilde{S}$.*

Proof We define a flow in $T(S)$ obtained from pinching the loop c to a cusp. Let x_0 be represented by S, let $\{x_i\} \subset T(S)$ be a sequence on the flow such that x_i tends to $\partial T(S)$ and let c_i be the loop on S_i corresponding to c. Let $y_i = \varphi^{-1}(x_i) \in F(\widetilde{S})$. $\{y_i\}$ lies in a single fiber $\mathbb{H}^2$ since the forgetful map ι sends each x_i to a fixed point (origin) in $T(\widetilde{S})$ represented by $\widetilde{S}$.

For each y_i there is an element g_i in G so that $\{g_i(y_i)\}$ lies in a fundamental domain $\Delta \subset \mathbb{H}^2$ of G. $\{g_i(y_i)\}$ cannot lie in a compact subset of Δ. Since there are only finitely many conjugacy classes of parabolic elements of G, by selecting a subsequence if necessary, we assume that $\{g_i(y_i)\}$ tends to a fixed point of a parabolic element T' of G. We have

$$\rho(g_i(y_i),\ T'(g_i(y_i))) \to 0, \quad \text{as } i \to \infty, \tag{4.1}$$

where ρ is the Poincaré metric on $\mathbb{H}^2$. Since $\varphi : F(\widetilde{S}) \to T(S)$ is holomorphic, the restriction $\varphi|_{\mathbb{H}^2} : \mathbb{H}^2 \to T(S)$ is holomorphic. From (4.1), we get

$$\langle \varphi^*(g_i)(x_i),\ \varphi^*(T')\varphi^*(g_i)(x_i) \rangle \to 0.$$

This implies that $\varphi^*(T')$ is induced by the Dehn twist along $\varphi^*(g_i)(c_i)$. Let $T_0 = g_i \circ T' \circ g_i^{-1}$. It follows that $\varphi^*(T_0)$ is the mapping class induced by the Dehn twist along c_i. Lemma 4.1 is proved.

Let c be a simple loop on S. From the same process described in Lemma 4.1, we obtain a sequence $\{x_i\} \subset T(S)$ as well as a sequence of loops $\{c_i\}$, $c_i \subset S_i$. Let $\widetilde{c_i}$ is the image loop of c_i on $\widetilde{S}_i$ under the forgetful map $\iota : T(S) \to T(\widetilde{S})$. Let $l(c_i)$ denote the hyperbolic length of the closed geodesic homotopic to c_i on S_i. Similarly, let $l(\tilde{c}_i)$ denote the hyperbolic length of the closed geodesic homotopic to $\tilde{c}_i$ on $\widetilde{S}_i$.

Lemma 4.2 *Under the situation as above, and in addition we assume that $\tilde{c}$ is non-contractible on $\widetilde{S}$, then $l(\tilde{c}_i) \to 0$. In particular, $\tilde{x}_i$ tends to the boundary $\partial T(\widetilde{S})$.*

Proof Let θ_0 be the mapping class of S induced by a simple Dehn twist along c. Since $l(c_i)$ tends to zero,

$$\langle x_i, \theta_0(x_i) \rangle \to 0, \quad \text{as } i \to \infty. \tag{4.2}$$

But θ_0 projects to a Dehn twist $t_{\tilde{c}_i}$ along the loop $\tilde{c}_i$. If $l(\tilde{c}_i)$ does not tend to zero, the Mumford-Bers theorem [10, 4] asserts that $\{\tilde{x}_i\}$ lies in a compact subset of $T(\widetilde{S})$. It follows that

$$\langle \tilde{x_i}, t_{\tilde{c}_i}(\tilde{x}_i) \rangle > M \tag{4.3}$$

for a constant $M > 0$.

Since $\pi \circ \varphi^{-1} : T(S) \to T(\widetilde{S})$ is holomorphic, it is distance non-increasing. From (4.2), as $i \to \infty$,

$$\begin{aligned}\langle \tilde{x}_i,\ t_{\tilde{c}_i}(\tilde{x}_i)\rangle &= \langle \pi \circ \varphi^{-1}(x_i),\ t_{\tilde{c}_i}(\pi \circ \varphi^{-1}(x_i))\rangle \\ &= \langle \pi \circ \varphi^{-1}(x_i),\ \pi \circ \varphi^{-1}(\theta_0(x_i))\rangle \le \langle x_i,\ \theta_0(x_i)\rangle \to 0.\end{aligned}$$

This contradicts (4.3). The lemma is proved.

5 Absolutely Extremal Teichmüller Mappings and Their Lifts

In this section we review some known facts about an absolutely extremal Teichmüller mapping on $\widetilde{S}$ and its lifts to $\mathbb{H}^2$. More details can be found in [8]. Let $f : \widetilde{S} \to \widetilde{S}$ be an absolutely extremal Teichmüller mapping that fixes a point ζ, ζ being either a puncture or a regular point. Let $f^* : \mathbb{H}^2 \to \mathbb{H}^2$ be the lift such that $f^*(\zeta^*) = \zeta^*$, where $\zeta^* \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$ be such that $\varrho(\zeta^*) = \zeta$ (ϱ is defined in (2.1)). $\zeta^* \in \partial\mathbb{H}^2$ if and only if ζ is a puncture of $\widetilde{S}$. Recall that E_ζ is defined in (3.1).

The differential q, associated to the absolutely extremal Teichmüller mapping f on $\widetilde{S}$, lifts to q^* via

$$q^*(z) = q(\varrho(z))\varrho'(z)^2. \tag{5.1}$$

q^* defined in (5.1) is a holomorphic function of $\mathbb{H}^2$. It is easy to check that q^* satisfies

$$q^*(g(z))g'(z)^2 = q^*(z), \quad \text{for all } g \in G.$$

From the analysis of Marden and Strebel [8], if $\zeta^* \in \mathbb{H}^2$, there are $2(r+2)$ critical rays (both horizontal and vertical) originated from ζ^* that divide $\mathbb{H}^2$ into $2(r+2)$ triangular regions $\Delta_1, \cdots, \Delta_{2(r+2)}$, where each Δ_i is bounded by a horizontal critical ray α_i, a vertical critical ray β_i, and the part $[\xi_i, \eta_i]$ of $\partial\mathbb{H}^2$ determined by the endpoints ξ_i and η_i of α_i and β_i, respectively. Note that both ξ_i and η_i are not fixed points of elements of G.

In the case that $\zeta^* \in \partial\mathbb{H}^2$, the situation is slightly different. Let T be a parabolic element of G that fixes ζ^*. All the critical rays originated from ζ^* constitute the set

$$\bigcup_{i\in\mathbb{Z}} T^i(\Sigma),$$

where Σ consists of $r+2$ horizontal critical rays and $r+2$ vertical critical rays. These $2(r+2)$ rays alternate and are labeled counterclockwise. By [8, Lemma 5.4] again, we know that except for ζ^*, all other end points of $T^i(\Sigma)$, $i \in \mathbb{Z}$, are not fixed by elements of G. All these rays divide $\mathbb{H}^2$ into countably many triangular regions that are relabeled as:

$$\cdots, \Delta_{-1}, \Delta_0, \Delta_1, \cdots. \tag{5.2}$$

We denote by $[\xi_i, \eta_i]$ the closed interval $\overline{\Delta_i} \cap \partial\mathbb{H}^2$. For each $i \in \mathbb{Z}$, $\overline{\Delta_i \cup \Delta_{i+1}}$ is a triangular region which is either bounded by horizontal critical rays α_i, α_{i+1}, or vertical critical rays β_i, β_{i+1}.

Let f^* be the lift of f that fixes ζ^*. Since f keeps each critical arc on $\widetilde{S}$ invariant and Δ_i is bounded by a horizontal critical ray and a vertical critical ray, f^* keeps each triangular region Δ_i (defined in (5.2)) invariant. It was shown in [8] that

$$(f^*)^n(z) \to \xi_i \quad \text{and} \quad (f^*)^{-n}(z) \to \eta_i, \tag{5.3}$$

whenever $n \to \infty$ and $z \in \Delta_i \cup (\xi_i, \eta_i)$ (where (ξ_i, η_i) denotes the open interval $(\overline{\Delta_i} \cap \partial\mathbb{H}^2)\setminus \{\xi_i, \eta_i\}$). The convergence is uniform if z is bounded away from $\beta_i \cup \{\eta_i\}$, or from $\alpha_i \cup \{\xi_i\}$, respectively.

Let $\Delta_i \cup \Delta_{i+1}$ be a triangular region bounded by α_i and α_{i+1}. (Similar discussion applies when $\Delta_i \cup \Delta_{i+1}$ is bounded by vertical critical rays.) $\Delta_i \cup \Delta_{i+1}$ may contain other pre-images of critical points of f. From each of these preimages we follow the same procedure to draw horizontal critical rays and we thus obtain all the pre-images of horizontal critical trajectories of q on $\widetilde{S}$. These rays are mutually disjoint in $\mathbb{H}^2$, and so are their end points on $\partial\mathbb{H}^2$.

The totality of these rays divide $\mathbb{H}^2$ into countably many simply connected regions Δ_1', Δ_2', $\cdots$. The tessellation $\{\Delta_i'\}$ satisfies the following properties:

(i) For $i \neq j$, either Δ_i' and Δ_j' are disjoint or they are adjacent along a horizontal critical ray.

(ii) Either f^* or any element $g \in G$ sends a Δ_i' to another Δ_j' and $\Delta_i' = \Delta_j'$ if and only if its closure contains a fixed point of f^*.

Note that for each $i \in \mathbb{Z}$, $f^*(\Delta_i \cup \Delta_{i+1}) = \Delta_i \cup \Delta_{i+1}$ and for every $g \in G$, either $g(\Delta_i \cup \Delta_{i+1})$ contains $\Delta_i \cup \Delta_{i+1}$, or is contained in $\Delta_i \cup \Delta_{i+1}$, or is disjoint from $\Delta_i \cup \Delta_{i+1}$. Note also that the endpoint ξ_i of α_i is a fixed point of f^* but not a fixed point of any element of G.

6 Proof of Theorem 3.1

Let $\hat{\theta} = \varphi^*(g \circ [f^*])$. Suppose that $g \circ [f^*]$ commutes with a parabolic element T of G. From [7, 11, Theorem 2], $\varphi^*(T)$ is induced by a spin whose inner loop is contractible to a puncture x of $\widetilde{S}$. So $\varphi^*(T)$ can be realized as a Dehn twist $t_{c'}$ along a loop c' on S, where c' bounds a topological disk that encloses a and x. Let $f_{\hat{\theta}}$ be a representative of $\hat{\theta}$ and consider $h = f_{\hat{\theta}} \circ t_{c'} \circ f_{\hat{\theta}}^{-1}$. Obviously, h leaves $f_{\hat{\theta}}(c')$ invariant. By hypothesis h is isotopic to $t_{c'}$. So $f_{\hat{\theta}}(c')$ is homotopic to c' and hence $\hat{\theta}$ is pseudo-hyperbolic.

Conversely, we assume that $\chi \in \mathrm{Mod}_{\widetilde{S}}$ is hyperbolic and $\hat{\theta}$, $i(\hat{\theta}) = \chi$, is pseudo-hyperbolic. There is a loop system

$$C = \{c_1, \ldots, c_k\}, \quad k \geq 1, \tag{6.1}$$

of non-contractible, homotopically independent and homotopically disjoint simple loops such that $f_{\hat{\theta}}(C) = C$, where again $f_{\hat{\theta}} : S \to S$ is a representative of $\hat{\theta}$. There is an integer N such that $f_{\hat{\theta}}^N$ stabilizes each component of $S\setminus C$ as well as each loop in C and that each component map is either the identity or irreducible. Let $P_1, \cdots, P_r$, $r \geq 1$, be the components of $S\setminus C$.

We define a smooth flow in $T(S)$ that is obtained from pinching the loops in (6.1) to cusps. Let $\{x_i\} \subset T(S)$ be any discrete instances on the flow such that $x_i \to \partial T(S)$ as $i \to \infty$. Let S_i represent x_i, let $P_{i,j}$, $j = 1, \cdots, r$, be the corresponding components on S_i.

The component maps $f_{\hat{\theta}}^N|_{P_{i,j}}$ on $P_{i,j}$, if not the identity, is isotopic to a Teichmüller mapping with dilatation $K_{i,j}$. Since $f_{\hat{\theta}}^N$ is reduced by (6.1), from [3], each $K_{i,j}$ remains bounded during the pinching process mentioned above. Hence there is a constant K such that $K_{i,j} \leq K$ as $i \to \infty$, $j = 1, \cdots, k$. It follows that

$$\langle x_i, \hat{\theta}^N(x_i) \rangle \leq \frac{1}{2} \log K, \quad i \to \infty. \tag{6.2}$$

Let $\tilde{x}_i = \pi \circ \varphi^{-1}(x_i) \in T(\widetilde{S})$. Since $\pi \circ \varphi^{-1} = \iota : T(S) \to T(\widetilde{S})$ is the forgetful map, The representative $\widetilde{S}_i$ of $\tilde{x}_i$ is obtained from S_i by filling in the puncture a. Let

$$R(\widetilde{S}) = T(\widetilde{S}) \;/\; \mathrm{Mod}_{\widetilde{S}}$$

denote the Riemann moduli space of $\widetilde{S}$, and let $\varpi : T(\widetilde{S}) \to R(\widetilde{S})$ be the natural projection.

Let $\Lambda \subset R(\widetilde{S})$ and $\Lambda' \subset T(\widetilde{S})$ denote the point sets $\{\varpi \circ \pi \circ \varphi^{-1}(x_i)\}$ and $\{\pi \circ \varphi^{-1}(x_i)\}$, respectively. There are two cases to consider.

Case 1 Λ is not compact in $R(\widetilde{S})$. From Mumford's theorem [10, 4], there are non-contractible loops denoted by α_i on $\widetilde{S}_i$ such that $l_i = l(\alpha_i) \to 0$ as $i \to \infty$. This implies that for sufficiently large i there is a small number δ_i, depending only on $\widetilde{S}_i$, such that

$$l_i < \delta_i < 2b_i,$$

where b_i is chosen so that

$$\sinh\, b_i = \frac{1}{2\,\sinh\,\frac{l_i}{2}}.$$

See [9] or [3]. Since χ is hyperbolic, it is induced by an extremal Teichmüller mapping f_i. By [3, Lemma 3], the dilatation

$$K(f_i) \geq \Big(\frac{\delta_i}{l_i}\Big)^{\frac{1}{3p-3+n}}. \tag{6.3}$$

Since $l_i \to 0$, and δ_i is chosen to be proportional to $\log\,(\frac{1}{l_i})$, from (6.3), there is a constant M such that

$$K(f_i) \geq \Big[\frac{M}{l_i}\Big(\log\frac{1}{l_i}\Big)\Big]^{\frac{1}{3p-3+n}} \to \infty, \quad \text{as } l_i \to 0. \tag{6.4}$$

But f_i is the extremal map that induces χ. It follows from (6.4) that

$$\langle \tilde{x}_i, \chi^N(\tilde{x}_i)\rangle \geq \langle \tilde{x}_i, \chi(\tilde{x}_i)\rangle = \frac{1}{2}\,\log\,K(f_i) \to \infty. \tag{6.5}$$

On the other hand, since $\pi \circ \varphi^{-1} : T(S) \to T(\widetilde{S})$ is holomorphic, it is distance non-increasing. So by (6.2),

$$\begin{aligned}\langle \tilde{x}_i, \chi^N(\tilde{x}_i)\rangle &= \langle \pi\circ\varphi^{-1}(x_i),\ \chi^N \circ (\pi\circ\varphi^{-1}(x_i))\rangle \\ &= \langle \pi\circ\varphi^{-1}(x_i),\ \pi\circ\varphi^{-1}(\hat{\theta}^N(x_i))\rangle \leq \langle x_i, \hat{\theta}^N(x_i)\rangle \leq \frac{1}{2}\log K.\end{aligned}$$

This contradicts (6.5). So this case cannot happen.

Remark 6.1 Case 1 can also be settled by observing that $f_{\hat{\theta}}^N$ projects to the map f^N on $\widetilde{S}$. We adopt a different approach here since the method can also be used to handle various situations including the case that f^N is trivial or a Dehn twist. See [14] for a similar argument.

Case 2 Λ is compact. In this case, there is a subsequence $\{\tilde{x}_i\} \subset \Lambda'$, which may tend to the boundary $\partial T(\widetilde{S})$, such that $\widetilde{S}_i$ has no short closed geodesics. Let $y_i = \varphi^{-1}(x_i)$ be such that $\pi(y_i) = \tilde{x}_i$. By construction, $\{x_i\}$ lies in the flow defined by a pinching process.

By Lemma 4.2, every loop $\tilde{c}_j$, $j = 1, \cdots, k$, is contractible on $\widetilde{S}$, and in this case all loops c_j in (6.1) bound disks $D_j \subset S$ that enclose a and another puncture x_j. However, this case does not occur when $\widetilde{S}$ is closed.

Now we assume that $\widetilde{S}$ is not closed and $k \geq 2$. In this case, x_j are also punctures of $\widetilde{S}$ and x_j is not contained in $D_{j'}$ whenever $j \neq j'$. But both D_j and $D_{j'}$ enclose a. This implies that c_j intersects with $c_{j'}$. This contradicts the fact that all loops in (6.1) are homotopically disjoint.

We conclude that if $\widetilde{S}$ is not closed and $\hat{\theta}$ is pseudo-hyperbolic, then $k = 1$ which says that (6.1) consists of a single loop, say c_1, that bounds a topological disk $D \subset S$ enclosing a and another puncture x. x can also be regarded as a puncture of $\widetilde{S}$. In particular, $f_{\hat{\theta}}$ is reduced by c_1 and is a component map with $f_{\hat{\theta}}(c_1) = c_1$. By Lemma 4.1, there is a parabolic element T_0 in G such that $\varphi^*(T_0) \in \mathrm{Mod}_S^a$ can be realized as the Dehn twist along c_1.

Since $\hat{\theta}$ is reduced by c_1, $\varphi^*(T_0)$ commutes with $\hat{\theta}$. Recall that $\hat{\theta} = \varphi^*(g \circ [f^*])$. We conclude that T_0 commutes with $g \circ [f^*]$. Theorem 3.1(2) is proved.

Since $\iota \circ \varphi^* = \pi$, the loop c_1 bounds only x on $\widetilde{S}$ when a is filled in. We see that T_0 corresponds to the puncture x of $\widetilde{S}$. By using the same argument of Corollary 3.1, we see that T_0 and $g \circ [f^*]$ share the same fixed point $x^* \in \partial\mathbb{H}^2$ with $\varrho(x^*) = x$. This implies that $f : \widetilde{S} \to \widetilde{S}$ fixes x, and this contradicts the hypothesis (1) of Theorem 3.1. This completes the proof of Theorem 3.1.

7 Existence of Hyperbolic Mapping Classes

The aim of this section is to prove Theorem 3.3. Suppose that $g \circ (f^*)^n$ has a fixed point z in $\partial\mathbb{H}^2$ for a large n. Certainly, for any i, $z \neq \xi_i$ since ξ_i is a fixed point of f^* but not fixed by g (see [8, Lemma 5.4]).

Let x and y be the attracting and repelling fixed points of g. If z lies in (ξ_k, ξ_{k+1}) with $x, y \notin (\xi_k, \xi_{k+1})$, then $g(\xi_k, \xi_{k+1})$ is disjoint from (ξ_k, ξ_{k+1}) while $f^*([\xi_k, \xi_{k+1}]) = [\xi_k, \xi_{k+1}]$. Hence, for every n, $g \circ (f^*)^n$ cannot have a fixed point in $[\xi_k, \xi_{k+1}]$. This is a contradiction.

If z lies in (ξ_k, ξ_{k+1}) and (ξ_k, ξ_{k+1}) contains the attracting fixed point x of g, without loss of generality we assume that $x \in (\xi_k, \eta_k)$. If $z = x$, then $g(z) = z$ while by (5.3), for large n, $(f^*)^n(z) \to \xi_k \neq z$. So $g \circ (f^*)^n$ cannot fix z. If $z \in (x, \eta_k)$, from (5.3) again, $(f^*)^n(z) \to \xi_k$. In particular, for n sufficiently large, $(f^*)^n(z)$ lies in a small neighborhood of ξ_k. Hence $(f^*)^n(z)$ lies in (ξ_k, x). Since x is the attracting fixed point of g, $g \circ (f^*)^n(z)$ also lies in (ξ_k, x). But z lies in (x, η_k). So z cannot be a fixed point of $g \circ (f^*)^n$. Finally, let $z \in (\eta_k, \xi_{k+1})$. Once again using (5.3), we get that $(f^*)^n(z) \to \xi_{k+1}$, as $n \to \infty$. Observe that f^* keeps the closed interval $[\eta_k, \xi_{k+1}]$ invariant. As a real Möbius transformation, g maps (η_k, ξ_{k+1}) to the open interval $g(\eta_k, \xi_{k+1})$ so that one of the following must hold:

(i) $g(\eta_k, \xi_{k+1})$ contains (η_k, ξ_{k+1}),

(ii) $g(\eta_k, \xi_{k+1})$ is contained in (η_k, ξ_{k+1}),

(iii) $g(\eta_k, \xi_{k+1})$ is disjoint from (η_k, ξ_{k+1}).

In case (iii), $g \circ (f^*)^n$ has no fixed point lying in (η_k, ξ_{k+1}). In case (i), (η_k, ξ_{k+1}) contains the repelling fixed point of g. From (5.3), $(f^*)^n(z)$ moves away from z. Since (η_k, ξ_{k+1}) contains the repelling fixed point of g, $g \circ (f^*)^n(z)$ moves farther away from z. In particular, $g \circ (f^*)^n$ has no fixed point in (η_k, ξ_{k+1}). In case (ii), (η_k, ξ_{k+1}) contains the attracting fixed point of g, and we must have $x \in (\eta_k, \xi_{k+1})$. This is impossible.

To see that for large n, $g \circ (f^*)^n$ has a fixed point lying in the open interval (ξ_k, x), we choose an open interval $U_{d_0} = (\xi_k, d_0)$, $d_0 < x$. Since x is the attracting fixed point of g, $g(U_{d_0}) = (d_1, d_2)$, where d_1 and $d_2 \in (\xi_k, x)$. Let $d = x - d_2$. Fix a small δ, $0 < \delta < d$. From (5.3), for large $n \geq N$, $g \circ (f^*)^n(x')$ lies in U_{d_0} whenever $x' \in U_\delta := \{x'';\ x - x'' < \delta\}$. It follows that

$$|(f^*)^n(x') - x'| > |(f^*)^n(x') - g \circ (f^*)^n(x')|.$$

But when x' lies in U_{d_0} for d_0 sufficiently small, from (5.3) again,

$$|(f^*)^n(x') - x'| < |(f^*)^n(x') - g \circ (f^*)^n(x')|.$$

Since $g \circ (f^*)^n$ is continuous on $\partial\mathbb{H}^2$, by the intermediate-value theorem, there is a point $z \in (\xi_k, x)$ such that $g \circ (f^*)^n(z) = z$.

Next we show that the fixed point z of $g \circ (f^*)^n$ does not lie in E_ζ (defined in (3.1)) for any puncture ζ of $\widetilde{S}$. Suppose that for some large n, $g \circ (f^*)^n$ has a fixed point $\zeta_0 \in (\xi_k, x) \cap E_\zeta$ that is a fixed point of a parabolic element T_0 of G. $T_0 \in \{T\}$ but T_0 is distinct from T. We have $(g \circ (f^*)^n) \circ T_0 \circ (g \circ (f^*)^n)^{-1} = T_0^m$ for some integer m, or

$$(f^*)^n \circ T_0 \circ (f^{*n})^{-1} = g^{-1} \circ T_0^m \circ g. \tag{7.1}$$

For a fixed horodisk D_0 of T_0 at ζ_0, the Euclidean area of the horodisk $h(D_0)$, $h \in G$, can be used to describe how far the parabolic element $h \circ T_0 \circ h^{-1} \in \{T_0\}$ is from T_0. Without loss of generality, we let g be defined by sending any point z' in $\mathbb{H}^2$ to $\lambda z'$, $\lambda > 1$. In this case, 0 is the repelling fixed point of g and ∞ is the attracting fixed point of g. The element $g^{-1} \circ T_0^m \circ g$ has a horodisk $g^{-1}(D_0)$ with the Euclidean area:

$$\sim O(\frac{1}{\lambda^2}).$$

On the other hand, from (5.3), for any $z' \in (\xi_k, x)$, $(f^*)^n(z') \to \xi_k$. Hence the (Euclidean) area of $(f^*)^n(D_0)$ can be made to be arbitrarily small, and this would contradict (7.1).

We conclude that for sufficiently large n, $g \circ (f^*)^n$ has no fixed point in E_ζ. Theorem 3.3 is proved.

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