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Bifurcation of Homoclinic Orbits with Saddle-Center Equilibrium**

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Abstract In this paper, the authors develop new global perturbation techniques for detecting the persistence of transversal homoclinic orbits in a more general nondegenerated system with action-angle variable. The unperturbed system is assumed to have saddle-center type equilibrium whose stable and unstable manifolds intersect in one dimensional manifold, and does not have to be completely integrable or near-integrable. By constructing local coordinate systems near the unperturbed homoclinic orbit, the conditions of existence of transversal homoclinic orbit are obtained, and the existence of periodic orbits bifurcated from homoclinic orbit is also considered.

Keywords Local coordinate system, Homoclinic orbit, Bifurcation
2000 MR Subject Classification 34C23, 34C37, 37C29

1 Introduction

In this paper we study the following singular perturbation system with action-angle variable

$$
\begin{aligned}
&\dot{z} = f(z, I) + \varepsilon g^z(z, I, \theta, \lambda, \varepsilon),\\
&\dot{I} = \varepsilon g^I(z, I, \theta, \lambda, \varepsilon),\\
&\dot{\theta} = \omega,
\end{aligned}
\tag{1.1}
$$

where $(z, I, \theta) \in R^n \times R^m \times T^l$, $\lambda \in R^k$, $0 \le \varepsilon \ll 1$, $|\lambda| \ll 1$, and g^z, g^I are 2π periodic in each component of their l dimensional θ variable. The existence and transversality of homoclinic orbits of the above systems have extensively been studied in recent years (see [1–6] and the references theirin), where they use geometrical singular perturbation theory and the theory of invariant manifolds to get conditions for the existence of transversal homoclinic orbit of completely or near-integrable Hamilton system. In this paper we develop a different method to solve the same problem for more general system (1.1). First we make a suitable C^r $(r \ge 3)$ transformation in a small neighborhood U of equilibrium to flatten the local stable , unstable and center manifolds, so that system has a simple normal form near the equilibrium, and moreover we can get the expression of its solutions near the equilibrium using Silnikov coordinate variables. Then we establish the local coordinate system in a small tangular neighborhood of unperturbed homoclinic orbit. Thus we can construct Poincaré map induced by system solutions which can be expressed as an identical transformation summing a Melnikov function

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approximately. Using the Melnikov function we give some sufficient conditions to guarantee the existence of transversal homoclinic orbit and the existence of periodic orbit bifurcated from homoclinic orbit. The main method used in this paper is initially employed in [7, 8], and then extended to the study of homoclinic bifurcation in fast variable space by [9].

2 The Geometrical Structure of Unperturbed and Perturbed System

Consider the C^r ($r \geq 3$) system (1.1) and the corresponding unperturbed system

$$\dot{z} = f(z, I), \tag{2.1a}$$
$$\dot{I} = 0, \tag{2.1b}$$
$$\dot{\theta} = \omega. \tag{2.1c}$$

We make the following assumptions:

(H_1) There exists $I_0 \in R^m$, such that system (2.1a) has a hyperbolic equilibrium $z_0 = z(I_0)$ and a homoclinic orbit $\Gamma = \{\hat{r}(t) \mid \hat{r}(\pm\infty) = z_0, t \in R\}$. The unstable manifold W^u and stable manifold W^s of z_0 are n_1-dimensional and n_2-dimensional, respectively, with $n_1 + n_2 = n$. Moreover the linearization $Df_z(z_0, I_0)$ at the equilibrium z_0 has simple real eigenvalues λ_1, $-\lambda_2$ such that the remaining eigenvalues of $Df_z(z_0, I_0)$ satisfy either $\operatorname{Re}\lambda > a > \lambda_1 > 0$, or $\operatorname{Re}\lambda < -b < -\lambda_2 < 0$ for some positive numbers a, b. For any $p \in \Gamma$,

(H_2)

$$\begin{aligned}
&\dim(W^u \cap W^s) = \dim(T_p W^u \cap T_p W^s) = 1.\\
&\operatorname{Span}\{T_{\hat{r}(t)}W^u, T_{\hat{r}(t)}W^s, e^-\} = R^n, \quad t \geq T \gg 1,\\
&\operatorname{Span}\{T_{\hat{r}(t)}W^u, T_{\hat{r}(t)}W^s, e^+\} = R^n, \quad t \leq -T \ll -1, \cdots
\end{aligned}$$

where $e^{\pm} = \lim\limits_{t\to\pm\infty} \frac{\dot{\hat{r}}(t)}{|\dot{\hat{r}}(t)|}$, $e^+ \in T_{x_0}W^s$ and $e^- \in T_{x_0}W^u$ are unit eigenvectors corresponding to $-\lambda_2$ and λ_1, respectively.

Hypothesis (H_2) is equivalent to the following strong inclination property:

$$\begin{aligned}
&T_{\hat{r}(t)}W^u \to T_{x_0}W^{uu} \oplus e^+, \quad t \to +\infty,\\
&T_{\hat{r}(t)}W^s \to T_{x_0}W^{ss} \oplus e^-, \quad t \to -\infty.
\end{aligned}$$

From the assumption (H_1), it is easy to see system (2.1) possesses an l dimensional invariant torus

$$M = \{(z, I, \theta) \mid z = z_0, I = I_0, \theta \in T^l\}$$

and a homoclinic manifold

$$\widetilde{\Gamma}_0 = \{(x, I, \theta) \mid x = \hat{r}(t), I = I_0, \theta = \theta_0 + \omega t, \theta_0 \in T^l, t \in R\},$$

$\widetilde{\Gamma}_0 \to M$ as $t \to \pm\infty$. Now we consider the bifurcations of homoclinic manifold $\widetilde{\Gamma}_0$ under small perturbations. For convenience, we first restrict system (1.1) into the $z - I$ space with $\theta = \theta_0 + \omega t$, and regard θ_0 as a parameter.

Make the transformation

$$z \to z + z_0, \quad I \to I + I_0,$$

so that system (1.1) is changed into

$$\begin{aligned} \dot z &= \tilde f(z,I) + \varepsilon \tilde g^z(z,I,\theta,\lambda,\varepsilon), \\ \dot I &= \varepsilon \tilde g^I(z,I,\theta,\lambda,\varepsilon). \end{aligned} \tag{2.2}$$

with $\tilde f(0,0)=0$, $\theta=\theta_0+\omega t$. We need a further assumption for (2.2):

(H$_3$) $\tilde g^z(0,0,\theta,\lambda,\varepsilon)=0$, $\tilde g^I(0,0,\theta,\lambda,\varepsilon)=0$, $\mathrm{Re}\,(\sigma(D_I\tilde g^I(0,0,\theta,\lambda,0)))\neq 0$,

$$\tilde g^I(z,I,\theta,\lambda,\varepsilon)=\tilde g_1^I(z,I)+\varepsilon\tilde g_2^I(z,I,\theta,\lambda,\varepsilon).$$

Based on center manifold theorem, system (2.2) has a local C^r center manifold $W^c_{\mathrm{loc}}: X = X^c(I,\theta,\lambda,\varepsilon)$ satisfying

$$X^c(0,0,0,0)=0, \quad X_I^c(0,0,0,0)=-\tilde f_X^{-1}(0,0)\tilde f_I(0,0).$$

Under the hypothesis (H_3), the autonomous part of system (2.2) is

$$\dot z=\tilde f(z,I), \quad \dot I=\varepsilon\tilde g_1^I(z,I). \tag{2.3}$$

The hyperbolicity of $\tilde g_1^I$ suggests that we can decompose I space into $I=(I_1,I_2)\in R^{m_1}\times R^{m_2}$, $m_1+m_2=m$, $D_I\tilde g^I(0,0,\theta,\lambda,0)=\mathrm{diag}(C_1(\lambda),-C_2(\lambda))$, where $\mathrm{Re}\,(\sigma(C_1(\lambda)))>0$, $\mathrm{Re}\,(\sigma(-C_2(\lambda)))<0$. Then the second equation of system (2.2) can be decomposed into

$$\begin{aligned} \dot I_1 &= \varepsilon g_{11}(z,I)+\varepsilon^2 g_{12}(z,I,\theta,\lambda,\varepsilon), \\ \dot I_2 &= \varepsilon g_{21}(z,I)+\varepsilon^2 g_{22}(z,I,\theta,\lambda,\varepsilon). \end{aligned} \tag{2.4}$$

In the following, we use $(z,I)=(x,y,I)$ to denote the variables belonging to the unstable, stable and center subspace, respectively, for the autonomous system (2.3). Taking a neighborhood U_0 of the origin small enough, then up to a linear transformation (see [9]), we can flatten the stable, unstable and center manifolds of (2.3) so that system (2.2) becomes the following C^{r-1} system

$$\begin{aligned} \dot x_1 &= f_{11}(x,y,I,\theta,\lambda,\varepsilon), \quad & \dot y_1 &= f_{21}(x,y,I,\theta,\lambda,\varepsilon), \\ \dot x_2 &= f_{12}(x,y,I,\theta,\lambda,\varepsilon), \quad & \dot y_2 &= f_{22}(x,y,I,\theta,\lambda,\varepsilon), \\ \dot I_1 &= \varepsilon g_1(x,y,I,\theta,\lambda,\varepsilon), \quad & \dot I_2 &= \varepsilon g_2(x,y,I,\theta,\lambda,\varepsilon) \end{aligned} \tag{2.5}$$

and the system (2.5) has the following form in U_0

$$\begin{aligned} \dot x_1 &= [\lambda_1+\cdots]x_1+O(|y|)\cdot O(|x_2|)+\varepsilon g_1^z(z,\theta,\lambda,\varepsilon), \\ \dot y_1 &= [-\lambda_2+\cdots]y_1+O(|x|)\cdot O(|y_2|)+\varepsilon g_2^z(z,\theta,\lambda,\varepsilon), \\ \dot x_2 &= [A_2+\cdots]x_2+(O(|y|))x_1+O(x_1^2)+\varepsilon g_3^z(z,\theta,\lambda,\varepsilon), \\ \dot y_2 &= [-B_2+\cdots]y_2+(O(|x|))y_1+O(y_1^2)+\varepsilon g_4^z(z,\theta,\lambda,\varepsilon), \\ \dot I_1 &= \varepsilon[(C_1(\lambda)+\cdots)I_1+(O(|x|)+O(|y|))\cdot O(|I_2|)]+\varepsilon^2 g_5^I, \\ \dot I_2 &= \varepsilon[(-C_2(\lambda)+\cdots)I_2+(O(|x|)+O(|y|))\cdot O(|I_1|)]+\varepsilon^2 g_6^I, \end{aligned} \tag{2.6}$$

where $\mathrm{Re}(\sigma(A_2))>a$, $\mathrm{Re}(\sigma(-B_2))<-b$, $z=(x,y,I)$, $\sigma(*)$ means all eigenvalues of the matrix $(*)$ and due to (H$_3$) we know $g_i^z=O(z)$, $g_j^I=O(z)$, $i=1,2,3,4$, $j=5,6$.

Remark 2.1 There is a technique problem in [9], the C^∞ bump functions h^u and h^{cs} defined on the open cones will cause the discontinuous of the vector field at O. Therefore, we should modify the definition of the bump functions so that h^u and h^{cs} are defined on balls centered at O.

3 Poincaré Map

Our study will be based on the analysis of the poincaré map defined on some local transversal section of Γ, so we need to set up a local coordinate system near Γ. Denote $f = (f_{11}, f_{21}, f_{12}, f_{22})$, $r(t) = (\hat{r}(t), 0) = (x_1(t), y_1(t), x_2(t), y_2(t), 0)$, where 0 means the origin of the I space. Choose a neighborhood U_0 small enough, and T large enough such that $r(\pm T) \in U$, $x_1(-T) = \delta > 0$, $y_1(T) = \delta > 0$, where δ is sufficiently small such that $\{(x, y, I) : |x|, |y|, |I| < \frac{3\delta}{2}\} \subset U_0$.

Denote

$$A(t) = \Big(\frac{\partial f}{\partial(x, y, I)}(r(t), 0),\ \varepsilon\frac{\partial(g_1, g_2)}{\partial(x, y, I)}(r(t), 0)\Big)^T\Big|_{\varepsilon=0}.$$

Consider the linear variational system of $(2.5)|_{\varepsilon=0}$

$$\dot{Z} = A(t)Z \tag{3.1}$$

and its adjoint system

$$\dot{Z} = -A^*(t)Z. \tag{3.2}$$

Now we choose a fundamental solution matrix of (3.1) $U(t) = (u_1(t), u_2(t), \cdots, u_6(t))$ satisfying

$$\begin{aligned}
&u_1(t) \in (T_{r(t)}W^u + T_{r(t)}W^s)^c \cap (T_{r(t)}W^c_{\rm loc})^c,\\
&u_2(t) = -\frac{\dot{r}(t)}{|\dot{r}(T)|} \in T_{r(t)}W^u \cap T_{r(t)}W^s,\\
&u_3(t) \in T_{r(t)}W^u, \quad u_4(t) \in T_{r(t)}W^s,\\
&u_5(t),\ u_6(t) \in T_{r(t)}W^c_{\rm loc}.
\end{aligned}$$

Notice that when $\varepsilon = 0$, $\dot{I} \equiv 0$. So similar to [7], we get

Lemma 3.1 *There exist suitable $u_1(t)$, $u_3(t)$, $\cdots$, $u_6(t)$ such that*

$$U(T) = \begin{pmatrix} u_{11} & 0 & u_{31} & 0 & u_{51} & 0\\ u_{12} & 1 & u_{32} & 0 & u_{52} & 0\\ 0 & 0 & u_{33} & 0 & u_{53} & 0\\ u_{14} & u_{24} & u_{34} & Id & u_{54} & 0\\ 0 & 0 & 0 & 0 & Id & 0\\ 0 & 0 & 0 & 0 & 0 & Id \end{pmatrix}, \quad U(-T) = \begin{pmatrix} 0 & u_{21} & 0 & u_{41} & 0 & u_{61}\\ 1 & 0 & 0 & u_{42} & 0 & u_{62}\\ u_{13} & u_{23} & Id & u_{43} & 0 & u_{63}\\ 0 & 0 & 0 & u_{44} & 0 & u_{64}\\ 0 & 0 & 0 & 0 & Id & 0\\ 0 & 0 & 0 & 0 & 0 & Id \end{pmatrix},$$

where $u_{21} < 0$, $\det u_{ii} \neq 0$, moreover for $T \gg 1$ and $j \neq i$, $|u_{ij}u_{ii}^{-1}| \ll 1$, $i = 1, 2, 3, 4$, $|u_{24}| \ll 1$, $|u_{23}| \ll |u_{21}|$.

Proof Let $Z = (z_1, \cdots, z_6)^*$ be the solution of (3.1). Based on $\frac{\partial(\varepsilon g_1, \varepsilon g_2)}{\partial(x,y,I)} = 0$ as $\varepsilon = 0$, we have

$$\dot{z}_5 = 0, \quad \dot{z}_6 = 0.$$

If we take $u_5(-T) = (0, 0, 0, 0, Id, 0)^*$, and $u_6(T) = (0, 0, 0, 0, 0, Id)^*$, then it follows that

$$u_5(T) = (u_{51}, \cdots, u_{54}, I, 0)^*, \quad u_6(-T) = (u_{61}, \cdots, u_{64}, 0, I)^*.$$

That $u_{21} < 0$ and the existence of $u_3(t)$ and $u_4(t)$ with the given expressions at $\pm T$ are clear if we notice that Γ leaves the origin along the positive x_1- axis and that $u_3(-T)$ and $u_4(T)$ can be taken as $(0,0,Id,0,0,0)^*$ and $(0,0,0,Id,0,0)^*$, respectively. The inequalities $\det u_{33} \neq 0$ and $u_{44} \neq 0$ follow directly from the strong inclination property.

Based on (H_2), we take $u_0(t) \in (T_{r(t)}W^u + T_{r(t)}W^s)^c \cup (T_{r(t)}W^c_{\mathrm{loc}})^c$ satisfying $u_0(-T) = (0,1,0,0,0,0)^*$, $u_0(T) = (\tilde{u}_{11}, \cdots, \tilde{u}_{14}, 0, 0)^*$. Now let $u_1(t) = u_0(t) - u_{13}u_3(t)$, $u_{13} = \tilde{u}_{13}u_{33}^{-1}$. Then we have $u_1(t) \in (T_{r(t)}W^u + T_{r(t)}W^s)^c \cup (T_{r(t)}W^c_{\mathrm{loc}})^c$, and $u_1(-T) = (0,1,u_{13},0,0,0)^*$, $u_1(T) = (u_{11}, u_{12}, 0, u_{14}, 0, 0)^*$, where $u_{1i} = \tilde{u}_{1i} - \tilde{u}_{13}u_{33}^{-1}u_{3i}$, $i = 1,2,4$. Since $u_1(T) \in (T_{r(T)}W^u + T_{r(T)}W^s)^c \cup (T_{r(T)}W^c_{\mathrm{loc}})^c$, we must have $u_{11} \neq 0$.

The others can be proved similarly to [7, 9].

Denote $\Psi(t) = (\psi_1^*, \cdots, \psi_6^*) = U^{-1*}(t)$. Then $\Psi(t)$ is a fundamental solution matrix of (3.2). By using $\Psi^*(t)U(t) = Id$, the invariance of $T_{r(t)}W^u + T_{r(t)}W^s$ and $T_{r(t)}W^c$, and the hypothesis (H_2), it is easy to see

$$\exp(\mu|t|)\psi_1(t) \in (T_{r(t)}W^u + T_{r(t)}W^s)^c \cap (T_{r(t)}W^c_{\mathrm{loc}})^c \to 0, \quad t \to \pm\infty, \ \forall\, 0 \leq \mu < \min\{\lambda_1, \lambda_2\}.$$

For easy application, instead of using u_i as coordinate vectors directly, we take

$$v_i(t) = u_i(t), \quad i = 1,2,3,4,$$

and

$$v_5(t) = (0,0,0,0,I_{m_1\times m_1},0), \quad v_6(t) = (0,0,0,0,0,I_{m_2\times m_2}).$$

Clearly the I components of v_1, v_2, v_3, v_4 are all zero. In the following , we regard

$$\Big(v_1, -\frac{\dot{r}}{|\dot{r}(T)|}, v_3, v_4, v_5, v_6\Big)$$

as a local coordinate system of (2.5) along Γ, which will be used to establish the poincaré map induced by the flow of (2.5) defined in some tubular neighborhood of homoclinic orbit Γ in the next section. Denote

$$s(t) = r(t) + \sum_{i\neq 2} v_i(t)n_i. \tag{3.3}$$

Let

$$S_0 = \{(x,y,I) = s(T) : |n_i| < \delta\}, \quad S_1 = \{(x,y,I) = s(-T) : |n_i| < \delta\}.$$

It is easy to see S_0 and S_1 are two poincaré sections at $r(T)$ and $r(-T)$, respectively. Let δ be small enough so that S_0, $S_1 \subset U$.

3.1 Establishment of the regular map F_1

First we use the flow of (2.5) to establish the regular map F_1: $S_1 \to S_0$. Set

$$(x_1, y_1, x_2, y_2, I_1, I_2)^* = s(t), \quad t \in [-T, T].$$

Then substituting it into (2.5) with the I component satisfying (2.4), we get

$$\begin{aligned}&(\dot{r}(t) + \sum_{i\neq 2} \dot{v}_i(t)n_i) + \sum_{i\neq 2} v_i(t)\dot{n}_i\\ &= \dot{r}(t) + A(t)\sum_{i\neq 2} v_i(t)n_i + \varepsilon(f_\varepsilon(r(t),\hat{\alpha},0), \tilde{g}^I(r(t),\hat{\alpha},0))^* + O(2),\end{aligned}$$

where $\hat{\alpha} = (\theta, \lambda)$, $\theta = \theta_0 + \omega t$. Multiplying the above equations by ψ_1^*, ψ_3^*, ψ_4^*, respectively, we have

$$\begin{aligned} \dot{n}_i &= \varepsilon \tilde{\psi}_i^* [f_\varepsilon(r(t), \hat{\alpha}, 0)] + O(2), \\ \dot{n}_j &= \varepsilon g(r(t), \hat{\alpha}, 0) + O(2), \end{aligned} \tag{3.4}$$

where $\tilde{\psi}_i$ is the $x - y$ component of ψ_i, $i = 1, 3, 4$, $j = 5, 6$, $g = (g_1, g_2)^*$. Now integrating two sides of (3.4) from $-T$ to T, we have

$$\begin{aligned} n_1(t) &= n_1(-T) + \varepsilon \int_{-T}^{t} \tilde{\psi}_1^* f_\varepsilon(r(s), \hat{\alpha}, 0)]ds + \text{h.o.t.}, \\ n_3(t) &= n_3(-T) + \varepsilon \int_{-T}^{t} \tilde{\psi}_3^* f_\varepsilon(r(s), \hat{\alpha}, 0)]ds + \text{h.o.t.}, \\ n_4(t) &= n_4(-T) + \varepsilon \int_{-T}^{t} \tilde{\psi}_4^* f_\varepsilon(r(s), \hat{\alpha}, 0)]ds + \text{h.o.t.}, \\ n_5(t) &= n_5(-T) + \varepsilon \int_{-T}^{t} g_1(r(s), \hat{\alpha}, 0)ds + \text{h.o.t.}, \\ n_6(t) &= n_6(-T) + \varepsilon \int_{-T}^{t} g_2(r(s), \hat{\alpha}, 0)ds + \text{h.o.t.}, \end{aligned}$$

where the higher order terms h.o.t. include $O(\varepsilon^2)$, $O(n_i^2(-T))$, $O(\varepsilon n_i(-T))$, etc. If we denote

$$\begin{aligned} M_i(T, \alpha) &= \int_{-T}^{T} \psi_i^* f_\varepsilon(r(t), \hat{\alpha}, 0)]dt, \quad i = 1, 3, 4, \\ M_5(T, \alpha) &= \int_{-T}^{T} g_1(r(t), \hat{\alpha}, 0)dt, \quad M_6(T, \alpha) = \int_{-T}^{T} g_2(r(t), \hat{\alpha}, 0)dt, \end{aligned}$$

where $\alpha = (\theta_0, \lambda)$, then the regular map $F_1 : S_1 \to S_0$,

$$q_1(n_1(-T), n_3(-T), n_4(-T), n_5(-T), n_6(-T)) \to q_2(n_1(T), n_3(T), n_4(T), n_5(T), n_6(T))$$

can be expressed in the following form

$$\begin{aligned} n_1(T) &= n_1(-T) + \varepsilon M_1(T, \alpha) + \text{h.o.t.}, \\ n_3(T) &= n_3(-T) + \varepsilon M_3(T, \alpha) + \text{h.o.t.}, \\ n_4(T) &= n_4(-T) + \varepsilon M_4(T, \alpha) + \text{h.o.t.}, \\ n_5(T) &= n_5(-T) + \varepsilon M_5(T, \alpha) + \text{h.o.t.}, \\ n_6(T) &= n_6(-T) + \varepsilon M_6(T, \alpha) + \text{h.o.t.} \end{aligned} \tag{3.5}$$

We call $(M_1(T, \alpha), M_3(T, \alpha), M_4(T, \alpha), M_5(T, \alpha), M_6(T, \alpha))$ the Melnikov functions.

3.2 Establishment of the singular map F_2

Now we consider the map induced by the flow (2.6) in U_0

$$F_2 : S_0 \to S_1, \quad q_0(x_{10}, y_{10}, x_{20}, y_{20}, I_{10}, I_{20}) \to q_1(x_{11}, y_{11}, x_{21}, y_{21}, I_{11}, I_{21}).$$

Notice that F_2 is not well defined at $W^s \cap S_0$, where the x-component is zero, but we can extend F_2 to $W^s \cap S_0$ continuously so that $F_2(W^s \cap S_0) \subset W^u \cap S_1$ and $F(0, \delta, 0, \delta_y, I_{10}, I_{20}) = (0, \delta, 0, \delta_x, I_{11}, I_{21})$ as $\varepsilon = 0$.

First we assume $\lambda_1 < \lambda_2$. Let $\tau(\rho)$ be the flying time from $q_0 \in S_0$ to $q_1 \in S_1$. We can get the following expression by variation of constants formula

$$
\begin{aligned}
x_1(t) &= e^{\lambda_1(\varepsilon)(t-T-\tau)}\Big[x_{11} + \varepsilon\int_{T+\tau}^{t} e^{\lambda_1(-s+T+\tau)}\tilde{g}_1 ds\Big] + \text{h.o.t.},\\
y_1(t) &= e^{-\lambda_2(\varepsilon)(t-T)}\Big[y_{10} + \varepsilon\int_{T}^{t} e^{\lambda_2(s-T)}\tilde{g}_2 ds\Big] + \text{h.o.t.},\\
x_2(t) &= e^{A_2(\varepsilon)(t-T-\tau)}x_{21} + \hat{x}_2(t) + O(\varepsilon) + \text{h.o.t.},\\
y_2(t) &= e^{-B_2(\varepsilon)(t-T)}y_{20} + \hat{y}_2(t) + O(\varepsilon) + \text{h.o.t.},\\
I_1(t) &= e^{\epsilon C_1(\lambda)(t-T-\tau)}I_{11} + \text{h.o.t.},\\
I_2(t) &= e^{-\varepsilon C_2(\lambda)(t-T)}I_{20} + \text{h.o.t.},
\end{aligned}
$$

where

$$
\begin{aligned}
T &\le t \le T+\tau,\\
\tilde{g}_1 &= [O(x_2) + O(y_1) + O(y_2) + O(I)],\\
\tilde{g}_2 &= [O(x_1) + O(x_2) + O(y_2) + O(I)],\\
\hat{x}_2(t) &= O(|x_{11}|)\cdot O(|y|) + O(x_{11}^2),\\
\hat{y}_2(t) &= O(|y_{10}|)\cdot O(|x|) + O(y_{10}^2),\\
O(|y|) &= O(|y_{10}|) + O(|y_{20}|),\\
O(|x|) &= O(|x_{11}|) + O(|x_{21}|).
\end{aligned}
$$

In order to guarantee the differentiability of the map at the origin, we set $s = e^{-\lambda_1(\varepsilon)\tau}$, which is called Silnikov time (see [10]). Notice that $s \to 0$ as $\tau \to +\infty$. Thus we obtain the singular map F_2 of (2.5) defined by Silnikov variables $(x_{21}, y_{20}, I_{11}, I_{20}, s)$ in U. The expression of F_2 is given by

$$
\begin{aligned}
x_{10} &= x_1(T) \approx s x_{11} + \varepsilon\frac{-b_1}{\lambda_1+\lambda_2}y_{10},\\
y_{11} &= y_1(T+\tau) \approx s^{\lambda_2/\lambda_1}y_{10} + \varepsilon\frac{b_2}{\lambda_1+\lambda_2}x_{11},\\
x_{20} &= x_2(T) \approx s^{A_2/\lambda_1}x_{21} + O(|x_{11}|)\cdot O(|y_0|) + O(x_{11}^2),\\
y_{21} &= y_2(T+\tau) \approx s^{B_2/\lambda_1}y_{20} + O(|y_{10}|)\cdot O(|x_1|) + O(y_{10}^2),\\
I_{10} &= I_1(T) \approx s^{\varepsilon C_1(\lambda)/\lambda_1}I_{11},\\
I_{21} &= I_2(T+\tau) \approx s^{\varepsilon C_2(\lambda)/\lambda_1}I_{20},
\end{aligned}
\tag{3.6}
$$

where $b_1 = \frac{\partial\tilde{g}_1}{\partial y_1}|_0$ and $b_2 = \frac{\partial\tilde{g}_2}{\partial x_1}|_0$.

Now we use (3.3) to seek the new coordinate of q_0 and q_1 in the new coordinate system. Let

$$
\begin{aligned}
q_0 &= (x_{10}, y_{10}, x_{20}^*, y_{20}^*, I_{10}^*, I_{20}^*)^* = r(T) + Z(T)(n_{10}, 0, n_{30}^*, n_{40}^*, n_{50}^*, n_{60}^*)^*,\\
q_1 &= (x_{11}, y_{11}, x_{21}^*, y_{21}^*, I_{11}^*, I_{21}^*)^* = r(-T) + Z(-T)(n_1, 0, n_3^*, n_4^*, n_5^*, n_6^*)^*,
\end{aligned}
$$

where $Z(t)=(v_1(t),v_2(t),v_3(t),\cdots,v_6(t))$. Then, based on $r(T)=(0,\delta,0^*,\delta_y^*,0^*,0^*)^*$, $r(-T)=(\delta,0,\delta_x^*,0^*,0^*,0^*)^*$, ($|\delta_x^*|,|\delta_y^*|\ll\delta$), we have

$$\begin{aligned}(n_{10},0,n_{30},n_{40},n_{50},n_{60})^*&=Z^{-1}(T)(x_{10},y_{10}-\delta,x_{20},y_{20}-\delta_y,I_{10},I_{20})^*,\\(n_1,0,n_3,n_4,n_5,n_6)^*(-T)&=Z^{-1}(-T)(x_{11}-\delta,y_{11},x_{21}-\delta_x,y_{21},I_{11},I_{21})^*,\end{aligned}\tag{3.7}$$

which are equivalent to

$$\begin{aligned}&n_{10}=u_{11}^{-1}(x_{10}-u_{31}u_{33}^{-1}x_{20}),\quad n_{30}=u_{33}^{-1}x_{20},\\&n_{40}=y_{20}-\delta_y-u_{14}u_{11}^{-1}x_{10}+(u_{14}u_{11}^{-1}u_{31}-u_{34})u_{33}^{-1}x_{20},\\&n_{50}=I_{10},\quad n_{60}=I_{20},\quad y_{10}\approx\delta\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&n_1(-T)=y_{11}-u_{42}u_{44}^{-1}y_{21},\\&n_3(-T)=x_{21}-\delta_x-u_{13}y_{11}+(u_{13}u_{42}-u_{43})u_{44}^{-1}y_{21},\\&n_4(-T)=u_{44}^{-1}y_{21},\\&n_5(-T)=I_{11},\quad n_6(-T)=I_{21},\quad x_{11}\approx\delta.\end{aligned}\tag{3.9}$$

Now based on (3.5)–(3.9), we can establish the poincaré map of (2.5) near Γ

$$\begin{aligned}&F=F_1\circ F_2:\quad q_0\in S_0\to q_2\in S_0,\\&F(n_{10},n_{30},\cdots,n_{60})\to(n_1(T),n_3(T),\cdots,n_6(T)),\end{aligned}$$

which is given by

$$\begin{aligned}&n_1(T)=\delta s^{\frac{\lambda_2}{\lambda_1}}+\varepsilon M_1(T,\alpha)+\varepsilon\frac{b_2}{\lambda_1+\lambda_2}x_{11}+\text{h.o.t.},\\&n_3(T)=x_{21}-\delta_x-\delta u_{13}s^{\frac{\lambda_2}{\lambda_1}}+\varepsilon M_3(T,\alpha)+O(\varepsilon)+\text{h.o.t.},\\&n_4(T)=u_{44}^{-1}s^{\frac{B_2}{\lambda_1}}y_{20}+\delta s^{\frac{\lambda_2}{\lambda_1}}O(|x_1|)+\varepsilon M_4(T,\alpha)+O(\varepsilon)+\text{h.o.t.},\\&n_5(T)=I_{11}+\varepsilon M_5(T,\alpha)+\text{h.o.t.},\\&n_6(T)=s^{\frac{\varepsilon C_2(T,\lambda)}{\lambda_1}}I_{20}+\varepsilon M_6(\alpha)+\text{h.o.t.}\end{aligned}\tag{3.10}$$

And its associated successor function

$$G(s,x_{21},y_{20},I_{11},I_{20})=(G_1,G_3,\cdots,G_6)=(F-I)(n_{10},n_{30},\cdots,n_{60})\tag{3.11}$$

is given by

$$\begin{aligned}&G_1=\delta s^{\frac{\lambda_2}{\lambda_1}}-u_{11}^{-1}\delta s+\varepsilon M_1(T,\alpha)+\varepsilon\frac{b_1+b_2}{\lambda_1+\lambda_2}\delta+\text{h.o.t.},\\&G_3=x_{21}-\delta_x-\delta u_{13}s^{\frac{\lambda_2}{\lambda_1}}-u_{33}^{-1}\delta sO(|y_0|)+\varepsilon M_3(T,\alpha)+O(\varepsilon)+\text{h.o.t.},\\&G_4=-y_{20}+\delta_y+u_{14}u_{11}^{-1}\delta s+\varepsilon M_4(T,\alpha)+O(\varepsilon)+\text{h.o.t.},\\&G_5=\Big(Id-s^{\frac{\varepsilon C_1(\lambda)}{\lambda_1}}\Big)I_{11}+\varepsilon M_5(T,\alpha)+\text{h.o.t.},\\&G_6=\Big(s^{\frac{\varepsilon C_2(\lambda)}{\lambda_1}}-Id\Big)I_{20}+\varepsilon M_6(T,\alpha)+\text{h.o.t.}\end{aligned}\tag{3.12}$$

It is easy to see system (2.5) has homoclinic orbit (resp. periodic orbit) near Γ if and only if $G=0$ has solution satisfying $s=0$ (resp. $s>0$).

4 Existence of Transversal Homoclinic Orbit

In this section, we use the above successor function to study the existence and transversality of homoclinic orbit. Consider the equation

$$G(s, x_{21}, y_{20}, I_{11}, I_{20}) = 0. \tag{4.1}$$

Due to (3.12), equation $(4.1)|_{s=0}$ is equivalent to

$$\begin{aligned}
&\varepsilon K(\alpha) + O(\varepsilon^2) = 0,\\
&x_{21} - \delta_x + \varepsilon M_3(T,\alpha) + O(\varepsilon) + \text{h.o.t.} = 0,\\
&-y_{20} + \delta_y + \varepsilon M_4(T,\alpha) + O(\varepsilon) + \text{h.o.t.} = 0,\\
&I_{11} + \varepsilon M_5(T,\alpha) + \text{h.o.t.} = 0,\\
&-I_{20} + \varepsilon M_6(T,\alpha) + \text{h.o.t.} = 0,
\end{aligned} \tag{4.2}$$

where $K(\alpha) = M_1(T,\alpha) + \frac{b_1+b_2}{\lambda_1+\lambda_2}\delta$. If there is an $\alpha = \alpha_0 = (\theta_0, \lambda_0)$ such that $K(\alpha_0) = 0, K_\lambda(\alpha_0) \neq 0$, then by the implicit function theorem, we can claim that there is a $(k-1)$ dimensional surface $\lambda = \lambda(\theta_0,\varepsilon)$, when $\lambda = \lambda(\theta_0,\varepsilon)$ and $0 < \varepsilon \ll 1$. System (2.2) has a homoclinic orbit $\widetilde{\Gamma}_\varepsilon$ satisfying $\widetilde{\Gamma}_\varepsilon \to \Gamma \times \{I = 0\}$ as $\varepsilon \to 0^+$. And the coordinates $\{(x_1(t), y_1(t), x_2(t), y_2(t), I_1(t), I_2(t))\}$ on $\widetilde{\Gamma}_\varepsilon$ satisfy $x_1(-T) \approx \delta$, $y_1(T) \approx \delta$, $x_2(-T) = \hat{x}_{21}$, $y_2(T) = \hat{y}_{20}$, $I_1(-T) = \widehat{I}_{11}$, $I_2(T) = \widehat{I}_{20}$, where $\hat{x}_{21}$, $\hat{y}_{20}$, $\widehat{I}_{11}$, $\widehat{I}_{20}$ are the unique solutions of the last four equations of (4.2) as $\lambda = \lambda(\theta_0,\varepsilon)$, respectively.

Remark 4.1 In case $\lambda_1 > \lambda_2$, we take $s = e^{-\lambda_2\tau}$, similar results can be obtained.

Then for system (1.1), we obtain

Theorem 4.1 *Suppose that hypotheses* (H$_1$)–(H$_3$) *are valid. If there is an* $\alpha = \alpha_0 = (\theta_0, \lambda_0)$ *such that* $K(\alpha_0) = 0, K_\lambda(\alpha_0) \neq 0$*, then there exists a* $(k-1)$*-dimensional parameter surface* $\lambda = \lambda(\theta_0,\varepsilon)$ *satisfying* $\lambda(\theta_0, 0) = \lambda_0$*, such that system* (1.1) *has a unique* 1*-homoclinic orbit* $\Gamma_\varepsilon : \widetilde{\Gamma}_\varepsilon \times T^l_{\theta_0}$ *near* $\Gamma \times \{I = I_0\} \times T^l$ *for* $\lambda = \lambda(\theta_0,\varepsilon)$ *and* $0 < \varepsilon \ll 1$*, where* $T^l_{\theta_0} = \{\theta \in T^l : \theta = \theta_0 + \omega t\}$.

Obviously, the orbit Γ_ε is homoclinic to the l dimensional invariant torus M.

Next we use the corresponding tangent map of F to study the transversality problem for the homoclinic orbit.

By the transversality theory, if we want to prove the stable and unstable manifolds intersect transversely, we only need to prove that their tangent spaces at the intersection points can span the whole space. It follows from the above discussion that, to show the transversality of Γ_ε, it suffices to show that the stable manifolds $W^s_\varepsilon(M)$ and center-unstable manifolds $W^{cu}_\varepsilon(M)$ intersect transversaly when they are restricted at the section $S_0 \times T^l$.

Owing to the discussion in Sections 2 and 3, we see that, at the intersection point (q_0, θ) of $W^s_\varepsilon(M)$ and $W^{cu}_\varepsilon(M)$ restricted in the section $S_0 \times T^l$, their tangent spaces have the following expressions

$$T_{(q_0,\theta)} W^s_\varepsilon(M) = \text{span}\{v_4(T) + O(\varepsilon), v_7(T)\}$$

and

$$T_{(q_0,\theta)} W^{cu}_\varepsilon(M) = \begin{pmatrix} \frac{\partial F_1}{\partial q_1}, & \frac{\partial F_1}{\partial \theta_0} \\ 0 & Id \end{pmatrix} T_{(q_1,\theta)} W^{cu}_\varepsilon(M),$$

where $v_7(T) = (0,0,0,0,0,0,Id)^*$ means the l dimensional unit line vectors, $T_{(q_1,\theta)}W_\varepsilon^{cu}(M)$ is the tangent space of $W_\varepsilon^{cu}(M)$ restricted in $S_1 \times T^l$ at the intersection point(q_1,θ) with $S_1 \times T^l$, which satisfies

$$T_{(q_1,\theta)}W_\varepsilon^{cu}(M) = \text{span}\{v_3(-T) + O(\varepsilon), v_5(-T), v_6(-T), v_7(T)\}.$$

In order to prove $W_\varepsilon^s(M)$ and $W_\varepsilon^{cu}(M)$ intersect transversaly, we need to modify the Poincaré map $F_1\colon S_1 \to S_0$ defined by (3.5), so that it can reflect the dependent relation to the coordinate of original variable q_1.

Based on (3.3) and (3.7), we know

$$\begin{aligned} q_2 &= r(T) + Z(T)(n_1(T), 0, n_3(T)^*, n_4(T)^*, n_5(T)^*, n_6(T)^*)^*,\\ q_1 &= r(-T) + Z(-T)(n_1(-T), 0, n_3(-T)^*, n_4(-T)^*, n_5(-T)^*, n_6(-T)^*)^*, \end{aligned}$$

which suggests the following expression of F_1 in the original coordinate system

$$q_2 = F_1(q_1) = r(T) + Z(T)[Z_*^{-1}(-T)(q_1 - r(-T)) + \varepsilon M(T,\alpha) + \text{h.o.t.}], \tag{4.3}$$

where

$$\begin{aligned} M &= (M_1, 0, M_3^*, M_4^*, M_5^*, M_6^*)^*,\\ r(T) &\equiv (0, \delta, 0^*, \delta_y^*, 0^*, 0^*)^*,\\ r(-T) &\equiv (\delta, 0, \delta_x^*, 0^*, 0^*, 0^*)^*,\\ Z_*^{-1}(-T) &= \begin{pmatrix} 0 & 1 & 0 & -u_{42}u_{44}^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -u_{23}u_{21}^{-1} & -u_{13} & Id & w & 0 & 0\\ 0 & 0 & 0 & u_{44}^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & Id & 0\\ 0 & 0 & 0 & 0 & 0 & Id \end{pmatrix}, \end{aligned}$$

where $w = (u_{13}u_{42} + u_{21}^{-1}u_{23}u_{41} - u_{43})u_{44}^{-1}$. Consequently we get

$$T_{(q_0,\theta)}W_\varepsilon^{cu}(M) = \begin{pmatrix} Z(T)Z_*^{-1}(-T) + O(\varepsilon) & \varepsilon Z(T)\frac{\partial M}{\partial \theta_0}\\ 0 & Id \end{pmatrix} T_{(q_1,\theta)}W_\varepsilon^{cu}(M).$$

Let

$$P = \begin{pmatrix} Z^{-1}(T) & 0\\ 0 & Id \end{pmatrix}.$$

It is obvious that the transversality of $T_{(q_0,\theta)}W_\varepsilon^s(M)$ and $T_{(q_0,\theta)}W_\varepsilon^{cu}(M)$ restricted in the section $S_0 \times T^l$ is invariant under the action of P. If we notice that $v_4(T)$ and $v_7(T)$ are also invariant under the action of P, while $v_3(-T)$, $v_5(-T)$, $v_6(-T)$ and $v_7(T)$ are invariant under the action of $P\begin{pmatrix} \frac{\partial F_1}{\partial q_1}, & \frac{\partial F_1}{\partial \theta_0}\\ 0 & Id \end{pmatrix}$, then by simple calculation we get

$$\begin{aligned} &\text{span}\{PT_{(q_0,\theta)}W_\varepsilon^s(M), PT_{(q_0,\theta)}W_\varepsilon^{cu}(M)\}\\ &= \text{span}\{\varepsilon v_0, v_3(-T) + O(\varepsilon), v_4(T) + O(\varepsilon), v_5(-T) + O(\varepsilon), v_6(-T) + O(\varepsilon), v_7(T)\}\\ &= \text{span}\{\varepsilon v_0, w_3 + O(\varepsilon), \cdots, w_{n+m} + O(\varepsilon), v_7(T)\}, \end{aligned}$$

where $v_0 = (\frac{\partial M_1}{\partial \theta_0}, 0, 0, 0, 0, 0, 0)^*$ means l line vectors, their first components are $\frac{\partial M_1}{\partial \theta_{01}}, \cdots, \frac{\partial M_1}{\partial \theta_{0l}}$ respectively; w_i means the i unit basis in the $n+m+l$ dimensional space, $3 \le i \le n+m$.

Due to $(0,1,0,0,0,0)^* \approx v_2(T) \notin S_0$, we get the following result immediately.

Theorem 4.2 *If the conditions of Theorem* 4.1 *hold, and* $\frac{\partial M_1(T,\alpha_0)}{\partial \theta_0}$ *has at least one nonzero component, then the homoclinic orbit* Γ_ε *obtained in Theorem* 4.1 *is transversal.*

Remark 4.2 The above method can be applied to study the existence of transversal homoclinic orbit of the more general system

$$\begin{aligned}
&\dot{x} = f(x, I) + \varepsilon g^x(x, I, \theta, \lambda, \varepsilon),\\
&\dot{I} = \varepsilon g^I(x, I, \theta, \lambda, \varepsilon),\\
&\dot{\theta} = \Omega(x, I) + \varepsilon g^\theta(x, I, \theta, \lambda, \varepsilon).
\end{aligned}$$

5 Existence of Multi-pulse Orbit

Now we consider the solutions of $G = 0$ satisfying $s > 0$. Since

$$\widetilde{G} = \frac{\partial G(s, x_{21}, y_{20}, I_{11}, I_{20})}{\partial(s, x_{21}, y_{20}, I_{11}, I_{20})}$$

is degenerated at $(s, x_{21}, y_{20}, I_{11}, I_{20}) = 0$ and $\varepsilon = 0$, thus we can not use implicit function theorem to get the solutions of $G = 0$. But it follows from (3.12) that the second and third equations of (4.1): $G_3 = 0$, $G_4 = 0$ always have a unique solution $\tilde{x}_{21} = x_{21}(s, \varepsilon)$, $\tilde{y}_{20} = y_{20}(s, \varepsilon)$ as ε, s sufficiently small. Substituting it into $G_1 = 0$, $G_5 = 0$ and $G_6 = 0$, we see $G = 0$ is equivalent to $G_1 = 0$, $G_5 = 0$ and $G_6 = 0$. In case $\lambda_2 > \lambda_1$ and $K(\alpha) \neq 0$, it suffices to consider the following equations

$$\begin{aligned}
\delta s^{\frac{\lambda_2}{\lambda_1}} - u_{11}^{-1}\delta s + \varepsilon K(\alpha) + \text{h.o.t.} &= 0,\\
(Id - s^{\frac{\varepsilon C_1(\lambda)}{\lambda_1}}) I_{11} + \varepsilon M_5(T, \alpha) + \text{h.o.t.} &= 0,\\
(s^{\frac{\varepsilon C_2(\lambda)}{\lambda_1}} - Id) I_{20} + \varepsilon M_6(T, \alpha) + \text{h.o.t.} &= 0.
\end{aligned} \tag{5.1}$$

Now if $u_{11}K(\alpha) > 0$, then $G_1 = 0$ has a unique solution $s = \varepsilon\delta^{-1}u_{11}K(\alpha) + \text{h.o.t.} > 0$, Which also means $s = O(\varepsilon)$. Using Taylor formula, $s^{\varepsilon C_1(\lambda)/\lambda_1}$ and $s^{-\varepsilon C_2(\lambda)/\lambda_1}$ can be expressed approximately as follows

$$Id - s^{\varepsilon C_1/\lambda_1} \approx -\varepsilon \lambda_1^{-1} C_1 \ln s, \quad Id - s^{-\varepsilon C_2/\lambda_1} \approx \varepsilon \lambda_1^{-1} C_2 \ln s.$$

Then from the last two equations of (5.1), we can get

$$\begin{aligned}
I_{11} &= \frac{\lambda_1 C_1^{-1} M_5(T, \alpha)}{\ln s} + O(\ln^{-2} s),\\
I_{20} &= -\frac{\lambda_1 C_2^{-1} M_6(T, \alpha)}{\ln s} + O(\ln^{-2} s).
\end{aligned}$$

Thus system (1.1) has an orbit near $\Gamma \times \{I_0\} \times T^l$ as following

$$\widetilde{\Gamma}_\varepsilon = \{(x_1(t), y_1(t), x_2(t), y_2(t), I_1(t), I_2(t), \theta(t)) : \ t \in R\}$$

and

$$\widetilde{\Gamma}_\varepsilon \to \Gamma \times \{I_0\} \times \{\theta_0 + \omega t : t \in R\}, \quad \varepsilon \to 0,$$

which satisfies

$$\begin{aligned}
&x_1(-T) \approx \delta, \quad y_1(T) \approx \delta, \quad x_2(-T) = \tilde{x}_{21}, \quad y_2(T) = \tilde{y}_{20},\\
&I_1(-T) = I_{11}, \quad I_2(T) = I_{20}, \quad \theta(t) = \theta_0 + \omega t,\\
&x_1(T_0) = x_1(0), \quad y_1(T_0) = y_1(0), \quad x_2(T_0) = x_2(0), \quad y_2(T_0) = y_2(0),\\
&I_1(T_0) = I_1(0), \quad I_2(T_0) = I_2(0),
\end{aligned}$$

where

$$T_0 = 2T + \tau = 2T - \lambda_1^{-1} \ln(\varepsilon \delta^{-1} u_{11} K(\alpha)) + O(\varepsilon).$$

When $\lambda_2 < \lambda_1$, we substitute s by s^{λ_1/λ_2} in (5.1). It is easy to see when $K(\alpha) < 0$, $G_1 = 0$ has a unique solution $s = -\varepsilon\delta^{-1}K(\alpha) + \text{h.o.t.} > 0$; while when $K(\alpha) > 0$, $G_1 = 0$ has no solution satisfying $s > 0$. Then we can discuss the existence of $\widetilde{\Gamma}_\varepsilon$ in a similar way as above.

In general, $\widetilde{\Gamma}_\varepsilon$ is not a periodic orbit or quasi-periodic orbit, it may be an orbit tangling several circle around $\Gamma \times \{I_0\} \times T^l$, we call it the multi-pulse orbit.

As a summation, we get the following conclusion

Theorem 5.1 *If the hypotheses* (H$_1$)–(H$_3$) *hold,* $\lambda_1 \neq \lambda_2$, $K(\alpha) \neq 0$, *then the following is true*

(1) *In case* $\lambda_2 > \lambda_1$, $u_{11}K(\alpha) < 0$ *or* $\lambda_2 < \lambda_1$, $K(\alpha) > 0$, *system* (1.1) *has no* 1*-homoclinic orbit and* 1*-periodic orbit near* $\Gamma \times \{I = I_0\} \times T^l$;

(2) *In case* $\lambda_2 > \lambda_1$, $u_{11}K(\alpha) > 0$ *or* $\lambda_2 < \lambda_1$, $K(\alpha) < 0$, *system* (1.1) *has a multi-pulse orbit* $\widetilde{\Gamma}_\varepsilon(\theta_0)$ *near* $\Gamma \times \{I = I_0\} \times T^l$.

Remark 5.1 When the l components of $\omega = (\omega_1, \omega_2, \cdots, \omega_l)$ are all not zero, it is easy to see $\widetilde{\Gamma}_\varepsilon$ is 1-periodic if and only if there exist positive numbers $k_1, \cdots, k_l$ such that

$$T_0 = \frac{2k_1\pi}{\omega_1} = \cdots = \frac{2k_l\pi}{\omega_l}. \tag{5.2}$$

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