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Boundedness of Commutators with Lipschitz Functions in Non-homogeneous Spaces****

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Abstract Under the assumption that the underlying measure is a non-negative Radon measure which only satisfies some growth condition, the authors prove that for a class of commutators with Lipschitz functions which include commutators generated by Calderón-Zygmund operators and Lipschitz functions as examples, their boundedness in Lebesgue spaces or the Hardy space $H^1(\mu)$ is equivalent to some endpoint estimates satisfied by them. This result is new even when the underlying measure μ is the d-dimensional Lebesgue measure.

Keywords Commutator, Lipschitz function, Lebesgue space, Hardy space, RBMO space, Non-doubling measure
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1 Introduction

Let μ be a non-negative measure on $\mathbb{R}^d$ which only satisfies the following growth condition that there exists a positive constant C_0 such that

$$\mu(B(x,r)) \le C_0 r^n \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$, n is a fixed number and $0 < n \le d$. We call the Euclidean space $\mathbb{R}^d$ endowed with the usual Euclidean distance and the measure satisfying (1.1) a non-homogeneous space, since the measure μ is not necessary to satisfy the doubling condition which is a key assumption in the analysis on spaces of homogeneous type. Here, we recall that μ is said to satisfy the doubling condition if there exists some positive constant C such that $\mu(B(x,2r)) \le C\mu(B(x,r))$ for all $x \in \operatorname{supp}\mu$ and $r > 0$. Recently, considerable attention has been paid to Calderón-Zygmund operator theory in non-homogeneous spaces and many classical results have been proved still valid in non-homogeneous spaces (see [2, 6–11]). The motivation for developing the analysis on non-homogeneous spaces and some examples of non-doubling measures can be found in [16]. We only point out that the analysis

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on non-homogeneous spaces played an essential role in solving the long-standing Painlevé's problem by Tolsa in [14].

The purpose of this paper is to investigate the relation between the boundedness of commutators with Lipschitz functions, which include commtators generated by Calderón-Zygmund operators and Lipschitz functions, in Lebesgue spaces or the Hardy space $H^1(\mu)$ and some endpoint estimates for them.

To this end, we first introduce the Lipschitz function in non-homogeneous spaces of García-Cuerva and Gatto in [1].

Definition 1.1 *Let $\beta > 0$ and $b \in L^1_{\rm loc}(\mu)$. We say that b belongs to the space* $\mathrm{Lip}\,(\beta,\,\mu)$ *if there is a constant $C > 0$ such that*

$$|b(x) - b(y)| \le C|x - y|^{\beta} \tag{1.2}$$

for μ-almost every x and y in the support of μ. The minimal constant C appeared in (1.2) *is the* $\mathrm{Lip}\,(\beta,\,\mu)$ *norm of b and is denoted simply by $\|b\|_{\mathrm{Lip}\,(\beta)}$.*

Let $b \in \mathrm{Lip}_{\beta}(\mu)$ for $0 < \beta \le 1$ and K be a function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,\,y) :\, x = y\}$ that satisfies

$$|K(x,\,y)| \le C|x - y|^{-n} \quad \text{for } x \ne y, \tag{1.3}$$

and if $|x - y| \ge 2|x - x'|$,

$$|K(x,\,y) - K(x',\,y)| + |K(y,\,x) - K(y,\,x')| \le C\frac{|x - x'|^{\delta}}{|x - y|^{n+\delta}}, \tag{1.4}$$

where $\delta \in (0,\,1]$ and $C > 0$ are positive constants independent of x, x' and y. We define the commutator T_b associated with the Lipschitz function b and the kernel K satisfying (1.3) and (1.4) as follows. For any bounded function f with compact support and μ-a.e. $x \notin \mathrm{supp}\,(f)$,

$$T_b f(x) = \int_{\mathbb{R}^d} [b(x) - b(y)]K(x,\,y)f(y)\,d\mu(y). \tag{1.5}$$

Obviously, the commutator generated by the Calderón-Zygmund operator and Lipschitz function satisfies (1.5) (see [5]). Moreover, the boundedness of Calderón-Zygmund commutators with Lipschitz functions in Lebesgue spaces and the Hardy space $H^1(\mu)$, and some endpoint estimates for them can also be found in [5]. In this paper, we will prove the boundedness of commutators defined by (1.5) in Lebesgue spaces and the Hardy space $H^1(\mu)$ is equivalent to some endpoint estimates satisfied by them. We point out that our result is new even when μ is the d-dimensional Lebesgue measure.

Before stating our result, we need to recall some necessary notation and definitions.

Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube with sides parallel to the axes and centered at some point of $\mathrm{supp}\,(\mu)$. For any cube $Q \subset \mathbb{R}^d$, we denote its length by $l(Q)$ and denote its center by x_Q. Let $\alpha > 1$ and $\beta > \alpha^n$. We say that Q is a $(\alpha,\,\beta)$-doubling cube if $\mu(\alpha Q) \le \beta\mu(Q)$, where αQ denotes the cube with the same center as Q and having the length $\alpha l(Q)$. It was pointed out by Tolsa in [12] that for any $x \in \mathrm{supp}\,(\mu)$ and $c > 0$, there exists some $(\alpha,\,\beta)$-doubling cube Q centered at x with $l(Q) \ge c$. On the other hand, if $\beta > \alpha^d$, then for μ-a.e. $x \in \mathbb{R}^d$, there exists a sequence of $(\alpha,\,\beta)$-doubling cubes $\{Q_i\}_{i\in\mathbb{N}}$ centered at

x with $l(Q_i) \to 0$ as $i \to \infty$. In the sequel, for definiteness, if α and β are not specified, by a doubling cube we mean a $(2,\, 2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\widetilde{Q}$ the smallest doubling cube in the family $\{2^iQ\}_{i\geq 0}$. Given two cubes $Q \subset R$ in $\mathbb{R}^d$, set

$$K_{Q,\,R} = 1 + \sum_{i=1}^{N_{Q,\,R}} \frac{\mu(2^iQ)}{l(2^iQ)^n},$$

where $N_{Q,\,R}$ is the smallest positive integer i such that $l(2^iQ) \geq l(R)$.

Using the coefficient $K_{Q,\,R}$, Tolsa in [12] introduced the function space $\mathrm{RBMO}(\mu)$ with the non-doubling measure μ.

Definition 1.2 *Let $\rho > 1$ be some fixed constant. We say that a function $f \in L^1_{\mathrm{loc}}(\mu)$ belongs to the space* $\mathrm{RBMO}(\mu)$ *if there exists some constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^d$,*

$$\frac{1}{\mu(\rho Q)} \int_Q \left| f(y) - m_{\widetilde{Q}}(f) \right| \, d\mu(y) \leq C,$$

and for any two doubling cubes $Q \subset R$,

$$|m_Q(f) - m_R(f)| \leq C K_{Q,\,R},$$

where for any cube $Q \subset \mathbb{R}^d$, $m_Q(f)$ denotes the mean of f over the cube Q, that is,

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(y)\, d\mu(y).$$

The minimal constant $C > 0$ as above is defined to be the $\mathrm{RBMO}(\mu)$ *norm of f and is denoted by $\|f\|_*$.*

Tolsa proved in [12] that the definition of $\mathrm{RBMO}(\mu)$ is independent of chosen constant ρ, and that the space $\mathrm{RBMO}(\mu)$ is the dual of the Hardy space $H^1(\mu)$. To state the definition of the Hardy space $H^1(\mu)$ of Tolsa in [12, 15], we first recall the definition of the "grand" maximal operator M_Φ of Tolsa in [15].

Definition 1.3 *Given $f \in L^1_{\mathrm{loc}}(\mu)$, we set*

$$M_\Phi f(x) = \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f(y)\varphi(y)\, d\mu(y) \right|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,

(ii) $0 \leq \varphi(y) \leq \frac{1}{|y-x|^n}$ *for all $y \in \mathbb{R}^d$, and*

(iii) $|\nabla\varphi(y)| \leq \frac{1}{|y-x|^{n+1}}$ *for all $y \in \mathbb{R}^d$, where $\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})$.*

Based on [12, Theorem 1.2], Tolsa defined the Hardy space $H^1(\mu)$ as follows.

Definition 1.4 *The Hardy space $H^1(\mu)$ is the set of all functions $f \in L^1(\mu)$ satisfying that $\int_{\mathbb{R}^d} f\, d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, the norm of $f \in H^1(\mu)$ is defined by*

$$\|f\|_{H^1(\mu)} = \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

Here is the main result of this paper.

Theorem 1.1 *Let $b \in \mathrm{Lip}(\beta, \mu)$ for $0 < \beta \le 1$. Let K be a function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\}$ satisfying (1.3) and (1.4) and the commutator T_b be as in (1.5). Then there exists a constant $C > 0$ such that for all bounded function f with compact support, the following statements are equivalent:*

(I) *if* $1 < p < \frac{n}{\beta}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$,

$$\|T_b f\|_{L^q(\mu)} \le C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)};$$

(II) *for all* $\lambda > 0$,

$$\mu(\{x \in \mathbb{R}^d : |T_b f(x)| > \lambda\}) \le C\|b\|_{\mathrm{Lip}(\beta)} \left\{\lambda^{-1}\|f\|_{L^1(\mu)}\right\}^{n/(n-\beta)};$$

(III)

$$\|T_b f\|_* \le C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^{n/\beta}(\mu)};$$

(IV)

$$\|T_b f\|_{L^{n/(n-\beta)}(\mu)} \le C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{H^1(\mu)}.$$

Throughout this paper, C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $\frac{1}{p} + \frac{1}{p}' = 1$. For $A \sim B$, we mean that there is a constant $C > 0$ such that $C^{-1}B \le A \le CB$. Similar is $A \lesssim B$.

2 Proof of Theorem 1.1

We begin with the atomic characterization of the Hardy space $H^1(\mu)$ (see [12, 15]).

Definition 2.1 *Let $\rho > 1$ and $1 < p \le \infty$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a p-atomic block if*

(1) *there exists some cube R such that* $\mathrm{supp}\,(b) \subset R$,

(2) $\int_{\mathbb{R}^d} b\, d\mu = 0$,

(3) *for $j = 1, 2$, there are functions a_j supported on cube $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and*

$$\|a_j\|_{L^p(\mu)} \le \left\{[\mu(\rho Q_j)]^{1-1/p}\, K_{Q_j, R}\right\}^{-1}.$$

Then we define

$$|b|_{H^{1,\,p}_{atb}(\mu)} = |\lambda_1| + |\lambda_2|.$$

We say that $f \in H^{1,\,p}_{atb}(\mu)$ if there are p-atomic blocks $\{b_i\}_{i \in \mathbb{N}}$ such that

$$f = \sum_{i=1}^{\infty} b_i \quad \textit{with} \quad \sum_{i=1}^{\infty} |b_i|_{H^{1,\,p}_{atb}(\mu)} < \infty.$$

The $H^{1,\,p}_{atb}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,\,p}_{atb}(\mu)} = \inf\Big\{\sum_i |b_i|_{H^{1,\,p}_{atb}(\mu)}\Big\},$$

where the infimum is taken over all the possible decompositions of f in atomic blocks.

It was proved by Tolsa in [12, 15] that the definition of $H^{1,\,p}_{atb}(\mu)$ is independent of chosen constant $\rho > 1$. For $1 < p \le \infty$, the atomic Hardy spaces $H^{1,\,p}_{atb}(\mu)$ are just the Hardy space $H^1(\mu)$ with equivalent norms.

To prove Theorem 1.1, we need to introduce the Calderón-Zygmund decomposition in [12, 13] as follows.

Lemma 2.1 *For $1 \le p < \infty$, consider $f \in L^p(\mu)$ with compact support. For any $\lambda > 0$ $\Big($with $\lambda > \frac{2^{d+1}\|f\|_{L^1(\mu)}}{\|\mu\|}$ if $\|\mu\| < \infty\Big)$, there exists a sequence of cubes $\{Q_j\}$ with bounded overlaps, that is, $\sum\limits_j \chi_{Q_j}(x) \le C < \infty$, such that*

(a) $\displaystyle \frac{1}{\mu(2Q_j)}\int_{Q_j}|f(x)|^p\,d\mu(x) > \frac{\lambda^p}{2^{d+1}}$;

(b) $\displaystyle \frac{1}{\mu(2\eta Q_j)}\int_{\eta Q_j}|f(x)|^p\,d\mu(x) \le \frac{\lambda^p}{2^{d+1}}$ *for any $\eta > 2$;*

(c) $|f(x)| \le \lambda$ *μ-a.e. on* $\mathbb{R}^d \Big\backslash \bigcup\limits_j Q_j$;

(d) *for each fixed j, let R_j be the smallest $(6,\,6^{n+1})$-doubling cube of the form 6^iQ_j, $i \ge 1$. Set $w_j = \frac{\chi_{Q_j}}{\sum_i \chi_{Q_i}}$. Then there is a function φ_j with $\operatorname{supp}\varphi_j \subset R_j$ and some positive constant C satisfying*

$$\int_{\mathbb{R}^d}\varphi_j(x)\,d\mu(x) = \int_{Q_j} f(x)w_j(x)\,d\mu(x) \quad \text{and} \quad \sum_j |\varphi_j(x)| \le C\lambda.$$

Moreover, if $p = 1$,

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(R_j) \le C\int_{Q_j}|f(x)|\,d\mu(x),$$

and if $1 < p < \infty$,

$$\Big\{\int_{R_j}|\varphi_j(x)|^p\,d\mu(x)\Big\}^{1/p}[\mu(R_j)]^{1/p'} \le \frac{C}{\lambda^{p-1}}\int_{Q_j}|f(x)|^p\,d\mu(x).$$

The following lemma plays an important role in the proof of Theorem 1.1 and its proof can be found in [4].

Lemma 2.2 *Let T be a linear operator which is bounded from $L^{p_0}(\mu)$ into $\mathrm{RBMO}(\mu)$ and from $H^1(\mu)$ into weak $L^{p_0'}(\mu)$. Then T extends boundedly from $L^p(\mu)$ into $L^q(\mu)$, where $1 < p < p_0 < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{p_0}$.*

Now we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1 By the homogeneity, we may assume that $\|b\|_{\mathrm{Lip}(\beta)} = 1$.

(I)⇒(II) Without loss of generality, we may assume that $\|f\|_{L^1(\mu)} = 1$.

It is easy to see that the conclusion (II) holds if $\lambda \le \frac{2^{d+1}\|f\|_{L^1(\mu)}}{\|\mu\|}$ when $\|\mu\| < \infty$. Then we assume that $\lambda > \frac{2^{d+1}\|f\|_{L^1(\mu)}}{\|\mu\|}$ if $\|\mu\| < \infty$. For f and any fixed $\lambda > \frac{2^{d+1}\|f\|_{L^1(\mu)}}{\|\mu\|}$, applying Lemma 2.1 with λ replaced by λ^{q_0} with $q_0 = \frac{n}{n-\beta}$, we obtain that with the same notation as in Lemma 2.1, $f = g + h$, where

$$g(x) = f(x)\chi_{\mathbb{R}^d\setminus\cup_j Q_j}(x) + \sum_j \varphi_j(x),$$

$$h(x) = f(x) - g(x) = \sum_j [w_j(x)f(x) - \varphi_j(x)] = \sum_j h_j(x).$$

By Lemma 2.1, we can obtain the following properties:

(A) $\displaystyle \frac{1}{\mu(2Q_j)}\int_{Q_j} |f(x)|\,d\mu(x) > \frac{\lambda^{q_0}}{2^{d+1}}$;

(B) $\displaystyle |f(x)| \le \lambda^{q_0},\ \ \mu\text{-a.e. } x \in \mathbb{R}^d\Big\backslash \bigcup_j Q_j$;

(C) $\displaystyle \int_{R_j} \varphi_j(x)\,d\mu(x) = \int_{Q_j} f(x)w_j(x)\,d\mu(x)$;

(D) $\displaystyle \|\varphi_j\|_{L^\infty(\mu)}\mu(R_j) \lesssim \int_{Q_j} |f(x)|\,d\mu(x)$;

(E) $\displaystyle \sum_j |\varphi_j(x)| \lesssim \lambda^{q_0}$.

By (B) and (D), we easily obtain

$$\|g\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)} \lesssim 1. \tag{2.1}$$

From (B) and (E), it follows that for μ-a. e. $x \in \mathbb{R}^d$,

$$|g(x)| \lesssim \lambda^{q_0}. \tag{2.2}$$

Choose $1 < p_1 < \frac{n}{\beta}$ and $\frac{1}{q_1} = \frac{1}{p_1} - \frac{\beta}{n}$. The boundedness of T_b from $L^{p_1}(\mu)$ into $L^{q_1}(\mu)$, (2.1) and (2.2) give us that

$$\begin{aligned}\mu\left(\left\{x \in \mathbb{R}^d : |T_b g(x)| > \lambda\right\}\right) &\lesssim \lambda^{-q_1}\int_{\mathbb{R}^d} |T_b g(x)|^{q_1}\,d\mu(x) \lesssim \lambda^{-q_1}\|g\|_{L^{p_1}(\mu)}^{q_1}\\ &\lesssim \lambda^{-q_1}\lambda^{q_0(p_1-1)q_1/p_1}\|f\|_{L^1(\mu)}^{q_1/p_1} \lesssim \lambda^{-q_0}.\end{aligned} \tag{2.3}$$

The facts (A) and $\sum_j \chi_{Q_j}(x) \lesssim 1$ tell us that

$$\mu\Big(\bigcup_j 2Q_j\Big) \lesssim \lambda^{-q_0}\int_{\mathbb{R}^d} |f(y)|\,d\mu(y) \lesssim \lambda^{-q_0}. \tag{2.4}$$

Noting that $f = g + h$, from (2.3) and (2.4), we deduce that the proof of (II) can be reduced to proving that

$$\mu\Big(\Big\{x \in \mathbb{R}^d\Big\backslash \bigcup_j 2Q_j : |T_b h(x)| > \lambda\Big\}\Big) \lesssim \lambda^{-q_0}. \tag{2.5}$$

Let θ be a bounded function satisfying $\|\theta\|_{L^{q_0'}(\mu)} \le 1$ and $\operatorname{supp}\theta \subset \mathbb{R}^d \Big\backslash \bigcup_j 2Q_j$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d \backslash \bigcup_j 2Q_j} |T_b h(x)\theta(x)|\, d\mu \\
&\le \sum_j \int_{\mathbb{R}^d\backslash 2R_j} |T_b h_j(x)\theta(x)|\; d\mu(x) + \sum_j \int_{2R_j\backslash 2Q_j} |T_b h_j(x)\theta(x)|\; d\mu(x) \\
&= \mathrm{F}_1 + \mathrm{F}_2.
\end{aligned}$$

Recall that $h_j = w_j f - \varphi_j$. This together with (C) gives us that

$$\int_{\mathbb{R}^d} h_j(x)\, d\mu(x) = 0.$$

By this fact, (1.2)–(1.4) and the Hölder inequality, we have

$$\begin{aligned}
\mathrm{F}_1 &\le \sum_j \int_{\mathbb{R}^d\backslash 2R_j} \int_{\mathbb{R}^d} |\theta(x)||[b(x)-b(y)]K(x,\, y) - [b(x)-b(x_{R_j})]K(x,\, x_{R_j})| \\
&\quad \times |h_j(y)|\, d\mu(y)\, d\mu(x) \\
&\le \sum_j \int_{\mathbb{R}^d\backslash 2R_j} \int_{\mathbb{R}^d} |\theta(x)||[b(x)-b(y)][K(x,\, y) - K(x,\, x_{R_j})]||h_j(y)|\, d\mu(y)\, d\mu(x) \\
&\quad + \sum_j \int_{\mathbb{R}^d\backslash 2R_j} \int_{\mathbb{R}^d} |\theta(x)||[b(x_{R_j}) - b(y)]K(x,\, x_{R_j})||h_j(y)|\, d\mu(y)\, d\mu(x) \\
&\lesssim \sum_j \int_{\mathbb{R}^d} |h_j(y)|\, d\mu(y) \sum_{i=1}^{\infty} \int_{2^{i+1}R_j\backslash 2^iR_j} \frac{l(R_j)^\delta}{l(2^iR_j)^{n+\delta-\beta}} |\theta(x)|\, d\mu(x) \\
&\quad + \sum_j \int_{\mathbb{R}^d} |h_j(y)|\, d\mu(y) \sum_{i=1}^{\infty} \int_{2^{i+1}R_j\backslash 2^iR_j} \frac{l(R_j)^\beta}{l(2^iR_j)^{n}} |\theta(x)|\, d\mu(x) \\
&\lesssim \|\theta\|_{L^{q_0'}(\mu)} \sum_j \int_{Q_j} |f(y)|\, d\mu(y) \Big[\sum_{i=1}^{\infty} 2^{-i\delta} + \sum_{i=1}^{\infty} 2^{-i\beta}\Big] \\
&\lesssim 1.
\end{aligned}$$

On the other hand, (1.2), (1.3), the Hölder inequality and (1.1) lead to

$$\begin{aligned}
\mathrm{F}_2 &\le \sum_j \int_{2R_j\backslash 2Q_j} |\theta(x)||T_b(w_jf)(x)|\, d\mu(x) + \sum_j \int_{2R_j} |\theta(x)|\, |T_b\varphi_j(x)|\; d\mu(x) \\
&\lesssim \sum_j \int_{2R_j\backslash 2Q_j} \frac{|\theta(x)|}{|x-x_{Q_j}|^{n-\beta}}\, d\mu(x) \int_{Q_j} |f(y)|\, d\mu(y) + \sum_j \Big\{ \int_{2R_j} |T_b\varphi_j(x)|^{q_0}\, d\mu(x)\Big\}^{1/q_0} \\
&\lesssim \sum_j \int_{Q_j} |f(y)|\, d\mu(y) \Big\{ \sum_{i=1}^{N_{2Q_j,\,2R_j}} \int_{2^{i+1}Q_j\backslash 2^iQ_j} \frac{1}{|x-x_{Q_j}|^{(n-\beta)q_0}}\, d\mu(x)\Big\}^{1/q_0} \\
&\quad + \sum_j \Big\{ \int_{2R_j} |T_b\varphi_j(x)|^{q_2}\, d\mu(x)\Big\}^{1/q_2} [\mu(2R_j)]^{1/q_0 - 1/q_2}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_j \int_{Q_j} |f(y)|\,d\mu(y)[K_{2Q_j,\,2R_j}]^{1/q_0} + \sum_j \|\varphi_j\|_{L^{p_2}(\mu)}[\mu(2R_j)]^{1/q_0-1/q_2}\\
&\lesssim \sum_j \int_{Q_j} |f(y)|\,d\mu(y) + \sum_j \|\varphi_j\|_{L^\infty(\mu)}\mu(2R_j)\\
&\lesssim 1,
\end{aligned}$$

where we have chosen p_2 and q_2 such that $1 < p_2 < \frac{n}{\beta}$ and $\frac{1}{q_2} = \frac{1}{p_2} - \frac{\beta}{n}$. And we have also used the following simply fact that

$$[K_{2Q_j,\,2R_j}]^{1/q_0} \le K_{2Q_j,\,2R_j} \lesssim 1.$$

The estimates for F_1 and F_2 indicate (2.5) and this finishes the proof of (I) $\Rightarrow$ (II).

(II)$\Rightarrow$(III) For any cube Q, let

$$h_Q = m_Q(T_b[f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}]).$$

To prove $T_b f \in \mathrm{RBMO}(\mu)$, we only need to verify that for any cube Q,

$$\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b f(x) - h_Q|\ d\mu(x) \lesssim \|f\|_{L^{n/\beta}(\mu)}, \tag{2.6}$$

and for any cubes $Q \subset R$,

$$|h_Q - h_R| \lesssim K_{Q,\,R}\|f\|_{L^{n/\beta}(\mu)}. \tag{2.7}$$

In fact, by (2.6), it is easy to see that if Q is doubling, then

$$|m_Q(T_b f) - h_Q| \lesssim \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b f(x) - h_Q|\ d\mu(x) \lesssim \|f\|_{L^{n/\beta}(\mu)}. \tag{2.8}$$

Moveover, for any cube Q, $K_{Q,\widetilde{Q}} \lesssim 1$, and then by (2.6), (2.7) and (2.8), we obtain that

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b f(x) - m_{\widetilde{Q}}(T_b f)|\,d\mu(x)\\
&\le \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b f(x) - h_Q|\,d\mu(x) + |h_Q - h_{\widetilde{Q}}| + |m_{\widetilde{Q}}(T_b f) - h_{\widetilde{Q}}|\\
&\lesssim \|f\|_{L^{n/\beta}(\mu)}.
\end{aligned} \tag{2.9}$$

On the other hand, for any doubling cubes $Q \subset R$, from (2.7) and (2.8), it follows that

$$|m_Q(T_b f) - m_R(T_b f)| \le |m_Q(T_b f) - h_Q| + |h_Q - h_R| + |h_R - m_R(T_b f)| \lesssim \|f\|_{L^{n/\beta}(\mu)},$$

which together with (2.9) indicates that $T_b f \in \mathrm{RBMO}(\mu)$ and

$$\|T_b f\|_* \lesssim \|f\|_{L^{n/\beta}(\mu)}.$$

Now we verify (2.6). Decompose

$$\begin{aligned}
&\frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b f(x) - h_Q|\,d\mu(x)\\
&\le \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b(f\chi_{\frac{4}{3}Q})(x)|\,d\mu(x) + \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |T_b(f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q})(x) - h_Q|\,d\mu(x)\\
&= \mathrm{H} + \mathrm{I}.
\end{aligned}$$

From the Kolmogorov inequality that for $0 < p < q$ and any function $f \geq 0$,

$$\|f\|_{L^{q,\,\infty}(\mu)} \leq \sup_E \frac{\|f\chi_E\|_{L^p(\mu)}}{\|\chi_E\|_{L^s(\mu)}} \lesssim \|f\|_{L^{q,\,\infty}(\mu)},$$

where $L^{q,\,\infty}(\mu)$ is just weak $L^q(\mu)$, $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, and the supremum is taken for all measurable sets E with $0 < \mu(E) < \infty$ (see [3, p. 485]), and the condition (II) of Theorem 1.1, it follows that

$$\mathrm{H} \lesssim \frac{1}{\mu(\frac{3}{2}Q)}\|\chi_Q\|_{L^{n/\beta}(\mu)}\|T_b(f\chi_{\frac{4}{3}Q})\|_{L^{q_0,\,\infty}(\mu)} \lesssim \frac{[\mu(Q)]^{\beta/n}}{\mu(\frac{3}{2}Q)}\|f\chi_{\frac{4}{3}Q}\|_{L^1(\mu)} \lesssim \|f\|_{L^{n/\beta}(\mu)}.$$

To estimate I, by (1.2)–(1.4), the Hölder inequality and (1.1), we first have that for any $x, y \in Q$,

$$\begin{aligned}
&|T_b(f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q})(x) - T_b(f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q})(y)| \\
&\leq \int_{\mathbb{R}^d\setminus\frac{4}{3}Q} |[b(x)-b(z)]K(x,\,z) - [b(y)-b(z)]K(y,\,z)||f(z)|\,d\mu(z) \\
&\leq \int_{\mathbb{R}^d\setminus\frac{4}{3}Q} |[b(x)-b(z)][K(x,\,z) - K(y,\,z)]||f(z)|\,d\mu(z) \\
&\quad + \int_{\mathbb{R}^d\setminus\frac{4}{3}Q} |[b(x)-b(y)]||K(y,\,z)||f(z)|\,d\mu(z) \\
&\lesssim \sum_{i=1}^{\infty}\int_{2^i\frac{4}{3}Q\setminus 2^{i-1}\frac{4}{3}Q} \frac{|x-y|^{\delta}}{|x-z|^{n+\delta-\beta}}|f(z)|\,d\mu(z) + \sum_{i=1}^{\infty}\int_{2^i\frac{4}{3}Q\setminus 2^{i-1}\frac{4}{3}Q} \frac{|x-y|^{\beta}}{|y-z|^{n}}|f(z)|\,d\mu(z) \\
&\lesssim \|f\|_{L^{n/\beta}(\mu)}\Big\{\sum_{i=1}^{\infty}\frac{2^{-i\delta}}{l(2^i\frac{4}{3}Q)^{n-\beta}}\Big[\mu\Big(2^i\frac{4}{3}Q\Big)\Big]^{1-\beta/n} + \sum_{i=1}^{\infty}\frac{l(Q)^{\beta}}{l(2^i\frac{4}{3}Q)^{n}}\Big[\mu\Big(2^i\frac{4}{3}Q\Big)\Big]^{1-\beta/n}\Big\} \\
&\lesssim \|f\|_{L^{n/\beta}(\mu)}.
\end{aligned}$$

Therefore,

$$\mathrm{I} \lesssim \|f\|_{L^{n/\beta}(\mu)}.$$

The estimates for H and I lead to (2.6) immediately.

Now we check (2.7) for chosen $\{h_Q\}_Q$. Let $N_1 = N_{Q,\,R} + 1$. Write

$$\begin{aligned}
|h_Q - h_R| &= |m_Q(T_b[f\chi_{\mathbb{R}^d\setminus\frac{4}{3}Q}]) - m_R(T_b[f\chi_{\mathbb{R}^d\setminus\frac{4}{3}R}])| \\
&\leq |m_Q(T_b[f\chi_{2Q\setminus\frac{4}{3}Q}])| + |m_Q(T_b[f\chi_{2^{N_1}Q\setminus 2Q}])| + |m_R(T_b[f\chi_{2^{N_1}Q\setminus\frac{4}{3}R}])| \\
&\quad + |m_Q(T_b[f\chi_{\mathbb{R}^d\setminus 2^{N_1}Q}]) - m_R(T_b[f\chi_{\mathbb{R}^d\setminus 2^{N_1}Q}])| \\
&= \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4.
\end{aligned}$$

An argument similar to the estimate for H tells us that

$$\mathrm{J}_1 \lesssim \|f\|_{L^{n/\beta}(\mu)} \quad \text{and} \quad \mathrm{J}_3 \lesssim \|f\|_{L^{n/\beta}(\mu)}.$$

Some calculations completely similar to the estimate for I lead to

$$\mathrm{J}_4 \lesssim \|f\|_{L^{n/\beta}(\mu)}.$$

Finally, we estimate J_2. By (1.2), (1.3) and the Hölder inequality, we obtain that for any $x \in Q$,

$$|T_b(f\chi_{2^{N_1}Q\setminus 2Q})(x)| \lesssim \Big\{ \sum_{i=1}^{N_1-1} \int_{2^{i+1}Q\setminus 2^iQ} \frac{1}{|x-z|^{(n-\beta)q_0}}\, d\mu(z) \Big\}^{1/q_0} \|f\|_{L^{n/\beta}(\mu)}$$
$$\lesssim \Big\{ \sum_{i=1}^{N_1-1} \frac{\mu(2^{i+1}Q)}{l(2^{i+1}Q)^n} \Big\}^{1/q_0} \|f\|_{L^{n/\beta}(\mu)} \lesssim K_{Q,\,R}\|f\|_{L^{n/\beta}(\mu)}.$$

Then

$$\mathrm{J}_2 \lesssim K_{Q,\,R}\|f\|_{L^{n/\beta}(\mu)}.$$

The estimates for J_1, J_2, J_3 and J_4 yield (2.7) and thus this completes the proof of (II)⇒(III).

(III)⇒(IV) We first verify that for any cube Q and any bounded function a supported on Q,

$$\int_Q |T_b a(x)|^{q_0}\, d\mu(x) \lesssim \|a\|^{q_0}_{L^\infty(\mu)}\, [\mu(2Q)]^{q_0}\,. \tag{2.10}$$

We consider the following two cases.

Case I $l(Q) \le \frac{\mathrm{diam}(\,\mathrm{supp}\,(\mu))}{20}$. By the condition (III) of Theorem 1.1 and [12, Corollary 3.5], we have

$$\int_Q |T_b a(x) - m_{\widetilde{Q}}(T_b a)|^{q_0}\, d\mu(x) \lesssim \|a\|^{q_0}_{L^{n/\beta}(\mu)}\mu(2Q) \lesssim \|a\|^{q_0}_{L^\infty(\mu)}[\mu(2Q)]^{q_0}.$$

Thus, to prove (2.10), it suffices to verify

$$|m_{\widetilde{Q}}(T_b a)| \lesssim \|a\|_{L^\infty(\mu)}[\mu(2Q)]^{\beta/n}. \tag{2.11}$$

Let $x_0 \in \mathrm{supp}\,(\mu)$ be the point (or one of the points) in $\mathbb{R}^d \setminus (5Q)^\circ$ which is closest to Q, where $(5Q)^\circ$ is the set of all interior points of $5Q$. We denote $\mathrm{dist}(x_0,\, Q)$ by d_0. Assume that x_0 is a point such that some cube with side length $2^{-i}d_0$ and centered at x_0, $i \ge 2$, is doubling. Otherwise, we choose y_0 in $\mathrm{supp}\,(\mu) \cap B\big(x_0,\, \frac{l(Q)}{100}\big)$ such that this is true for y_0, and we interchange x_0 with y_0 (see [12, pp. 136–137]). We denote by R a cube concentric with Q with side length $\max\{10d_0,\, l(\widetilde{Q})\}$. It is easy to check $K_{\widetilde{Q},\,R} \lesssim 1$. Let Q_0 be the biggest doubling cube centered at x_0 with side length $2^{-i}d_0$, $i \ge 2$. Then $Q_0 \subset R$ with $K_{Q_0,\,R} \lesssim 1$, and it is easy to check that

$$|m_{Q_0}(T_b a) - m_{\widetilde{Q}}(T_b a)| \lesssim \|T_b a\|_* \lesssim \|a\|_{L^{n/\beta}(\mu)} \lesssim \|a\|_{L^\infty(\mu)}[\mu(Q)]^{\beta/n}. \tag{2.12}$$

Note that $\mathrm{dist}(Q_0,\, Q) \sim d_0$ and $l(Q) < d_0$. This together with (1.2) and (1.3) tells us that for $y \in Q_0$,

$$|T_b a(y)| \lesssim \frac{\mu(Q)}{d_0^{n-\beta}}\|a\|_{L^\infty(\mu)} \lesssim [\mu(Q)]^{\beta/n}\|a\|_{L^\infty(\mu)}.$$

Therefore,

$$|m_{Q_0}(T_b a)| \lesssim [\mu(Q)]^{\beta/n}\|a\|_{L^\infty(\mu)}. \tag{2.13}$$

The estimates (2.12) and (2.13) lead to (2.11) in this case.

Case II $l(Q) > \frac{\operatorname{diam}(\operatorname{supp}(\mu))}{20}$. We may assume Q is centered at some point of $\operatorname{supp}(\mu)$ and $l(Q) \leq 4\operatorname{diam}(\operatorname{supp}(\mu))$. Then $Q \cap \operatorname{supp}(\mu)$ can be covered by a finite number of cubes, $\{Q_j\}_{j=1}^J$, centered at points of $\operatorname{supp}(\mu)$ with side length $\frac{l(Q)}{200}$. It is quite easy to check that J only depends on d. We set

$$a_j = \frac{\chi_{Q_j}}{\sum\limits_{i=1}^{J} \chi_{Q_i}} a.$$

Since (2.11) is true, if we replace Q by $2Q_j$ which contains the support of a_j, by (1.2) and (1.3), we have

$$\begin{aligned}
\int_Q |T_b a(x)|^{q_0}\, d\mu(x) &\lesssim \sum_{j=1}^{J} \int_{Q \setminus 2Q_j} |T_b a_j(x)|^{q_0}\, d\mu(x) + \sum_{j=1}^{J} \int_{2Q_j} |T_b a_j(x)|^{q_0}\, d\mu(x) \\
&\lesssim \sum_{j=1}^{J} \int_{Q \setminus 2Q_j} \Big[\int_{Q_j} \frac{|a_j(y)|}{|x-y|^{n-\beta}}\, d\mu(y) \Big]^{q_0} d\mu(x) + \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} [\mu(4Q_j)]^{q_0} \\
&\lesssim \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} \frac{[\mu(Q_j)]^{q_0}}{l(Q_j)^n} \mu(Q) + \sum_{j=1}^{J} \|a_j\|_{L^\infty(\mu)}^{q_0} [\mu(4Q_j)]^{q_0} \\
&\lesssim J \|a\|_{L^\infty(\mu)}^{q_0} \mu(2Q)^{q_0}.
\end{aligned}$$

Thus (2.11) also holds in this case.

To prove (IV), by the standard argument, it is enough to verify that

$$\|T_b h\|_{L^{q_0}(\mu)} \lesssim |h|_{H^{1,\infty}_{atb}(\mu)} \tag{2.14}$$

for any atomic block h with $\operatorname{supp}(h) \subset R$, $h = \sum\limits_{j=1}^{2} \lambda_j a_j$, where the a_j's are functions as in Definition 2.1 satisfying the following size condition that

$$\|a_j\|_{L^\infty(\mu)} \leq [\mu(4Q_j)]^{-1} K_{Q_j, R}^{-1}. \tag{2.15}$$

Write

$$\int_{\mathbb{R}^d} |T_b h(x)|^{q_0}\, d\mu(x) = \int_{2R} |T_b h(x)|^{q_0}\, d\mu(x) + \int_{\mathbb{R}^d \setminus 2R} |T_b h(x)|^{q_0}\, d\mu(x) = \mathrm{L}_1 + \mathrm{L}_2.$$

To estimate L_1, further decompose

$$\mathrm{L}_1 \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \int_{2Q_j} |T_b a_j(x)|^{q_0}\, d\mu(x) + \sum_{j=1}^{2} |\lambda_j|^{q_0} \int_{2R \setminus 2Q_j} |T_b a_j(x)|^{q_0}\, d\mu(x) = \mathrm{L}_{11} + \mathrm{L}_{12}.$$

From (2.10) and (2.15), it follows that

$$\mathrm{L}_{11} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0} \|a_j\|_{L^\infty(\mu)}^{q_0} [\mu(4Q_j)]^{q_0} \lesssim \sum_{j=1}^{2} |\lambda_j|^{q_0}.$$

For L_{12}, by (1.2), (1.3) and (2.15), we have

$$\begin{aligned}
L_{12} &\lesssim \sum_{j=1}^{2}|\lambda_j|^{q_0}\sum_{i=1}^{N_{Q_j,R}}\int_{2^{i+1}Q_j\setminus 2^iQ_j}\left\{\int_{Q_j}\frac{|[b(x)-b(y)]|}{|x-y|^n}|a_j(y)|\,d\mu(y)\right\}^{q_0}d\mu(x)\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|^{q_0}\sum_{i=1}^{N_{Q_j,R}}\int_{2^{i+1}Q_j\setminus 2^iQ_j}\left\{\int_{Q_j}\frac{|a_j(y)|}{|x-y|^{n-\beta}}\,d\mu(y)\right\}^{q_0}d\mu(x)\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|^{q_0}\sum_{i=1}^{N_{Q_j,R}}\frac{\mu(2^{i+1}Q_j)}{l(2^{i+1}Q_j)^{(n-\beta)q_0}}\|a_j\|_{L^\infty(\mu)}^{q_0}[\mu(Q_j)]^{q_0}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|^{q_0}K_{Q_j,R}\|a_j\|_{L^\infty(\mu)}^{q_0}[\mu(Q_j)]^{q_0}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|^{q_0}.
\end{aligned}$$

The estimates for L_{11} and L_{12} tell us that

$$L_1 \lesssim |h|_{H_{atb}^{1,\infty}(\mu)}^{q_0}.$$

On the other hand, from the fact $\int_{\mathbb{R}^d} h\,d\mu = 0$, (1.2), (1.3) and (1.4), it follows that

$$\begin{aligned}
L_2 &\lesssim \sum_{k=1}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\left|[b(x)-m_R(b)]\int_R\Big[K(x,y)-K(x,x_R)\Big]h(y)\,d\mu(y)\right|^{q_0}d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\left|\int_R[m_R(b)-b(y)]K(x,y)h(y)\,d\mu(y)\right|^{q_0}d\mu(x)\\
&\lesssim \sum_{k=1}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\left|l(2^kR)^{\beta}\int_R\frac{|y-x_R|^{\delta}}{|x-y|^{n+\delta}}\Big(\sum_{i=1}^{2}|\lambda_i||a_i(y)|\Big)\,d\mu(y)\right|^{q_0}d\mu(x)\\
&\quad+\sum_{k=1}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\left|\frac{l(R)^{\beta}}{l(2^kR)^n}\int_R\Big(\sum_{i=1}^{2}|\lambda_i||a_i(y)|\Big)\,d\mu(y)\right|^{q_0}d\mu(x)\\
&\lesssim \Big(\sum_{i=1}^{2}|\lambda_i|\Big)^{q_0}\sum_{k=1}^{\infty}l(2^kR)^{q_0(\beta-n-\delta)}l(R)^{\delta q_0}\mu(2^{k+1}R)\\
&\quad+\Big(\sum_{i=1}^{2}|\lambda_i|\Big)^{q_0}\sum_{k=1}^{\infty}l(2^kR)^{-q_0n}l(R)^{\beta q_0}\mu(2^{k+1}R)\\
&\lesssim \Big(\sum_{i=1}^{2}|\lambda_i|\Big)^{q_0}.
\end{aligned}$$

Combining the estimates for L_1 and L_2 yields (2.14) and this completes the proof of (III)⇒(IV).

(IV)⇒(I) First we claim that for any cube Q and any function $f\in L^1(\mu)$ with $\operatorname{supp}(f)\subset \frac{4}{3}Q$ and any $x\in Q$,

$$\frac{1}{\mu(\frac{3}{2}Q)}\int_Q|T_bf(y)|\,d\mu(y)\lesssim \|f\|_{L^{n/\beta}(\mu)}. \tag{2.16}$$

We also consider two cases.

Case A $l(Q) \leq \frac{\operatorname{diam}(\operatorname{supp}(\mu))}{20}$. We consider the same construction as the one in (III)⇒(IV). Let Q, Q_0 and R be the same as there. We have known that $Q,\, Q_0 \subset R$, $K_{Q,\,R} \lesssim 1$, $K_{Q_0,\,R} \lesssim 1$ and $\operatorname{dist}(Q_0,\, Q) \geq l(Q)$. Recall also that Q_0 is doubling.

Let

$$g = f + C_{Q_0}\chi_{Q_0},$$

where C_{Q_0} is a constant such that $\int_{\mathbb{R}^d} g\, d\mu = 0$. Then g is an atomic block supported in R. It is easy to check

$$\|g\|_{H^{1,\,n/\beta}_{atb}(\mu)} \lesssim \Big[\mu\Big(\frac{3}{2}Q\Big)\Big]^{1/q_0} \|f\|_{L^{n/\beta}(\mu)}.$$

This and the fact that $H^{1,\,n/\beta}_{atb}(\mu) = H^1(\mu)$ imply that

$$\|g\|_{H^1(\mu)} \lesssim \Big[\mu\Big(\frac{3}{2}Q\Big)\Big]^{1/q_0} \|f\|_{L^{n/\beta}(\mu)}. \tag{2.17}$$

For $y \in Q$, we have

$$|T_b\left(C_{Q_0}\chi_{Q_0}\right)(y)| \lesssim \frac{|C_{Q_0}|\mu(Q_0)}{\operatorname{dist}(Q,\, Q_0)^{n-\beta}} \lesssim \|f\|_{L^{n/\beta}(\mu)}. \tag{2.18}$$

Then by the Hölder inequality, the condition (V) of Theorem 1.1 and (2.17), we obtain that

$$\begin{aligned}\int_Q |T_b g(y)|\, d\mu(y) &= \Big\{\int_Q |T_b g(y)|^{q_0}\, d\mu(y)\Big\}^{1/q_0} \mu(Q)^{1-1/q_0}\\ &\lesssim \mu(Q)^{1-1/q_0}\|g\|_{H^1(\mu)} \lesssim \mu\Big(\frac{3}{2}Q\Big)\|f\|_{L^{n/\beta}(\mu)}.\end{aligned} \tag{2.19}$$

The estimates (2.18) and (2.19) indicate (2.16).

Case B $l(Q) > \frac{\operatorname{diam}(\operatorname{supp}(\mu))}{20}$. By an argument similar to the proof of (2.10) in the case of $l(Q) > \frac{\operatorname{diam}(\operatorname{supp}(\mu))}{20}$, we can prove that (2.16) also holds.

Now we turn to prove (I). By Lemma 2.2, we only need to verify that T_b is bounded from $L^{n/\beta}(\mu)$ into RBMO(μ). Repeating the proof of (2.6) and (2.7) step by step with replacing the weak $(L^1(\mu),\, L^{n/(n-\beta)}(\mu))$ type estimate of T_b by (2.16) when estimating H, we can prove that T_b is bounded from $L^{n/\beta}(\mu)$ into RBMO(μ). This finishes the proof of (V)⇒(I) and, therefore, the proof of Theorem 1.1.

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